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NAGELL'S TOTIENT FUNCTION

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1. Introduction.

Let r and n represent integers, $r > 0$. In [5] Nagell evaluated the function $\theta(n, r)$, defined to be the number of integers a (mod$r$) such that $(a, r) = (n-a, r) = 1$. More recently, [3, (7.6)] the author of the present paper obtained the following simple representation for this function:

$$\theta(n, r) = \varphi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{\mu(d)}{\varphi(d)}, \tag{1.1}$$

where $\varphi(r)$ is the Euler function, $\mu(r)$ the Möbius function, and the summation is over divisors d of r which are prime to n. For an interesting discussion of the function $\theta(n, r)$ along other lines, we mention Alder [1].

It is the purpose of this note to give a new proof of (1.1) as an illustration of an arithmetical inversion principle proved elsewhere [4, Theorem 2.3]. Before stating this principle we introduce some definitions and notation. Let $\gamma(r)$ denote the core of r, that is, the greatest square-free divisor of r. Further, let (n, r) denote the greatest common divisor of n and r, and place $\gamma(n, r) = \gamma\big((n, r)\big)$. A complex-valued function $f(n, r)$ will be termed *primitive* (mod$r$) if $f(n, r) = f\big(\gamma(n, r), r\big)$ for all integers n.

We now state the inversion relation referred to above. *Let* $r = r_1 r_2$ *where* r_1 *is square-free, and assume* $f(n, r)$ *to be primitive* (mod$r$); *then*

$$f(n, r) = \sum_{\substack{d \mid \gamma(r) \\ (d,n)=1}} F\left(d, \frac{r}{d}\right) \rightleftarrows F(r_1, r_2) = \frac{\gamma(r)\mu(r_1)}{r} \sum_{d \mid \frac{rr_1}{\gamma(r)}} f\left(\frac{r}{d}, r\right) c(r_2, d), \tag{1.2}$$

where $c(n, r)$ *denotes Ramanujan's trigonometric sum.*

Remark. Actually we prove somewhat more than (1.1) in this paper; a complete statement of the result proved is contained in the theorem in § 3.

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2. Preliminary formulas.

We list for convenient reference a number of known properties of $\varphi(r)$, $\mu(r)$, and $c(n, r)$.

$$\sum_{d|r} \mu(d) = \begin{cases} 1 & (r = 1) \\ 0 & (r > 1) \end{cases}, \quad \varphi(r) = \sum_{d|r} d\mu\left(\frac{r}{d}\right); \tag{2.1}$$

$$\varphi(r) = \frac{r\varphi(\gamma(r))}{\gamma(r)}, \qquad \frac{r}{\varphi(r)} = \sum_{d|r} \frac{\mu^2(d)}{\varphi(d)}; \tag{2.2}$$

$$c(n, r) = \sum_{d|(n,r)} d\mu\left(\frac{r}{d}\right) = \frac{\varphi(r)\mu(e)}{\varphi(e)} \qquad \left(e = \frac{r}{(n, r)}\right). \tag{2.3}$$

It is also recalled that $c(n, r)$, $\mu(r)$, $\varphi(r)$ are multiplicative in the argument r.

In addition, we shall require the following lemma.

LEMMA 2.1. *If r, r_1, and k are positive integers, r square-free, and r_1 a divisor of r, then*

$$\sum_{d|r_1} \mu(d) c\left(\frac{r}{d}, k\right) = \begin{cases} r_1\varphi\left(\dfrac{k}{r_1}\right) & \text{if } r_1 \mid k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

PROOF. Denote the left of (2.4) by $S(k, r_1)$. Then by (2.3),

$$S(k, r_1) = \sum_{d|r_1} \mu(d) \sum_{\delta|\left(\frac{r}{d}, k\right)} \delta\mu\left(\frac{k}{\delta}\right) = \sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right) \sum_{d|\left(r_1, \frac{r}{\delta}\right)} \mu(d).$$

By (2.1a), the inner sum of the last expression is 0 for all δ for which $(r_1, r/\delta) \neq 1$. By the square-free property of r and the hypothesis that $r_1|r$, the conditions $(r_1, r/\delta) = 1$ and $r_1|\delta$ are equivalent. But the conditions $r_1|\delta$ and $\delta|k$ are incompatible unless $r_1|k$; hence $S(k, r_1) = 0$ if $r_1 \nmid k$. This proves the second case of the lemma.

