MATH. SCAND. 8 (1960), 39–42

SOME REMARKS ON AN ALGEBRAIC IDENTITY

HARALD BERGSTRÖM

There is a well-known connection between the convergence of an infinite product $\prod a_k$ and the corresponding series $\sum (a_k - 1)$. One usually establishes this connection using the exponential function. When dealing with products of abstract elements we do not always have an exponential function in the algebraic systems which we may consider. For such cases I have given an identity, which can be used to find relations of the mentioned form. In its first presentation [2] (it was used but not explicitly given in [1]) it probably seemed more complicated than it really is and the proof was also rather difficult since it reflected the way in which I had found the result. In the present paper I will present a deduction which makes full use of the symmetry properties of the identity.

Consider polynomials of a special form in variables $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ in a ring with a unit element, which we denote by 1. We use the notation

$$\sum! \, y_1 y_2 \ldots y_k x_{k+1} \ldots x_n$$

for the sum of all monomials

$$y_1 y_2 \ldots y_k x_{k+1} \ldots x_n$$

corresponding to the different permutations of the numbers $1, 2, \ldots, n$, the first k factors being always y's, the other factors x's. The number of such monomials is obviously $n!$. Now we define the polynomial

$$(1) \qquad f^{(n)} = \frac{1}{(n+1)!}\{\sum! \, x_1 x_2 \ldots x_n + \sum! \, y_1 x_2 \ldots x_n + \\ + \sum! \, y_1 y_2 x_3 \ldots x_n + \ldots + \sum! \, y_1 y_2 \ldots y_n\}$$

and the polynomials $f_k^{(n)}$, $\bar{u}_{kj}$, $\bar{v}_{kj}$ and

$$(2) \qquad \bar{f}^{(n)} = \frac{1}{n} \sum_{k=1}^{n} f_k^{(n)},$$

where $f_k^{(n)}$ is obtained from $f^{(n)}$ when x_k and y_k are specialized to 1, $\frac{1}{2}\bar{u}_{kj}^{(n)}$

Received November 6, 1959.

is obtained from $f_k^{(n)}$ when x_j is specialized to 1 and y_j to 0, and $\frac{1}{2}\bar{v}_{kj}^{(n)}$ is obtained from $f_k^{(n)}$ when x_j is specialized to 0 and y_j to 0, that is

$$(3)\qquad \begin{cases} f_k^{(n)} = (f^{(n)})_{x_k=y_k=1}\,, \\ \frac{1}{2}\bar{u}_{kj}^{(n)} = (f_k^{(n)})_{x_j=1,\ y_j=0}\,, \\ \frac{1}{2}\bar{v}_{kj}^{(n)} = (f_k^{(n)})_{x_j=0,\ y_j=1}\,. \end{cases}$$

The identity to be proved then has the following form:

$$(4)\qquad \prod_{k=1}^{n} x_k - \prod_{k=1}^{n} y_k$$
$$= \bar{f}^{(n)} \sum_{k=1}^{n} (x_k - y_k) + \frac{1}{2n} \sum_{k=1}^{n} \sum_{j=1}^{n} \{\bar{u}_{kj}^{(n)}(x_k - y_k)(x_j - x_k) + \bar{v}_{kj}^{(n)}(x_k - y_k)(y_j - y_k)\}\,.$$

It may be observed that $\bar{f}^{(n)}$, $\bar{u}_{kj}^{(n)}$ and $\bar{v}_{kj}^{(n)}$ are arithmetic means of monomials in the variables. This is of importance for the applications. If for instance all x_k and y_k belong to a convex commutative semi-group, then $\bar{f}^{(n)}$, $\bar{u}_{kj}^{(n)}$ and $\bar{v}_{kj}^{(n)}$ belong to the same semi-group. (Concerning such applications, see [1].)

When we specialize all x_k to x and all y_k to y, the identity (4) reduces to

$$(5)\qquad x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})\,,$$

of which (4) is a generalization.

Before proving the identity we observe that $f^{(n)}$ is symmetrical in the variables $x_1, x_2, \ldots, x_n$ as well as in the variables $y_1, y_2, \ldots, y_n$ and invariant under the transformation $x_k \to y_k$, $y_k \to x_k$, $k = 1, 2, \ldots, n$. Moreover $f^{(n)}$ is linear in each variable and every monomial contains either y_k or x_k as a factor, but never both of them.

