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HOMOGENEOUS UNIVERSAL RELATIONAL SYSTEMS

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In [5] it was proved that under suitable conditions on the class $\boldsymbol{K}$ of relational systems, and under the assumption that the Generalized Continuum Hypothesis holds, there exists for each ordinal $\alpha > 0$ an $(\aleph_\alpha, \boldsymbol{K})$ universal system $\mathfrak{A}$. This system $\mathfrak{A}$ is in general not unique up to isomorphism, and R. Baer raised the question whether it is possible to impose additional conditions upon $\mathfrak{A}$ which would make it unique. In this note we shall show that if α is not a limit ordinal, then the condition of homogeneity introduced below will serve this purpose. We also take this opportunity to point out two ways in which the principal results in [5] can be generalized: by allowing the relational systems to consist of denumerably many relations, and by replacing the amalgamation property IV in [5] by a somewhat weaker, but closely related, property. Both these generalizations were discovered independently by R. Vought.

The changes needed in the arguments of [5] in order to obtain the principal results of that paper under the modified conditions in $\boldsymbol{K}$, for the case when $\alpha > 0$, are of a very trivial nature and will be left to the reader. Only where the condition of homogeneity is involved will we therefore give the arguments in detail.

As indicated, we shall be concerned with classes of relational systems,

$$\mathfrak{A} = \langle A, R_0, R_1, \ldots, R_\tau, \ldots \rangle_{\tau<\varkappa} ,$$

where A is a non-empty set, $\varkappa$ is a finite or denumerable ordinal, and each R_τ with $\tau < \varkappa$ is a relation of some finite rank μ_τ over A, that is $R_\tau \subseteq A^{\mu_\tau}$. The weakened form of the amalgamation property IV is:

IV′. *If f_0 and f_1 are isomorphisms of $\mathfrak{A} \in \boldsymbol{K}$ into $\mathfrak{B}_0 \in \boldsymbol{K}$ and $\mathfrak{B}_1 \in \boldsymbol{K}$ respectively, then there exist isomorphisms g_0 and g_1 of $\mathfrak{B}_0$ and $\mathfrak{B}_1$ respectively into some system $\mathfrak{C} \in \boldsymbol{K}$, such that $g_0 f_0(x) = g_1 f_1(x)$ for all elements x of $\mathfrak{A}$.*

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This condition, as well as IV, seems to have been first considered by Fraïssé (cf. [1] and [2]). There is only one argument in [5] where the condition IV cannot be replaced by IV′ (with some very minor changes in reasoning), this is in the proof of Lemma 2.4. We cannot conclude on the basis of the conditions I, II, III, IV′, V and $\mathrm{VI}_{\alpha+1}$ that there exists $\mathfrak{A} \in \boldsymbol{K}$ with $^\star\mathfrak{A} = \aleph_\alpha$. We therefore replace I by a somewhat stronger condition:

I′. *For each ordinal ξ there exists $\mathfrak{A} \in \boldsymbol{K}$ with $^\star\mathfrak{A} \geqq \aleph_\xi$.*

The condition of homogeneity, mentioned above, can be stated as follows:

Definition. *Given a class $\boldsymbol{K}$ of relational systems, a system $\mathfrak{A}$ is said to be $\boldsymbol{K}$ homogeneous if and only if $\mathfrak{A} \in \boldsymbol{K}$ and, for any subsystem $\mathfrak{B} \in \boldsymbol{K}$ of $\mathfrak{A}$ with $^\star\mathfrak{B} < {}^\star\mathfrak{A}$, every isomorphism of $\mathfrak{B}$ into $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$.*

It is interesting to note that Hausdorff's generalizations of the rational numbers, the η-types considered in [3, chapter 6] are precisely the $\aleph_\alpha$ universal, homogeneous simply ordered systems. Another special case of this concept was considered by Fraïssé in [2], namely the case when $\boldsymbol{K}$ has the property that every subsystem of a member of $\boldsymbol{K}$ is again a member of $\boldsymbol{K}$.

We now prove the existence and the uniqueness of $(\aleph_\alpha, \boldsymbol{K})$ universal $\boldsymbol{K}$ homogeneous systems.

