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SOME ARITHMETIC SUMS CONNECTED WITH THE GREATEST INTEGER FUNCTION

L. CARLITZ

1.

Jacobsthal [3] has introduced the sum

$$(1) \qquad S(a,b,m;\,r) = \sum_{k=0}^{r-1} D(k)\,,$$

where

$$(2) \qquad D(k) = D(a,b,m;\,k) = \left[\frac{a+b+k}{m}\right] - \left[\frac{a+k}{m}\right] - \left[\frac{b+k}{m}\right] + \left[\frac{k}{m}\right];$$

here a, b are arbitrary integers while $m \geqq 1$, $r \geqq 1$. Jacobsthal proved the inequality

$$(3) \qquad S(a,b,m;\,r) \geqq 0\,.$$

The writer [1] has given another proof of (3) making use of the representation

$$(4) \qquad S(a,b,m;\,r) = -\frac{1}{m}\sum_{s=1}^{m-1} \frac{(\zeta^{-sa}-1)(\zeta^{-sb}-1)(\zeta^{-sr}-1)}{(\zeta^{s}-1)(\zeta^{-s}-1)},$$

where $\zeta = e^{2\pi i/m}$. If we put

$$(5) \qquad J(a,b,c) = -\frac{1}{m}\sum_{s=1}^{m-1} \frac{(\zeta^{-sa}-1)(\zeta^{-sb}-1)(\zeta^{-sc}-1)}{(\zeta^{s}-1)(\zeta^{-s}-1)}$$

for arbitrary a, b, c, then clearly $J(a,b,c)$ is symmetric in a,b,c; also it is evident that

$$S(a,b,m;\,r) = J(a,b,r)\,.$$

It follows easily from (5) that

$$(6) \qquad J(a,b,c) = J(-a,\,-b,\,-c)\,.$$

Since $J(a,b,c)$ has period m in each variable and

$$J(a,b,c) = 0 \qquad (abc = 0)\,,$$

we may assume that

$$(7) \qquad 1 \leqq a \leqq m-1, \qquad 1 \leqq b \leqq m-1, \qquad 1 \leqq c \leqq m-1\,.$$

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Moreover in view of the symmetry and (6) we may also assume that

(8) $$b \leqq m-a \leqq c$$

and

(9) $$b+c \leqq m\ .$$

It is proved in [1] that when (7), (8), (9) hold, then

(10) $$J(a,b,c) = b\ .$$

We recall that the Bernoulli function $\bar{B}_p(x)$ is defined for $0 \leqq x \leqq 1$ by

$$\bar{B}_p(x) = B_p(x), \qquad \bar{B}_p(x+1) = \bar{B}_p(x)\ ,$$

where $B_p(x)$, the Bernoulli polynomial of degree p, is defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{p=0}^{\infty} B_p(x)\frac{t^p}{p!}\ .$$

Now put

(11) $$D_p(k) = D_p(a,b,m;k)$$
$$= -\bar{B}_p\left(\frac{a+b+k}{m}\right) + \bar{B}_p\left(\frac{a+k}{m}\right) + \bar{B}_p\left(\frac{b+k}{m}\right) - \bar{B}_p\left(\frac{k}{m}\right)\ .$$

(12) $$S_p(a;b,m;r) = \sum_{k=0}^{r-1} D_p(k)\ .$$

Making use of the formula [2, p. 521]

$$\bar{B}_p\left(\frac{r}{m}\right) = \frac{B_p}{m^p} + \frac{p}{m^p}\sum_{s=0}^{m-1}\frac{\zeta^{-rs}}{\zeta^s-1}H_{p-1}(\zeta^{-s})\ ,$$

which holds for all integral r and $p \geqq 1$, we get the representation

(13) $$S_p(a,b,m;r) = -\frac{p}{m^p}\sum_{s=1}^{m-1} H_{p-1}(\zeta^{-s})\frac{(\zeta^{-as}-1)(\zeta^{-bs}-1)(\zeta^{-rs}-1)}{(\zeta^s-1)(\zeta^{-s}-1)}\ ,$$

where $H_n(\lambda)$ is defined by

$$\frac{1-\lambda}{e^t-\lambda} = \sum_{n=0}^{\infty} H_n(\lambda)\frac{t^n}{n!} \qquad (\lambda \neq 1)\ .$$

If we put

(14) $$J_p(a,b,c) = -\frac{p}{m^p}\sum_{s=1}^{m-1} H_{p-1}(\zeta^{-s})\frac{(\zeta^{-as}-1)(\zeta^{-bs}-1)(\zeta^{-cs}-1)}{(\zeta^s-1)(\zeta^{-s}-1)}$$

then $J_p(a,b,c)$ has period m in each variable and is symmetric in a,b,c; also

$$S_p(a,b,m;\, r) = J_p(a,b,r)\,.$$

Comparing (14) with (5) it is clear that

$$J(a,b,c) = J_1(a,b,c)\,.$$

2.

