MATH SCAND. 8 (1960), 97–104

A NOTE ON COMPLETION AND COMPACTIFICATION

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The aim of this paper is to present a unified treatment of completion and compactification of completely regular spaces. We shall point out how such extensions of the spaces may be performed by means of the *uniformly regular filters* introduced in Section 1. In Section 2 we shall classify the various completions and compactifications of a given completely regular space. The investigation is based on purely set-theoretic methods. Auxiliary spaces, such as spaces of continuous, real valued functions, will not be applied. The main results are summed up in Theorem 3.

1. Uniformly regular filters.

A filter $\mathscr{F}$ on a uniform space S, will be termed *uniformly regular*, or briefly *regular*, if it has a base consisting of the sets $V(F)$, V being an *entourage*, and $F \in \mathscr{F}$. With the notation of [1], $\mathscr{F}$ is regular if and only if

$$(1.1) \qquad F \in \mathscr{F} \quad \Rightarrow \quad (\exists G \in \mathscr{F})[G \Subset F] .$$

We shall say that two uniform structures, $\mathscr{U}_1$, $\mathscr{U}_2$, on the same set S are p-*equivalent* if they possess the same uniform neighbourhoods of sets, i.e. if

$$(1.2) \qquad A \Subset B \ \ (\mathscr{U}_1) \quad \Leftrightarrow \quad A \Subset B \ \ (\mathscr{U}_2) .$$

Clearly, p-equivalent structures possess the same regular filters.

From the Theorems 1.2 of [1], we know that each p-equivalence class $\mathscr{P}$ contains a coarsest structure, $\mathscr{U}_{\mathscr{P}}$ (in [1] denoted $\mathscr{U}_{\omega}$), which is also the unique totally bounded structure of the class. Moreover, there is a $1-1$ correspondence between p-equivalence classes of uniform structures and proximity structures (cf. [1]). Hence any result concerning p-equivalence classes, and in particular any result concerning regular filters, will admit an alternative formulation in terms of proximity concepts.

The notion of "a regular filter" has occurred in the literature in vari-

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Received November 17, 1959.

ous contexts and with different names. Thus, P. Samuel defines the *envelope* of a filter, $\mathscr{F}$, on a uniform space to be the filter with base $V(F)$, V being an *entourage*, and $F \in \mathscr{F}$ [6, p. 121]. In our terminology this is the finest regular filter coarser than $\mathscr{F}$. Yu. M. Smirnov's theory on the compactification of proximity spaces is based essentially on the theory of *maximal, regular filters*, by Smirnov termed "maximal centered δ-systems" or "ends" [7, p. 552].

The notion of a maximal, regular filter seems to be an advantageous substitute for ultrafilters in various contexts. In particular it should be noticed that it is possible to define non-trivial maximal, regular filters of importance for applications, without use of the axiom of choice (cf. § 2).

We shall need some elementary properties of regular filters on a uniform space S. For a fixed set A, the collection of sets B such that $A \Subset B$ is a regular filter which we denote $\mathscr{F}_A$. If $A = \{x\}$, we briefly write $\mathscr{F}_x$, and we notice that $\mathscr{F}_x$ is the neighbourhood-filter of x, and that $\mathscr{F}_x$ is *maximal* as a regular filter. Every regular filter may be extended by Zorn's lemma to a maximal, regular filter. The set F_0 of condensation points of a regular filter, $\mathscr{F}$, is

$$F_0 = \bigcap_{F \in \mathscr{F}} F ,$$

and generally $\mathscr{F}_{F_0}$ is finer than $\mathscr{F}$. A regular filter, $\mathscr{F}$, is convergent with the limit x if and only if $\mathscr{F} = \mathscr{F}_x$. If the given uniform structure is separating, a maximal regular filter $\mathscr{F}$ has at most one condensation point x, and in this case $\mathscr{F} = \mathscr{F}_x$.

PROPOSITION 1. *A separated uniform space, S, is compact if and only if every maximal regular filter is convergent, or equivalently if and only if every maximal regular filter is of the form $\mathscr{F}_x$, where $x \in S$.*

PROOF. If S is compact, every maximal regular filter, $\mathscr{F}$, has a condensation point x, and then $\mathscr{F} = \mathscr{F}_x$. Conversely, if $\mathscr{F}$ is some ultrafilter on S, then the envelope $\mathscr{F}'$ of $\mathscr{F}$ is a maximal regular filter. Now, convergence of $\mathscr{F}'$ implies the convergence of $\mathscr{F}$, and hence compactness.