Theorem A. *Let α be a positive ordinal with the following two properties:*

(i) *If $\lambda < \omega_\alpha$ and if $\mathfrak{n}_\nu < \aleph_\alpha$ whenever $\nu < \lambda$, then $\Sigma_{\nu<\lambda}\mathfrak{n}_\nu < \aleph_\alpha$.*

(ii) *If $\mathfrak{n} < \aleph_\alpha$, then $2^{\mathfrak{n}} \leqq \aleph_\alpha$.*

If $\boldsymbol{K}$ is a class of systems which satisfies the conditions I′, II, III, IV′, V, *and* VI_α, *then there exists an $(\aleph_\alpha, \boldsymbol{K})$ universal $\boldsymbol{K}$ homogeneous system.*

Proof. By trivial changes in the proof of Theorem 2.9 of [5] we obtain, as there, systems $\mathfrak{A}_\xi, \mathfrak{A}_{\xi,\eta} \in \boldsymbol{K}$, associated with all the ordinals $\xi < \omega_\alpha$ and $\eta < \omega_\alpha$, satisfying the following conditions:

(1) $^\star\mathfrak{A}_{\xi,\eta} < \aleph_\alpha$ for all $\xi < \omega_\alpha$ and $\eta < \omega_\alpha$.

(2) $\mathfrak{A}_{\xi,\eta} \prec \mathfrak{A}_{\xi',\eta'}$ whenever $\xi \leqq \xi' < \omega_\alpha$ and $\eta \leqq \eta' < \omega_\alpha$.

(3) $\mathfrak{A}_\xi = \bigcup_{\eta<\omega_\alpha} \mathfrak{A}_{\xi,\eta}$ for all $\xi < \omega_\alpha$.

(4) $\mathfrak{A}_{\xi+1}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_{\xi,\xi}$ for all $\xi < \omega_\alpha$.

Letting

$$\mathfrak{A} = \bigcup_{\xi<\omega_\alpha} \mathfrak{A}_\xi \tag{5}$$

we infer as in [5] that $\mathfrak{A}$ is $(\aleph_\alpha, \boldsymbol{K})$ universal. We shall prove that $\mathfrak{A}$ is also $\boldsymbol{K}$ homogeneous.

Suppose $\mathfrak{B}_0 \in \boldsymbol{K}$ is a subsystem of $\mathfrak{A}$ with $*\mathfrak{B}_0 < \aleph_\alpha$, and assume that f_0 is an isomorphism of $\mathfrak{B}_0$ onto another subsystem $\mathfrak{C}_0$ of $\mathfrak{A}$. Consider any positive ordinal $\lambda < \omega_\alpha$, and suppose we have associated with each ordinal $\xi < \lambda$ two subsystems $\mathfrak{B}_\xi, \mathfrak{C}_\xi \in \boldsymbol{K}$ of $\mathfrak{A}$ with $*\mathfrak{B}_\xi < \aleph_\alpha$, as well as a function f_ξ mapping $\mathfrak{B}_\xi$ isomorphically onto $\mathfrak{C}_\xi$, in such a way that the following conditions are satisfied:

(6) $\mathfrak{B}_\xi \prec \mathfrak{B}_\eta$, $\mathfrak{A}_{\xi,\xi} \prec \mathfrak{B}_\eta$, $\mathfrak{C}_\xi \prec \mathfrak{C}_\eta$ and $\mathfrak{A}_{\xi,\xi} \prec \mathfrak{C}_\eta$ whenever $\xi < \eta < \lambda$.
(7) $f_\xi(x) = f_\eta(x)$ whenever $\xi < \eta < \lambda$ and x is a member of $\mathfrak{B}_\xi$.

If λ is a limit ordinal we let

$$\mathfrak{B}_\lambda = \bigcup_{\xi<\lambda} \mathfrak{B}_\xi \qquad \text{and} \qquad \mathfrak{C}_\lambda = \bigcup_{\xi<\lambda} \mathfrak{C}_\xi ,$$

and observe that there exists a unique function f_λ mapping $\mathfrak{B}_\lambda$ isomorphically onto $\mathfrak{C}_\lambda$ in such a way that

$$f_\lambda(x) = f_\xi(x) \qquad \text{whenever} \qquad \xi < \lambda \text{ and } x \text{ is a member of } \mathfrak{B}_\xi .$$