It is by no means evident how to extend (10) to the case of arbitrary $p \geqq 1$, or, in particular, to frame a theorem that will reduce to (3) when $p=1$. In the present note we limit ourselves to the special cases $p=2$ and 3. It is easily verified that

$$H_1(\lambda) = (\lambda-1)^{-1};$$

thus (14) becomes

$$J_2(a,b,c) = -\frac{2}{m^2}\sum_{s=1}^{m-1}\frac{(\zeta^{-as}-1)(\zeta^{-bs}-1)(\zeta^{-cs}-1)}{(\zeta^{s}-1)(\zeta^{-s}-1)^2}\,. \tag{15}$$

It follows easily from (15) and (5) that

$$J_2(a,b,c)+J_2(-a,\,-b,\,-c) = -\frac{2}{m}\,J_1(a,b,c)\,. \tag{16}$$

As above there is no loss in generality in assuming that (7), (8), (9) hold. We rewrite (15) as

$$-\tfrac{1}{2}m^2 J_2(a,b,c)+(m-a)bc$$
$$= \sum_{s=0}^{m-1}(1+\zeta^{s}+\ldots+\zeta^{(m-a-1)s})(1+\zeta^{-s}+\ldots+\zeta^{-(b-1)s})(1+\zeta^{-s}+\ldots+\zeta^{-(c-1)s})$$

and apply the familiar formula

$$\sum_{s=0}^{m-1}\zeta^{rs} = \begin{cases} m & (m \mid r) \\ 0 & (m \nmid r)\,. \end{cases} \tag{17}$$

Making use of (7), (8) and (9) we get

$$\begin{aligned} -\frac{m^2}{2}J_2(a,b,c)+(m-a)bc &= m\sum_{i=0}^{m-a-1}\sum_{\substack{j=0\\ j+k=i}}^{b-1}\sum_{k=0}^{c-1}1 \\ &= m\sum_{i=0}^{b-1}\sum_{j+k=i}1 + m\sum_{i=b}^{m-a-1}\sum_{\substack{j<b,\,k<c\\ j+k=i}}1 \\ &= m\sum_{i=0}^{b-1}(i-1) + m\sum_{i=b}^{m-a-1}b \\ &= \tfrac{1}{2}mb(b+1) + mb(m-a-b) \\ &= mb(m-a) - \tfrac{1}{2}mb(b-1)\,. \end{aligned}$$

It therefore follows without much trouble that

(18) $$\tfrac{1}{2}m^2 J_2(a,b,c) = -(m-a)(m-c)b + \tfrac{1}{2}mb(b-1) .$$

Combining (18) with (16) and using (10) we get also

(19) $$\tfrac{1}{2}m^2 J_2(-a,-b,-c) = (m-a)(m-c)b - \tfrac{1}{2}mb(b+1) .$$

Thus by means of (18) and (19), J_2 is evaluated for all a, b, c, the notation being such that (7), (8), (9) are satisfied. In particular note that (18) and (19) imply

(20) $$-(m-a)(m-c)b \leqq \tfrac{1}{2}m^2 J_2(a,b,c) \leqq \tfrac{1}{2}mb(b-1) ,$$

(21) $$-\tfrac{1}{2}mb(b-1) \leqq \tfrac{1}{2}m^2 J_2(-a,-b,-c) \leqq (m-a)(m-c)b;$$

these inequalities may be compared with (3).

It is also clear from (18) and (19) that $\frac{1}{2}m^2J_2$ is integral; indeed we have

$$\tfrac{1}{2}m^2J_2(a,b,c) \equiv -abc \pmod{m}, \qquad \tfrac{1}{2}m^2J_2(-a,-b,-c) \equiv abc \pmod{m} .$$

3.

For $p=3$, since

$$H_2(\lambda) = \frac{\lambda+1}{(\lambda-1)^2},$$

we find that

(22) $$J_3(a,b,c) = -\frac{3}{m^3}\sum_{s=1}^{m-1}(\zeta^{-s}+1)\frac{(\zeta^{-as}-1)(\zeta^{-bs}-1)(\zeta^{-cs}-1)}{(\zeta^s-1)(\zeta^{-s}-1)^3}.$$

Then

(23) $$\tfrac{1}{3}m^3J_3(a,b,c) = \tfrac{1}{2}m^2J_2(a,b,c) - 2K ,$$

where

(24) $$K = \sum_{s=1}^{m-1}\frac{(\zeta^{-as}-1)(\zeta^{-bs}-1)(\zeta^{-cs}-1)}{(\zeta^s-1)(\zeta^{-s}-1)^3}.$$

Now it follows from the identity

$$\sum_{s=1}^{m-1}\frac{\zeta^{-rs}}{\zeta^{-s}-x} = \frac{mx^{r-1}}{1-x^m} - \frac{1}{1-x} \qquad (1 \leqq r \leqq m)$$

that

(25) $$\sum_{s=1}^{m-1}\frac{\zeta^{-rs}}{\zeta^{-s}-1} = \tfrac{1}{2}(m+1)-r \qquad (1 \leqq r \leqq m) .$$

Rewrite (24) as

$$K = \sum_{s=1}^{m-1}\frac{1}{\zeta^s-1}(1+\zeta^s+\dots+\zeta^{(m-a-1)s})\cdot$$
$$\cdot(1+\zeta^{-s}+\dots+\zeta^{-(b-1)s})(1+\zeta^{-s}+\dots+\zeta^{-(c-1)s}) .$$