The next proposition is an analogue of a well-known property of ultrafilters exposed by H. Cartan [3] [2, p. 38]. The actual version of this proposition is essentially equivalent to the third property (property "B") of "ends" in Yu. M. Smirnov's work [7, p. 552].

PROPOSITION 2. *A regular filter $\mathscr{F}$ is maximal if and only if for every*

$A \in \mathscr{F}$ and every p-*covering* $\{A_i\}_{1 \leq i \leq n}$ *of A (cf.* [1]), *at least one of the sets A_i belongs to $\mathscr{F}$.*

PROOF. 1) Let $\mathscr{F}$ be a maximal regular filter, $A \in \mathscr{F}$ and $\{A_i\}_{1 \leq i \leq n}$ a p-covering of A. If $n=1$, the set $A_1 \supset A$ will belong to $\mathscr{F}$. Then we assume the proposition valid for $n-1$. We define $A'_n = \bigcup_{i=1}^{n-1} A_i$. Now, $\{A'_n, A_n\}$ is a p-covering of A, and the inductive proof will be accomplished if we can prove that either $A'_n \in \mathscr{F}$ or $A_n \in \mathscr{F}$.

We assume the converse, i.e. $A'_n \notin \mathscr{F}$ and $A_n \notin \mathscr{F}$. Then the sets

$$F \cap \complement A_n, \qquad F \in \mathscr{F},$$

generate a filter whose envelope we call $\mathscr{G}$. The filter $\mathscr{G}$ is evidently finer than $\mathscr{F}$, and by the maximality, $\mathscr{G} = \mathscr{F}$. By the definition of envelope, we have

$$(1.3) \qquad G \in \mathscr{G} \quad \Leftrightarrow \quad (\exists F \in \mathscr{F})[F \cap \complement A_n \Subset G].$$

As $\{A'_n, A_n\}$ is a p-covering of A, there exist sets B', B such that

$$A \subset B' \cup B, \qquad B' \Subset A'_n, \qquad B \Subset A_n.$$

Now, $A \in \mathscr{F}$, and

$$A \cap \complement A_n \subset A \cap \complement B \subset B' \Subset A'_n.$$

Hence by (1.3), $A'_n \in \mathscr{G} = \mathscr{F}$, contrary to the hypothesis.

2) Suppose $\mathscr{F}$ to be a regular filter with the property stated in the proposition, and assume $\mathscr{F}$ non-maximal. Let $\mathscr{G}$ be some maximal regular filter finer than $\mathscr{F}$, and $G \in \mathscr{G}$, $G \notin \mathscr{F}$. Then there exists a set $H \Subset G$ such that $H \in \mathscr{G}$. Hence $\{G, \complement H\}$ is a p-covering of S. Since $G \notin \mathscr{F}$, we must have $\complement H \in \mathscr{F}$, and hence $\complement H \in \mathscr{G}$. But this is impossible, since $H \in \mathscr{G}$ as well.

The introduction of regular filters enables us to obtain an interesting connection between maximal filters and Cauchy filters. This connection will be at the heart of the subsequent discussion. We first recall that two Cauchy filters are said to be *equivalent* if their intersection is also a Cauchy filter [2, p. 153].

THEOREM 1. *A regular Cauchy filter on a uniform space is a maximal regular filter, and every equivalence class of Cauchy filters contains exactly one (maximal) regular filter, which is the common envelope of all the other filters of the class, and hence it is the coarsest of them.*

PROOF. 1) Let $\mathscr{F}$ be a regular Cauchy filter on S, and let $\{A_i\}_{1 \leq i \leq n}$ be some p-covering of S. Then the set $V = \bigcup_{i=1}^{n} (A_i \times A_i)$ is an *entourage* of the coarsest structure in the p-equivalence class of the original uniform structure (cf. Th. 1 of [1]). Thus V is an *entourage* of the given uniform structure as well, and hence there must exist a set $F \in \mathscr{F}$ such that $F \times F \subset V$. This means that for some i, $1 \leq i \leq n$, we have $F \subset A_i$, and so $A_i \in \mathscr{F}$. Application of Proposition 2 proves $\mathscr{F}$ to be maximal.

