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# A NEW VERSION OF SOME CONSIDERATIONS OF A. THUE

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Among the articles of the late Norwegian mathematician A. Thue there are a few which treat some rational approximations to $r$th roots of rationals. In particular he wrote [1] and [2], which are rather troublesome to read. In this paper I give a modified exposition which is much shorter. My method is a little different from Thue's.

I first find polynomials $f_n(x)$ and $g_n(x)$ of degree $n$ with rational coefficients such that the function $f_n(x)-(1+x)^{1/r}g_n(x)$ has $x=0$ as a zero of multiplicity $2n+1$. The coefficients of $f_n$ and $g_n$ turn out to be very similar to those of the polynomials with which Thue was working. One might therefore think that a transition from one treatment to the other ought to be possible, but I don't know how to carry that out.

I prove again Thue's theorem for the $r$th roots. As is well known, stronger theorems have been proved by Siegel, Roth and others, but the proofs are rather long and complicated. The purpose of this paper is only to give a shorter proof of Thue's theorem in the case of $r$th roots.

## 1.

I shall prove the existence of polynomials $f_n(x)$, $g_n(x)$ of degree $n$ with rational coefficients such that

$$f_n(x)-(1+x)^{1/r}g_n(x) \;=\; c_{1,n}x^{2n+1}+c_{2,n}x^{2n+2}+\dots .$$

Such polynomials exist for $n=0$ namely $f_0(x)=g_0(x)=1$. I shall show how to find $f_{n+1}$ and $g_{n+1}$, provided that $f_n$ and $g_n$ are known.

We get

$$\int f_n(x)\,dx \;-\; \int g_n(x)\,(1+x)^{1/r}dx \;=\; \frac{c_{1,n}}{2n+2}\,x^{2n+2}+\frac{c_{2,n}}{2n+3}\,x^{2n+3}+\dots$$

so that

$$(2n+2)\int f_n(x)\,dx \;-\; (2n+2)\int g_n(x)(1+x)^{1/r}dx \;-$$
$$-\; x\big(f_n(x)-(1+x)^{1/r}g_n(x)\big) \;=\; c_{1,\,n+1}x^{2n+3}+\dots .$$

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Hence we may put

$$(2n+2)\int f_n(x)dx - xf_n(x) = f_{n+1}(x)$$

and

$$(2n+2)\int g_n(x)(1+x)^{1/r}dx - x(1+x)^{1/r}g_n(x) = (1+x)^{1/r}g_{n+1}(x) .$$

Putting

$$g_n(x) = a_{n,n}(1+x)^n + a_{n-1,n}(1+x)^{n-1} + \ldots + a_{0,n}$$

we obtain the following expression for $g_{n+1}(x)$:

$$g_{n+1}(x) = (2n+2)\left(a_{n,n}\frac{r}{r(n+1)+1}(1+x)^{n+1} + a_{n-1,n}\frac{r}{rn+1}(1+x)^n + \ldots\right.$$

$$\left. + \frac{r}{r+1}a_{0,n}(1+x)\right) - a_{n,n}(1+x)^{n+1} - a_{n-1,n}(1+x)^n - \ldots - a_{0,n}(1+x) +$$

$$+ a_{n,n}(1+x)^n + \ldots + a_{1,n}(1+x) + a_{0,n} .$$

Thus we obtain the following equations

$$a_{n+1,n+1} = \frac{r(n+1)-1}{r(n+1)+1}a_{n,n} ,$$

$$a_{n,n+1} = \frac{r(n+2)-1}{rn+1}a_{n-1,n} + a_{n,n} ,$$

$$a_{n-1,n+1} = \frac{r(n+3)-1}{r(n-1)+1}a_{n-2,n} + a_{n-1,n} ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{1,n+1} = \frac{r(2n+1)-1}{r+1}a_{0,n} + a_{1,n} ,$$

$$a_{0,n+1} = a_{0,n} .$$

The last equation yields $a_{0,n}=1$ for all $n$, since $a_{0,0}=1$. Putting

$$h_n = (r-1)(2r-1)\ldots(nr-1), \quad k_n = (r+1)(2r+1)\ldots(nr+1) ,$$

$$h_0 = 1, \quad k_0 = 1 ,$$

I assert that quite generally we have

$$a_{n-m,n} = \binom{n}{m}\frac{h_n}{h_m k_{n-m}} .$$

Indeed this is seen at once by induction with regard to $n$ to be true for $m=0$ and $m=n$. We may also use induction on $n$ to prove it generally because it is correct for $u=0$. We have

$$a_{n-m+1,\,n+1} = \frac{r(n+m+1)-1}{r(n-m+1)+1}\, a_{n-m,\,n} + a_{n-m+1,\,n}$$

so that assuming the correctness of the formula for $n$ and $m=0, 1, \ldots n$ we obtain

$$a_{n-m+1,\,n+1} = \big(r(n+m+1)-1\big)\binom{n}{m}\frac{h_n}{h_m k_{n-m+1}} + (rm-1)\binom{n}{m-1}\frac{h_n}{h_m k_{n-m+1}}$$

$$= \frac{h_n}{h_m k_{n-m+1}}\binom{n}{m-1}\left[\frac{\big(r(n+m+1)-1\big)(n-m+1)}{m} + rm-1\right].$$