If λ is not a limit ordinal, say $\lambda = \nu + 1$, we observe that there exists $\zeta < \omega_\alpha$ such that $\mathfrak{B}_\nu$, $\mathfrak{C}_\nu$ and $\mathfrak{A}_{\nu,\nu}$ are subsystems of $\mathfrak{A}_{\zeta,\zeta}$. Since $\mathfrak{A}_{\zeta+1}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{A}_{\zeta,\zeta}$, it is also an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{C}_\nu$. Consequently there exists a function g which maps the extension $\mathfrak{A}_{\zeta,\zeta}$ of $\mathfrak{C}_\nu$ isomorphically onto a subsystem $\mathfrak{B}_\nu'$ of $\mathfrak{A}_{\zeta+1}$ in such a way that $g(x) = f_\nu^{-1}(x)$ whenever x is a member of $\mathfrak{C}_\nu$. Next we choose $\zeta' < \omega_\alpha$ so that $\mathfrak{A}_{\zeta,\zeta}$ and $\mathfrak{B}_\nu'$ are subsystems of $\mathfrak{A}_{\zeta',\zeta'}$, and infer that there exists a function f_λ which maps $\mathfrak{A}_{\zeta',\zeta'}$ isomorphically onto a subsystem $\mathfrak{C}_\lambda$ of $\mathfrak{A}_{\zeta'+1}$ in such a way that $f_\lambda(x) = g^{-1}(x)$ whenever x is a member of $\mathfrak{B}_\nu'$. We let $\mathfrak{B}_\lambda = \mathfrak{A}_{\zeta',\zeta'}$, and it is easy to check that (6) and (7) hold with λ replaced by $\lambda + 1$.

Having shown that we can always continue the process of picking out the systems $\mathfrak{B}_\xi$ and $\mathfrak{C}_\xi$, and the functions f_ξ, we conclude that we can so choose $\mathfrak{B}_\xi$, $\mathfrak{C}_\xi$ and f_ξ for all $\xi < \omega_\alpha$ that (6) and (7) hold with $\lambda = \omega_\alpha$. It clearly follows that

$$\mathfrak{A} = \bigcup_{\xi<\omega_\alpha} \mathfrak{B}_\xi = \bigcup_{\xi<\omega_\alpha} \mathfrak{C}_\xi ,$$

and that there exists a unique automorphism f of $\mathfrak{A}$ such that $f(x) = f_\xi(x)$ whenever $\xi < \omega_\alpha$ and x is a member of $\mathfrak{B}_\xi$. In particular, f is an extension of the isomorphism f_0 of $\mathfrak{B}_0$ into $\mathfrak{A}$. Thus $\mathfrak{A}$ is $\boldsymbol{K}$ homogeneous.

THEOREM B. *If α is any ordinal and if $\boldsymbol{K}$ is a class of systems which satisfies the conditions* II, V *and* VI_α, *then any two $(\aleph_\alpha, \boldsymbol{K})$ universal, $\boldsymbol{K}$ homogeneous systems are isomorphic.*

PROOF. We first observe that if $\mathfrak{A}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal, $\boldsymbol{K}$ homogeneous system, and if $\mathfrak{B}\in\boldsymbol{K}$ is a subsystem of $\mathfrak{A}$ with $^\star\mathfrak{B}<\aleph_\alpha$, then $\mathfrak{A}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{B}$. In fact, suppose $\mathfrak{B}\prec\mathfrak{C}\in\boldsymbol{K}$ and $^\star\mathfrak{C}<\aleph_\alpha$. Then there exists a function f which maps $\mathfrak{C}$ isomorphically onto a subsystem $\mathfrak{C}'$ of $\mathfrak{A}$. Since f maps $\mathfrak{B}$ isomorphically onto a subsystem of $\mathfrak{A}$, there exists an automorphism g of $\mathfrak{A}$ such that $g(x)=f(x)$ for all x in $\mathfrak{B}$. It follows that the function $g^{-1}f$ maps $\mathfrak{C}$ isomorphically into $\mathfrak{A}$, and that $g^{-1}f(x)=x$ for all x in $\mathfrak{B}$.

Now suppose $\mathfrak{A}$ and $\mathfrak{B}$ are $(\aleph_\alpha, \boldsymbol{K})$ universal, $\boldsymbol{K}$ homogeneous systems. If β is the smallest ordinal such that ω_β is cofinal with ω_α, then we can associate with each ordinal $\xi<\omega_\beta$ a subsystem $\mathfrak{A}_\xi\in\boldsymbol{K}$ of $\mathfrak{A}$ and a subsystem $\mathfrak{B}_\xi\in\boldsymbol{K}$ of $\mathfrak{B}$ in such a way that

(1) $^\star\mathfrak{A}_\xi<\aleph_\alpha$ and $^\star\mathfrak{B}_\xi<\aleph_\alpha$ for all $\xi<\omega_\beta$.

(2) $\mathfrak{A}_\xi\prec\mathfrak{A}_\eta$ and $\mathfrak{B}_\xi\prec\mathfrak{B}_\eta$ whenever $\xi<\eta<\omega_\beta$.

(3) $\mathfrak{A}=\bigcup_{\xi<\omega_\beta}\mathfrak{A}_\xi$ and $\mathfrak{B}=\bigcup_{\xi<\omega_\beta}\mathfrak{B}_\xi$.