2) Let $\mathscr{F}$ be a Cauchy filter, and let us prove that its envelope $\mathscr{F}'$ is Cauchy as well. Let V be an *entourage* and choose another *entourage* V_0 such that $\overset{3}{V_0} \subset V$. Let $F \in \mathscr{F}$, $F \times F \subset V_0$. Then we shall have

$$(1.4) \qquad V_0(F) \times V_0(F) \subset \overset{3}{V_0} \subset V ,$$

which is the desired inclusion.

Clearly, $\mathscr{F}$ and $\mathscr{F}'$ are equivalent, and hence the equivalence class of $\mathscr{F}$ contains at least *one* regular Cauchy filter, namely $\mathscr{F}'$.

Now, assume $\mathscr{F}''$ to be some regular Cauchy filter equivalent to $\mathscr{F}'$. Then $\mathscr{G} = \mathscr{F}' \cap \mathscr{F}''$ is a Cauchy filter, but it is obviously also regular, hence maximal regular, and in virtue of

$$\mathscr{G} \subset \mathscr{F}' \quad \text{and} \quad \mathscr{G} \subset \mathscr{F}'' ,$$

we can conclude

$$\mathscr{F}' = \mathscr{G} = \mathscr{F}'', \qquad \text{q.e.d.}$$

Our next theorem gives a necessary and sufficient condition for the validity of the reverse implication.

THEOREM 2. *A necessary and sufficient condition that every maximal regular filter on a uniform space S be a Cauchy filter, is that the uniform structure is totally bounded.*

PROOF. 1) Assume every maximal regular filter to be Cauchy. Without lack of generality we may assume S to be a separated uniform space, (since total boundedness is preserved by the standard passage to the associated separated space [2, p. 137]). Let ξ be the embedding function mapping S into its completion $\hat{S}$. If $\mathscr{F}$ is some maximal regular filter on $\hat{S}$, then $\xi^{-1}(\mathscr{F})$ is a maximal regular filter on S, and hence $\xi^{-1}(\mathscr{F})$ is Cauchy. But then $\mathscr{F}$ is Cauchy as well, and so $\mathscr{F}$ has a limit in the complete space $\hat{S}$. By Prop. 1 this means that the completion $\hat{S}$ of S is compact, and by a well-known theorem, this implies that the uniform structure on S must be totally bounded [2, p. 161].

2) If the uniform structure on S is totally bounded, then by [1] we have the fundamental system of *entourages*

$$(1.5) \qquad V = \bigcup_{i=1}^{n} (A_i \times A_i) ,$$

where $\{A_i\}_{1 \leq i \leq n}$ is a p-covering of S.

Now let $\mathscr{F}$ be a maximal regular filter. Then by Prop. 2, there exists for each *entourage* V of the form (1.5) at least one i, $i=1,2,\ldots,n$, such that $A_i \in \mathscr{F}$. But as A_i is small of order V, that is $A_i \times A_i \subset V$, this proves $\mathscr{F}$ to be Cauchy, q.e.d.

2. Completion and compactification by maximal regular filters.

Theorem 1 of Section 1 informs us that the standard completion of a separated uniform space $(S, \mathscr{U})$ may be performed by means of regular Cauchy filters. These filters will then be the points of the extended space $\hat{S}$. The embedding function ξ will assign to each point $x \in S$ its own neighbourhood filter $\mathscr{F}_x$, or in symbols, $\mathscr{F}_x = \xi(x)$. The uniform structure $\hat{\mathscr{U}}$ on $\hat{S}$ will possess a fundamental system of *entourages* $\hat{V}$ defined by

$$(2.1) \qquad (\mathscr{F}, \mathscr{G}) \in \hat{V} \iff (\exists A \in \mathscr{F} \cap \mathscr{G})[A \times A \subset V],$$

where V runs through the *entourages* of $\mathscr{U}$, and $\mathscr{F}, \mathscr{G}$ are regular Cauchy filters [2, p. 152].