Now the reader will easily verify that identically

$$\big(r(n+m+1)-1\big)(n-m+1)+m(rm-1) = r(n+1)^2-n-1 = \frac{h_{n+1}}{h_n}(n+1)$$

so that we get

$$a_{n-m+1,\,n+1} = \frac{h_n}{h_m k_{n-m+1}}\binom{n}{m-1}\frac{h_{n+1}}{h_n}\frac{n+1}{m} = \frac{h_{n+1}}{h_m k_{n-m+1}}\binom{n+1}{m},$$

that is, the formula is valid for $n+1$ and $m=0, 1, \ldots, n$. However, as already noticed it is also valid for $m=n+1$. Therefore the formula is generally correct.

Writing

$$g_n(x) = a'_{n,\,n}x^n + a'_{n-1,\,n}x^{n-1} + \ldots + a'_{0,\,n}$$

we have

$$a'_{n,\,n} = a_{n,\,n},$$

$$a'_{n-1,\,n} = a_{n,\,n}\binom{n}{1} + a_{n-1,\,n},$$

$$a'_{n-2,\,n} = a_{n,\,n}\binom{n}{2} + a_{n-1,\,n}\binom{n-1}{2} + a_{n-2,\,n},$$

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$$a'_{0,\,n} = a_{n,\,n} + a_{n-1,\,n} + a_{n-2,\,n} \ldots + a_{0,\,n}.$$

If we write

$$f_n(x) = b_{n,\,n}x^n + b_{n-1,\,n}x^{n-1} + \ldots + b_{0,\,n},$$

the identity

$$f_{n+1}(x) = (2n+2)\int f_n(x)\,dx - xf_n(x)$$

yields all $b_{n,\,n}=1$ and for $m=0, 1, \ldots, n-1$ the equations

$$b_{n-m,\,n+1} = \frac{n+m+2}{n+m}\, b_{n-m-1,\,n},$$

whence

$$b_{n-m,\,n} = \binom{n+m}{2m} b_{0,\,m}.$$

Clearly $b_{0,\,m} = a'_{0,\,m}$.

Further it may be suitable to mention here that

$$\Delta(x) = f_{n+1}(x)g_n(x) - f_n(x)g_{n+1}(x)$$
$$= (a_{n,n} - a_{n+1,n+1})x^{2n+1} = \frac{2}{r(n+1)+1} a_{n,n}x^{2n+1}.$$

Indeed the elimination of $(1+x)^{1/r}$ between the expressions for $f_n(x)-(1+x)^{1/r}g_n(x)$ and $f_{n+1}(x)-(1+x)^{1/r}g_{n+1}(x)$ yields that $\Delta(x)$ must equal a series beginning with a term $cx^{2n+1}$. On the other hand $\Delta(x)$ is a polynomial of degree $2n+1$. It is important that $\Delta(x) \neq 0$ when $x \neq 0$.

Now let $d_n$ be the least positive integer such that all $d_n a_{m,n}$, $m=0, 1, \ldots, n$, are integers. Then all $d_n a'_{m,n}$ and all $d_n b_{m,n}$ will be integers as well. In order to estimate $d_n$ we may proceed as Thue does in [1, pp. 21–24]. Putting

$$\frac{(r+1)(2r+1)\ldots(nr+1)}{1\cdot 2\ldots n} = \frac{a}{b}, \qquad (a, b) = 1,$$

it turns out that $a \cdot a_{m,n}$ is an integer for $m=0, 1, \ldots, n$. This can be shown in the following way. We have

$$a_{m,n}a = \binom{n}{m}\frac{\big(r(n-m+1)-1\big)\ldots(rn-1)}{(r+1)(2r+1)\ldots(rm+1)}\cdot\frac{(r+1)(2r+1)\ldots(nr+1)}{1\cdot 2\ldots n}b$$
$$= \frac{\big(r(n-m+1)-1\big)\ldots(rn-1)}{1\cdot 2\ldots m}\cdot\frac{\big(r(m+1)+1\big)\ldots(rn+1)}{1\cdot 2\ldots(n-m)}b = \frac{A}{\alpha}\cdot\frac{B}{\beta}\cdot b,$$