Since $f^{(n)}$ doesn't change if j changes into k and k into j we obviously have

$$(6)\qquad \bar{u}_{kj}^{(n)} = \bar{u}_{jk}^{(n)}, \qquad \bar{v}_{kj}^{(n)} = \bar{v}_{jk}^{(n)}\,.$$

We are now going to prove the identity (4). For this purpose we use the following notations:

$$\prod_{\lambda,\mu} x_k = \begin{cases} 1 & \text{for } \lambda \geqq \mu\,, \\ x_{\lambda+1} \ldots x_\mu & \text{for } \mu > \lambda\,, \end{cases}$$

$$\sum_{\lambda,\mu} x_k = \begin{cases} 0 & \text{for } \lambda \geqq \mu\,, \\ x_{\lambda+1} + x_{\lambda+2} \ldots + x_\mu & \text{for } \mu > \lambda\,. \end{cases}$$

Writing

$$\prod_{0,n} x_k - \prod_{0,n} y_k = x_1 \prod_{1,n} x_\nu - y_1 \prod_{1,n} x_\nu + \left(\prod_{0,1} y_\nu\right) x_2 \prod_{2,n} x_\nu -$$
$$- \left(\prod_{0,1} y_\nu\right) y_2 \prod_{2,n} x_\nu + \dots + \left(\prod_{0,n-1} y_\nu\right) x_n - \left(\prod_{0,n-1} y_\nu\right) y_n ,$$

we get the identity (still holding true when the letters denote elements in a non-commutative ring)

$$\prod_{0,n} x_k - \prod_{0,n} y_k = \sum_{k=1}^{n} \left(\prod_{0,k-1} y_\nu\right) (x_k - y_k) \left(\prod_{k,n} x_\nu\right) . \tag{7}$$

Since $x_1, \dots, x_n, y_1, \dots, y_n$ are arbitrary, (7) still holds if we permute $x_1, \dots, x_n$ and correspondingly $y_1, \dots, y_n$ in all possible ways; the permutation, of course, leaves the left hand side unaltered. Adding the identities ($n!$ in number) which correspond to the different permutations, and dividing the obtained equality by $n!$, we get an identity of the form

$$\prod_{0,n} x_k - \prod_{0,n} y_k = \sum_{0,n} f_k^{(n)} (x_k - y_k) , \tag{8}$$

where obviously the coefficients $f_k^{(n)}$ are the polynomials defined above. In fact, the coefficient of $x_n - y_n$ is

$$\frac{1}{n!} \left\{ \sum ! \, x_1 x_2 \dots x_{n-1} + \sum ! \, y_1 x_2 \dots x_{n-1} + \dots + \sum ! \, y_1 y_2 \dots y_{n-1} \right\}$$

and the coefficient of $x_k - y_k$ is obtained from the last polynomial by interchanging k and n. Introducing the mean value (2), we may write

$$f_k^{(n)} = \bar{f}^{(n)} + f_k^{(n)} - \bar{f}^{(n)} , \tag{9}$$

where

$$f_k^{(n)} - \bar{f}^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \{ f_k^{(n)} - f_j^{(n)} \} . \tag{10}$$

Now we observe that

$$f_k^{(n)} = A x_j + B y_j$$

where A and B are polynomials independent of x_j and y_j. Hence we recognize A and B as the polynomials $\frac{1}{2}\bar{u}_{kj}^{(n)}$ and $\frac{1}{2}\bar{v}_{kj}^{(n)}$, respectively, and get

$$f_k^{(n)} = \tfrac{1}{2} \{ \bar{u}_{kj}^{(n)} x_j + \bar{v}_{kj}^{(n)} y_j \} .$$

According to (6) we may also write

$$f_j^{(n)} = \tfrac{1}{2} \{ \bar{u}_{kj}^{(n)} x_k + \bar{v}_{kj}^{(n)} y_k \} .$$

Thus we obtain

$$f_k^{(n)} - f_j^{(n)} = \tfrac{1}{2}\{\bar{u}_{kj}^{(n)}(x_j - x_k) + \bar{v}_{kj}^{(n)}(y_j - y_k)\}\,. \tag{11}$$

Combining (8), (9), (10) and (11) we end up with the identity (4).

REFERENCES

1. H. Bergström, *On the limit theorems for convolutions of distribution functions* II, J. Reine Angew. Math. 199 (1958), 1–22.
2. H. Bergström, *Konvergenzsätze über unendliche Produkte in abstrakten Halbgruppen*, Abh. Math. Sem. Univ. Hamburg 23 (1959), 228–256.