Assuming then that $r_1|k$, it follows by (2.1a) that

$$S(k, r_1) = \sum_{\substack{\delta|k \\ \left(r_1, \frac{r}{\delta}\right) = 1}} \delta\mu\left(\frac{k}{\delta}\right) = \sum_{\substack{\delta|k \\ r_1|\delta}} \delta\mu\left(\frac{k}{\delta}\right);$$

placing $\delta = r_1E$ and applying (2.1b), one obtains

$$S(k, r_1) = r_1 \sum_{E|\frac{k}{r_1}} E\mu\left(\frac{k/r_1}{E}\right) = r_1\varphi\left(\frac{k}{r_1}\right), \qquad (r_1|k).$$

This completes the proof.

3. Evaluation of $\theta(n, r)$.

We first note the following special case of the inversion relation (1.2).

LEMMA 3.1. *If $r = r_1 r_2$, r square-free, and if $f(n, r)$ is primitive* (mod r), *then*

$$(3.1)\quad f(n, r) = \sum_{\substack{d \mid r \\ (d, n)=1}} F\left(d, \frac{r}{d}\right) \rightleftarrows F(r_1, r_2) = \mu(r_1) \sum_{d \mid r_1} f\left(\frac{r}{d}, r\right) \mu(d) .$$

PROOF. This formula results from (1.2), because $(r_1, r_2) = 1$ and, by (2.3), $c(m, r) = \mu(r)$ in case $(m, r) = 1$.

We now prove our main result.

THEOREM. *The function $\theta(n, r)$ is primitive* (mod r), *and has the following unique representation of the form* (1.2),

$$(3.2)\quad \theta(n, r) = \varphi(r) \sum_{\substack{d \gamma \mid (r) \\ (d, n)=1}} \frac{\mu(d)}{\varphi(d)} .$$

PROOF. The function $\theta(n, r)$ has the Fourier expansion [2, Theorem 7, $(s = 2)$],

$$(3.3)\quad \theta(n, r) = \frac{1}{r} \sum_{d \mid r} c^2\left(\frac{r}{d}, r\right) c(n, d) .$$

By (2.3) one obtains therefore,

$$(3.4)\quad \theta(n, r) = \frac{\varphi^2(r)}{r} \sum_{d \mid r} \left(\frac{\mu(d)}{\varphi(d)}\right)^2 c(n, d) .$$

In view of the presence of the factor $\mu^2(d)$ in (3.4), the divisors d of r may be replaced by the divisors of $\gamma(r)$. Since $c(n, r) = c\big((n, r), r\big)$, it therefore follows that $\theta(n, r)$ is primitive (mod r), and moreover, by (2.2a), that

$$(3.5)\quad \theta(n, r) = \frac{\varphi^2(r)}{r} \cdot \frac{\gamma(r)}{\varphi^2\big(\gamma(r)\big)} \theta\big(n, \gamma(r)\big) = \frac{\varphi(r)\, \theta\big(n, \gamma(r)\big)}{\varphi\big(\gamma(r)\big)} .$$

We consider two cases.

Case 1 (r square-free). In this case, it follows by Lemma 3.1, that

$$(3.6)\quad \theta(n, r) = \sum_{\substack{d \mid r \\ (d, n)=1}} F\left(d, \frac{r}{d}\right),$$

where

$$(3.7)\quad F(r_1, r_2) = \mu(r_1) \sum_{d \mid r_1} \theta\left(\frac{r}{d}, r\right) \mu(d), \qquad r = r_1 r_2 .$$

Hence by (3.4), Lemma 2.1, and the multiplicative property of $\varphi(r)$, one obtains

$$F(r_1, r_2) = \frac{\mu(r_1)\varphi^2(r)}{r} \sum_{d|r_1} \mu(d) \sum_{D|r} \frac{\mu^2(D)}{\varphi^2(D)} c\left(\frac{r}{d}, D\right)$$

$$= \frac{\mu(r_1)\varphi^2(r)}{r} \sum_{D|r} \frac{\mu^2(D)}{\varphi^2(D)} \sum_{d|r_1} \mu(d) c\left(\frac{r}{d}, D\right) = \frac{\mu(r_1)\varphi^2(r)}{r_2\varphi(r_1)} \sum_{\substack{D|r \\ r_1|D}} \frac{\mu^2(D)}{\varphi(D)}.$$

Placing $D = r_1 E$ in the latter summation and using the multiplicativity of $\mu(r)$ and $\varphi(r)$, in connection with (2.2b), it follows that

$$F(r_1, r_2) = \frac{\mu(r_1)\varphi^2(r)}{r_2\varphi^2(r_1)} \sum_{E|r_2} \frac{\mu^2(E)}{\varphi(E)} = \frac{\varphi(r)\mu(r_1)}{\varphi(r_1)}.$$

This proves (3.2) for square-free r, on the basis of (3.6).

Case 2 (r arbitrary). The general case of (3.2) results from the case r square-free, in connection with (3.5).

The uniqueness of the representation (3.2) is a direct consequence of the inversion theorem (1.2). This completes the proof.

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