where

$$\frac{A}{\alpha} = \frac{\big(r(n-m+1)-1\big)\ldots(rn-1)}{1\cdot 2\ldots m}, \qquad \frac{B}{\beta} = \frac{\big(r(m+1)+1\big)\ldots(rn+1)}{1\cdot 2\ldots(n-m)},$$

and every prime divisor of $\alpha\beta$ divides $r$. Further we may write

$$1\cdot 2\cdot 3\ldots n = bC,$$
$$1\cdot 2\cdot 3\ldots m = \alpha D,$$
$$1\cdot 2\cdot 3\ldots(n-m) = \beta E,$$

where $C$, $D$, $E$ are integers, and we know that

$$\binom{n}{m} = \frac{bC}{\alpha D\beta E}$$

is an integer. Also

$$a = (r+1)(2r+1)\ldots(rn+1)/C$$

so that $C$ and $r$ must be coprime. Hence $\alpha\beta$ must divide $b$, so that $a_{m,n}a$ is an integer. Therefore $d_n \leqq a$.

Further, it can be shown that $a<((r+1)2^l)^n$, $l$ denoting the number of different primes dividing $r$. Indeed the exponent for the power of a prime $p$ dividing $n!$ is at most $n/(p-1)$, so that we get

$$b < \prod_p p^{n/(p-1)},$$

the product extended over the mentioned $l$ primes. Since $p^{1/(p-1)}<2$, we have

$$b < 2^{ln}.$$

Further

$$\frac{(r+1)(2r+1)\dots(nr+1)}{1\cdot 2\dots n} = (r+1)(r+\tfrac{1}{2})\dots\left(r+\frac{1}{n}\right) < (r+1)^n$$

so that

$$a < ((r+1)2^l)^n.$$

In the particular case $l=1$ it is seen that we may also write

$$a < ((r+1)r^{1/(r-1)})^n.$$

In the sequel I write

$$d_n < \varkappa^n$$

with $\varkappa=(r+1)2^l$ and $\varkappa=(r+1)r^{1/(r-1)}$, respectively, if $r$ is a prime. Putting

$$d_nf_n(x) = F_n(x), \qquad d_ng_n(x) = G_n(x),$$

$F_n$ and $G_n$ have integer coefficients and

$$F_n(x)-(1+x)^{1/r}G_n(x) = d_n(c_{1,n}x^{2n+1}+c_{2,n}x^{2n+2}+\dots).$$

The reader will easily verify that

$$c_{\nu,n+1} = -\frac{\nu}{2n+\nu+2}c_{\nu+1,n}$$

and that

$$c_{\nu,0} = -\binom{\frac{1}{r}}{\nu}$$

which is absolute $<1$ while for every $n$ the successive $c_{\nu,n}$ have alternating signs and are all absolute $<1$. Therefore we have the following rude estimate for $0<x<1$

$$|c_{1,n}+c_{2,n}x+\dots| \leqq \max(|c_{1,n}+c_{3,n}x^2+\dots|,\ x|c_{2,n}+c_{4,n}x^2+\dots|),$$

whence because of

$$|c_{1,n}+c_{3,n}x^2+\ldots| < 1+x^2+\ldots = \frac{1}{1-x^2},$$

$$|c_{2,n}+c_{4,n}x^2+\ldots| < 1+x^2+\ldots = \frac{1}{1-x^2},$$

in any case

$$|c_{1,n}+c_{2,n}x+\ldots| < \frac{1}{1-x^2}.$$

Hence

$$F_n(x)-(1+x)^{1/r}G_n(x) < d_n\frac{x^{2n+1}}{1-x^2}.$$

**2.**

Let the positive integers $a$, $b$, $c$, $u$, $v$ satisfy the equation

$$au^r-bv^r = c$$

or

$$\frac{a}{b}\frac{u^r}{v^r} = 1+\frac{c}{bv^r}$$

assuming $c<bv^r$. Putting

$$x = \frac{c}{bv^r}$$

we have $0<x<1$ and for every $n$

$$F_n(x)-\left(\frac{a}{b}\right)^{1/r}\frac{u}{v}G_n(x) = d_nx^{2n+1}(c_{1,n}+c_{2,n}x+\ldots).$$