Remark. The completion by regular Cauchy filters may be performed also for non-separated uniform spaces, but then the embedding function ξ mapping x into $\mathscr{F}_x$, is many-one, taking the same value for non-separable points. Hence, this procedure is equivalent to the passage to the associated separated space [2, p. 137], followed by the standard completion process [2, p. 151].

Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be two uniform structures on the same set S, and let $\mathscr{U}_1$ be coarser than $\mathscr{U}_2$. Then if $\mathscr{F}$ is a regular filter with respect to $\mathscr{U}_1$, it will also be regular with respect to $\mathscr{U}_2$. But on the other hand, if $\mathscr{F}$ is a Cauchy filter with respect to $\mathscr{U}_2$, it will also be Cauchy with respect to $\mathscr{U}_1$. Hence $\hat{S}_1$ and $\hat{S}_2$ need not be (set-theoretically) comparable although $\mathscr{U}_1$ and $\mathscr{U}_2$ are comparable.

If, however, the above structures $\mathscr{U}_1$ and $\mathscr{U}_2$ belong to the same p-equivalence class, then they will possess the same regular filters, and hence in this case $\hat{S}_1 \supset \hat{S}_2$. Moreover, the $\hat{S}_2$-restriction of a $\hat{\mathscr{U}}_1$-*entourage* of the form (2.1) will be the corresponding $\hat{\mathscr{U}}_2$-*entourage* determined by the same *entourage* V of the original structure $\mathscr{U}_1$, and hence of $\mathscr{U}_2$. In other words, $\hat{\mathscr{U}}_2$ is *finer* than the structure induced on $\hat{S}_2$ from $\hat{\mathscr{U}}_1$.

Let $(S, \mathscr{T})$ be a completely regular topological space, and let us use the therm *completion of* $(S, \mathscr{T})$ in the meaning "completion of S with respect to some uniform structure $\mathscr{U}$ compatible with the topology $\mathscr{T}$", It is natural to say that a completion $(\hat{S}_1, \hat{\mathscr{U}}_1)$ of $(S, \mathscr{T})$ is *larger* than another completion $(\hat{S}_2, \hat{\mathscr{U}}_2)$, provided $\hat{S}_2$ may be mapped uniformly continuously and $1-1$ into $\hat{S}_1$ in such a way that every point of S is invariant. Then we may conclude: *Within each* p-*equivalence class of uniform structures compatible with a given topology on* S, *coarser structures determine larger completions.*

Particular attention should be devoted to the completion determined by the coarsest uniform structure $\mathscr{U}_{\mathscr{P}}$ of a given p-equivalence class $\mathscr{P}$ of uniform structures compatible with a given (completely regular) topology $\mathscr{T}$ on S. As $\mathscr{U}_{\mathscr{P}}$ is totally bounded, this completion is a compactification, and every (dense) compactification may be obtained in this way (cf. [1]). Thus there is a 1–1 correspondence between p-equivalence classes and compactifications, and by Theorem 2 the extended space $\bar{S}$ of the compactification determined by $\mathscr{P}$, consists of *all* maximal regular filters with respect to $\mathscr{P}$ (i.e. with respect to an arbitrary structure of the class $\mathscr{P}$).

We shall say that a p-equivalence class $\mathscr{P}_1$, of uniform structures compatible with $\mathscr{T}$, is *finer* (resp. *coarser*) than another class $\mathscr{P}_2$, provided that $\mathscr{U}_{\mathscr{P}_1}$ is finer (resp. coarser) than $\mathscr{U}_{\mathscr{P}_2}$. We remember that a compactification $(\bar{S}_1, \bar{\mathscr{T}}_1)$ of $(S, \mathscr{T})$ is said to be *larger* than another compactification $(\bar{S}_2, \bar{\mathscr{T}}_2)$, provided that $\bar{S}_1$ may be mapped continuously (but not necessarily $1-1$) *onto* $\bar{S}_2$ in such a way that each point of S is invariant [5, p. 151]. It should be noticed that this definition is not a complete analogue of the preceding definition of order for completions. In both cases the larger extension has the higher set theoretic power, but the topological requirement is of opposite nature in the two cases. (This distinction is not as artificial as it may appear. Completeness is connected with the convergence of Cauchy filters, and compactness with the convergence of ultrafilters. Now, passage from one uniform structure to a coarser structure compatible with the same topology may yield some new non-convergent Cauchy filters. Hence the space has become "less complete" than it originally was. On the other hand, passage from one topology to a coarser topology may render convergent some ultrafilters which were not originally convergent. Hence the space has become "more compact" than it originally was.)