Then by (25) we get

$$K = \sum_{i=0}^{m-a-1} \sum_{j=0}^{b-1} \sum_{k=0}^{c-1} \{\tfrac{1}{2}(m+1) - R(j+k-i)\}, \tag{26}$$

where $R(k)$ is defined by

$$R(k) \equiv k \pmod{m}, \qquad 1 \leqq R(k) \leqq m .$$

Next, assuming that (7), (8) and (9) are satisfied, we have

$$\begin{aligned}\sum_{i=0}^{m-a-1} \sum_{j=0}^{b-1} \sum_{k=0}^{c-1} R(j+k-i) &= \sum_{\substack{i,j,k \\ i<j+k}} (j+k-i) + \sum_{\substack{i,j,k \\ i \geqq j+k}} (m+j+k-i) \\ &= \sum_{i,j,k} (j+k-i) + m \sum_{\substack{i,j,k \\ i \geqq j+k}} 1 = S_1 + mS_2 ,\end{aligned}$$

say. Clearly

$$\begin{aligned} S_1 &= -\tfrac{1}{2}(m-a)(m-a-1)bc + \tfrac{1}{2}(m-a)bc(b-1) + \tfrac{1}{2}(m-a)bc(c-1) \\ &= \tfrac{1}{2}(m-a)bc(a+b+c-m-1) ,\end{aligned}$$

while

$$\begin{aligned} S_2 &= \sum_{i=0}^{b=1} \sum_{j=i} \sum_{k \leqq i-j} 1 + \sum_{i=b}^{m-a-1} \sum_{j<b} \sum_{k \leqq i-j} 1 \\ &= \sum_{i=0}^{b-1} \sum_{j \leqq i} (i-j+1) + \sum_{i=b}^{m-a-1} \sum_{j<b} (i-j+1) \\ &= \sum_{i=0}^{b-1} \tfrac{1}{2}(i+1)(i+2) + \sum_{i=b}^{m-a-1} \{(i+1)b - \tfrac{1}{2}b(b-1)\} \\ &= \tfrac{1}{6}b(b+1)(b+2) + \tfrac{1}{2}b(m-a)(m-a+1) \\ &\quad - \tfrac{1}{2}b^2(b+1) - \tfrac{1}{2}b(b-1)(m-a-b) \\ &= \tfrac{1}{6}b(b-1)(b-2) + \tfrac{1}{2}b(m-a)(m-a-b+2) .\end{aligned}$$

Since by (26)

$$K = \tfrac{1}{2}(m+1)(m-a)bc - S_1 - mS_2 ,$$

a little manipulation now yields

$$\begin{aligned} K = (m+1)(m-a)bc - \tfrac{1}{2}(m-a)(a+b+c)bc - \tfrac{1}{2}m(m-a) \cdot \\ \cdot (m-a-b+2)b - \tfrac{1}{6}mb(b-1)(b-2) .\end{aligned} \tag{27}$$

We may rewrite (27) as

$$2K = (m-a)b(m-c)(a+b+c-m-2) - \tfrac{1}{3}mb(b-1)(b-2) .$$

Finally, using (18), (23), we get

$$(28)\quad \tfrac{1}{3}m^3J_3(a,b,c) = -b(m-a)(m-c)(a+b+c-m-1) + \tfrac{1}{6}mb(b-1)(2b-1)\,.$$

Since, from (22),

$$m^3J_3(-a,\,-b,\,-c) = m^3J_3(a,b,c) + 3m^2J_2(a,b,c) + 3mJ_1(a,b,c)\,,$$

it follows that

$$(29)\quad \tfrac{1}{3}m^3J_3(-a,\,-b,\,-c) = -b(m-a)(m-c)(a+b+c-m+1) + \tfrac{1}{6}mb(b+1)(2b+1)\,.$$

It is evident that (28) and (29) imply the inequalities

$$(30)\quad -b(m-a)(m-c)(a+b+c-m-1) \leqq \tfrac{1}{3}m^3J_3(a,b,c) \leqq \tfrac{1}{6}mb(b-1)(2b-1)\,,$$

$$(31)\quad -b(m-a)(m-c)(a+b+c-m+1) \leqq \tfrac{1}{3}m^3J_3(-a,\,-b,\,-c) \leqq \tfrac{1}{6}mb(b+1)(2b+1)\,.$$

These inequalities may be compared with (20) and (21).

It is clear from (28) and (29) that $\tfrac{1}{3}m^3J_3$ is integral; indeed we have

$$\tfrac{1}{3}m^3J_3(a,b,c) \equiv -abc(a+b+c-1) \pmod{m}\,,$$

$$\tfrac{1}{3}m^3J_3(-a,\,-b,\,-c) \equiv -abc(a+b+c+1) \pmod{m}\,.$$

The evaluation of J_p by the above method for $p>3$ becomes very cumbersome.

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DUKE UNIVERSITY, DURHAM, NORTH CAROLINA, U.S.A.