Multiplying by $(bv^r)^nv$ we obtain

$$d_n\big(b_{n,n}c^n+b_{n-1,n}c^{n-1}(bv^r)+\ldots+b_{0,n}(bv^r)^n\big)v - \left(\frac{a}{b}\right)^{1/r}d_n\big(a'_{n,n}c^n +$$

$$+ a'_{n-1,n}c^{n-1}(bv^r)+\ldots+a'_{0,n}(bv^r)^n\big)u = d_n\frac{c^{2n+1}v}{(bv^r)^{n+1}}\left(c_{1,n}+c_{2,n}\frac{c}{bv^r}+\ldots\right).$$

Writing

$$p_n = d_nv\big(b_{n,n}c^n+b_{n-1,n}c^{n-1}(bv^r)+\ldots\big),$$
$$q_n = d_nu\big(a'_{n,n}c^n+a'_{n-1,n}c^{n-1}(bv^r)+\ldots\big),$$

$p_n$ and $q_n$ are integers such that

$$p_n-q_n\left(\frac{a}{b}\right)^{1/r} < \frac{d_nc^{2n+1}v}{(bv^r)^{n+1}}\cdot\frac{1}{1-c^2(bv^r)^{-2}}.$$

Writing

$$bv^r = t \qquad \text{and} \qquad \varkappa c^2 = t^\alpha,$$

we assume in the sequel $t$ so great in comparison with $c$ that $\alpha < 1$. Having seen that $d_n < \varkappa^n$ we obtain

$$p_n - q_n \left(\frac{a}{b}\right)^{1/r} = \delta_n t^{(\alpha-1)n},$$

where

$$|\delta_n| < \delta = \frac{c}{bv^{r-1}} \cdot \frac{1}{1 - \dfrac{c^2}{(bv^r)^2}}.$$

Since we shall use this for large $v$ we may further assume $\delta < 1$. An estimate of $q_n$ is

$$q_n < ud_n 4^n t^n \left(1 + \left(1 + \frac{c}{t}\right) + \ldots + \left(1 + \frac{c}{t}\right)^n\right) = ud_n 4^n \frac{t^{n+1}}{c} \left(\left(1 + \frac{c}{t}\right)^{n+1} - 1\right)$$

and

$$\frac{t}{c}\left(\left(1+\frac{c}{t}\right)^{n+1} - 1\right) = \binom{n+1}{1} + \binom{n+1}{2}\frac{c}{t} + \ldots + \frac{c^n}{t^n}$$

$$< 2^{n+1}\left(1 + \frac{1}{\varkappa c} + \ldots + \frac{1}{\varkappa^n c^n}\right)$$

$$= 2^{n+1} \frac{\left(1 - \dfrac{1}{\varkappa^{n+1} c^{n+1}}\right)}{1 - \dfrac{1}{\varkappa c}} < 2^{n+1} \frac{\varkappa c}{\varkappa c - 1},$$

since $c/t < 1/(\varkappa c)$. Therefore

$$q_n < u \varkappa^n 4^n t^n \frac{2^{n+1} \varkappa c}{\varkappa c - 1},$$

that is,

$$q_n < w t^{(1+\beta)n}$$

when

$$\frac{2\varkappa c u}{\varkappa c - 1} = w \quad \text{and} \quad 8\varkappa = t^\beta.$$

Let $\varrho = (1+\beta)/(1-\alpha) + s$, $s > 0$. Then I assert that if

$$(1) \qquad x - y\left(\frac{a}{b}\right)^{1/r} = \frac{\varepsilon}{y^\varrho}, \qquad |\varepsilon| < 1,$$

we have

$$y \leqq (2w)^{\frac{1}{s}} 2^{\frac{1+\beta}{1-\alpha} \cdot \frac{1}{s}} t^{\frac{2(1+\beta)}{s}}.$$

For let us assume that

$$y^s > 2w2^{\frac{1+\beta}{1-\alpha}}t^{2(1+\beta)}.$$

Then taking the logarithms on both sides one obtains

$$s\log y - \log 2w - 2(1+\beta)\log t > \frac{1+\beta}{1-\alpha}\log 2$$

which can be written

$$\varrho\log y - \log 2w - 2(1+\beta)\log t > \frac{1+\beta}{1-\alpha}\log 2y,$$

which again yields

$$\frac{\varrho\log y - \log 2w}{(1+\beta)\log t} - 2 > \frac{\log 2y}{(1-\alpha)\log t}.$$

Then an integer $n$ exists such that

$$\frac{\varrho\log y - \log 2w}{(1+\beta)\log t} > n+1, \qquad \frac{\log 2y}{(1-\alpha)\log t} < n$$

or

$$(2) \qquad \frac{y}{t^{(1-\alpha)n}} < \tfrac{1}{2}, \qquad \frac{wt^{(1+\beta)(n+1)}}{y^{\varrho}} < \tfrac{1}{2}.$$