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two p-equivalence classes of uniform structures compatible with the same (completely regular) topology $\mathscr{T}$ on S, and let $\mathscr{P}_1$ be finer than $\mathscr{P}_2$. We shall consider the mapping η of $\bar{S}_1$ which assigns to each point of $\bar{S}_1$, i.e. to each maximal regular filter $\mathscr{F}$ with respect to $\mathscr{P}_1$, its envelope $\eta(\mathscr{F})$ with respect to (an arbitrary) uniform structure of) the coarser class $\mathscr{P}_2$. In order to see that η maps $\bar{S}_1$ *onto* $\bar{S}_2$, we notice that each filter $\mathscr{F}' \in \bar{S}_2$ is also regular with respect to $\mathscr{P}_1$, and hence may be extended by Zorns lemma to a filter $\mathscr{F} \in \bar{S}_1$. Then, clearly, $\eta(\mathscr{F}) \supset \mathscr{F}'$, that is $\eta(\mathscr{F}) = \mathscr{F}'$. To verify that η is continuous, let V be an *entourage* of $\mathscr{U}_{\mathscr{P}_2}$, and let V_0 be an *entourage* of $\mathscr{U}_{\mathscr{P}_2}$, and hence also of $\mathscr{U}_{\mathscr{P}_1}$, such that $\overset{3}{V_0} \subset V$. Let $\mathscr{F}, \mathscr{G} \in \bar{S}_1$, and $(\mathscr{F}, \mathscr{G}) \in \hat{V}_0$ (cf. (2.1)). Then there exists a set $A \in \mathscr{F} \cap \mathscr{G}$ such that $A \times A \subset V_0$. Hence

$$V_0(A) \times V_0(A) \subset \overset{3}{V_0} \subset V .$$

But

$$V_0(A) \in \eta(\mathscr{F}) \cap \eta(\mathscr{G}) ,$$

and so $(\eta(\mathscr{F}), \eta(\mathscr{G})) \in \hat{V}$. This proves the (uniform) continuity of η. Thus, *the* $\mathscr{P}_1$ *compactification is larger than the* $\mathscr{P}_2$ *compactification.*

The conclusions of the preceding discussion may be summed up in the following theorem:

THEOREM 3. *There is a biunique and order preserving correspondence between the* p-*equivalence classes of uniform structures compatible with the topology on a completely regular space and the compactifications of the space (i.e. finer classes determine larger compactifications). Within each class, however, there is a biunique and order reversing correspondence between the uniform structures and their completions. In particular, the compactification determined by a given* p-*equivalence class is the completion determined by the coarsest structure of the class.*

We shall briefly indicate how the well-known theorems on maximal and minimal compactifications (Stone-Čech, Alexandroff), may be obtained by the methods outlined above. The former is the compactification determined by the p-equivalence class of the finest uniform structure compatible with the given topology. (This finest structure is the supremum of all structures compatible with the topology [2, p. 135]. The "universal mapping property" may be obtained by application of the extension theorem of uniformly continuous functions [5, p. 151] [2, p. 151]). The minimal compactification may be performed for locally compact spaces (and no others, cf. [6, p. 129] [7, p. 561]. It is the compactification (and completion) determined by the totally bounded uniform structure having a fundamental system of entourages of the form

$$V = \bigcup_{i=1}^{n} (U_i \times U_i) , \tag{2.2}$$

where $\{U_i\}_{1 \leq i \leq n}$ is an open covering of S, and all U_i except one, are relatively compact. Here we have exactly one maximal regular filter which is not of the trivial form $\mathscr{F}_x$, namely the filter of complements of the relatively compact sets.

By Theorem 3 there always exists a maximal completion among the completions determined by the structures of a given p-equivalence class, but will there always exist a minimal completion among those?

This question is equivalent to the problem on the existence of a *finest* uniform structure of a given p-equivalence class. This question is answered affirmatively in the case where there exists a metric uniform structure of the class. (Yu. M. Smirnov [7, p. 570].) But the general problem is still unsolved, as far as we know.

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