Since $p_nq_{n+1}-p_{n+1}q_n$ is $\neq 0$, either $p_ny-q_nx$ or $p_{n+1}y-q_{n+1}x$ is $\neq 0$ and thus absolutely $\geqq 1$. On the other hand, we get by elimination of $(a/b)^{1/r}$ between the equations

$$p_n - q_n\left(\frac{a}{b}\right)^{1/r} = \frac{\delta_n}{t^{(1-\alpha)n}} \qquad \text{and} \qquad p_{n+1} - q_{n+1}\left(\frac{a}{b}\right)^{1/r} = \frac{\delta_{n+1}}{t^{(1-\alpha)(n+1)}},$$

and

$$x - y\left(\frac{a}{b}\right)^{1/r} = \frac{\varepsilon}{y^{\varrho}}$$

the equations

$$p_ny-q_nx = \frac{\delta_n y}{t^{(1-\alpha)n}} - \frac{\varepsilon q_n}{y^{\varrho}} \qquad \text{and} \qquad p_{n+1}y-q_{n+1}x = \frac{\delta_{n+1}}{t^{(1-\alpha)(n+1)}} - \frac{\varepsilon q_{n+1}}{y^{\varrho}},$$

respectively. But it follows from (2) that $|p_ny-q_nx|$ and $|p_{n+1}y-q_{n+1}x|$ are both $<1$, which is a contradiction.

## 3.

Now let us assume that an infinity of positive integers $u$, $v$ exist such that

$$(3) \qquad v - u\left(\frac{a}{b}\right)^{1/r} = \frac{\varepsilon}{u^{m-1}}, \qquad |\varepsilon| < 1, \quad m > 1.$$

Then we get successively with $\lim_{u\to\infty}\zeta_u=0$ and $\lim_{v\to\infty}\eta_v=0$

$$v^r-u^r\frac{a}{b} = ru^{r-1}\left(\frac{a}{b}\right)^{(r-1)/r}\frac{\varepsilon}{u^{m-1}}(1+\zeta_u) = \frac{\varepsilon r}{v^{m-r}}\left(\frac{a}{b}\right)^{(m-1)/r}(1+\eta_v),$$

whence

$$c = au^r-bv^r = -\varepsilon r a^{(m-1)/r}b^{(r+1-m)/r}v^{r-m}(1+\eta_v).$$

To every sufficiently large solution $u, v$ of $au^r-bv^r=c$ there exist, as shown in section 2, for every positive integer $n$ positive integers $p_n$ and $q_n$ such that

$$\left|p_n-q_n\left(\frac{a}{b}\right)^{1/r}\right| < \frac{1}{t^{(1-\alpha)n}},$$

where

$$\alpha = \frac{\log \varkappa c^2}{\log t}, \qquad t = bv^r.$$

It is obvious that by choosing a pair $u, v$ with $v$ sufficiently large we may get the number $\beta=\log 8\varkappa/\log t$ arbitrarily small. As to $\alpha$ we have

$$\alpha < \frac{\log\varkappa+2\big(\log r+((m-1)/r)\log a+((r+1-m)/r)\log b+\log(1+\eta_v)\big)+2(r-m)\log v}{\log b+r\log v}$$

so that

$$\alpha < \frac{2(r-m)}{r}+\zeta,$$

where $\zeta$ becomes as small as we please by choosing $v$ sufficiently large.

Thus, according to a result in § 2, by choosing $v$ large enough we can find an upper bound for the $y$ in (1) for any $\varrho>1/(1-\alpha')$ with $\alpha'>2(r-m)/r$, that means, when $\varrho>r/(2m-r)$. Now let $m$ be $>\frac{1}{2}r+1$. Then $r/(2m-r)<m-1$ so that after supposition there are infinitely many solutions $u, v$ of

$$\left|v-u\left(\frac{a}{b}\right)^{1/r}\right| < \frac{1}{u^{m-1}},$$

On the other hand, we have only a finite number of solutions $x, y$ of

$$\left|x-y\left(\frac{a}{b}\right)^{1/r}\right| < \frac{1}{y^{\varrho}}$$

with $\varrho$ chosen such that

$$r/(2m-r) < \varrho < m-1.$$

This is a contradiction and we have proved Thue's theorem: There is only a finite number of integral solutions $u, v$ of the inequality

$$\left|v-u\left(\frac{a}{b}\right)^{1/r}\right| < \frac{1}{u^{\frac{1}{2}r+\varepsilon}}, \qquad \varepsilon > 0.$$

## REFERENCES

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