Since $\mathfrak{A}$ is $(\aleph_\alpha, \boldsymbol{K})$ universal, $\mathfrak{B}$ is isomorphic to a subsystem of $\mathfrak{A}$, and in particular there exists a function g_0 which maps $\mathfrak{B}_0$ isomorphically onto a subsystem $\mathfrak{C}_0$ of $\mathfrak{A}$. By VI_α there exists a subsystem $\mathfrak{A}_0'\in\boldsymbol{K}$ of $\mathfrak{A}$ such that $\mathfrak{A}_0\prec\mathfrak{A}_0'$, $\mathfrak{C}_0\prec\mathfrak{A}_0'$, and $^\star\mathfrak{A}_0'<\aleph_\alpha$. Since $\mathfrak{B}$ is an $(\aleph_\alpha, \boldsymbol{K})$ universal extension of $\mathfrak{B}_0$, there exists a function f_0 mapping $\mathfrak{A}_0'$ isomorphically onto a subsystem $\mathfrak{B}_0'$ of $\mathfrak{B}$ in such a way that $f_0(x)=g_0^{-1}(x)$ for all x in $\mathfrak{C}_0$.

Suppose $0<\lambda<\omega_\beta$, and assume that we have associated with each ordinal $\xi<\lambda$ a subsystem $\mathfrak{A}_\xi'\in\boldsymbol{K}$ of $\mathfrak{A}$, a subsystem $\mathfrak{B}_\xi'\in\boldsymbol{K}$ of $\mathfrak{B}$, and a function f_ξ mapping $\mathfrak{A}_\xi'$ isomorphically onto $\mathfrak{B}_\xi'$, in such a way that the following conditions are satisfied:

(4) $^\star\mathfrak{A}_\xi'<\aleph_\alpha$ and $^\star\mathfrak{B}_\xi'<\aleph_\alpha$ for all $\xi<\lambda$.

(5) $\mathfrak{A}_\xi'\prec\mathfrak{A}_\eta'$ and $\mathfrak{B}_\xi'\prec\mathfrak{B}_\eta'$ whenever $\xi<\eta<\lambda$.

(6) $\mathfrak{A}_\xi\prec\mathfrak{A}_\xi'$ and $\mathfrak{B}_\xi\prec\mathfrak{B}_\xi'$ for all $\xi<\lambda$.

(7) $f_\xi(x)=f_\eta(x)$ whenever $\xi<\eta<\lambda$ and x is a member of $\mathfrak{A}_\xi'$.

Then the subsystems

$$\mathfrak{A}_\lambda''=\bigcup_{\xi<\lambda}\mathfrak{A}_\xi' \quad\text{and}\quad \mathfrak{B}_\lambda''=\bigcup_{\xi<\lambda}\mathfrak{B}_\xi'$$

of $\mathfrak{A}$ and $\mathfrak{B}$ are members of $\boldsymbol{K}$ with $^\star\mathfrak{A}_\lambda''<\aleph_\alpha$ and $^\star\mathfrak{B}_\lambda''<\aleph_\alpha$. Furthermore, there exists a unique function h_λ mapping $\mathfrak{A}_\lambda''$ isomorphically onto $\mathfrak{B}_\xi''$ in such a way that $h_\lambda(x)=f_\xi(x)$ whenever $\xi<\lambda$ and x is a member of $\mathfrak{A}_\xi'$. By VI_α there exists a subsystem $\mathfrak{D}_\lambda\in\boldsymbol{K}$ of $\mathfrak{B}$ such that $\mathfrak{B}_\lambda''\prec\mathfrak{D}_\lambda$, $\mathfrak{B}_\lambda\prec\mathfrak{D}_\lambda$, and $^\star\mathfrak{D}_\lambda<\aleph_\alpha$. There exists a function g_λ mapping $\mathfrak{D}_\lambda$ isomorphically onto a subsystem $\mathfrak{C}_\lambda$ of $\mathfrak{A}$ in such a way that $g_\lambda(x)=h_\lambda^{-1}(x)$ whenever x is a member of $\mathfrak{B}_\lambda''$. We can then find a subsystem $\mathfrak{A}_\lambda'\in\boldsymbol{K}$ of $\mathfrak{A}$

such that $\mathfrak{C}_\lambda \prec \mathfrak{A}_\lambda'$, $\mathfrak{A}_\lambda \prec \mathfrak{A}_\lambda'$, and $*\mathfrak{A}_\lambda' < \aleph_\alpha$, and there exists a function f_λ mapping $\mathfrak{A}_\lambda'$ isomorphically onto a subsystem $\mathfrak{B}_\lambda'$ of $\mathfrak{B}$ in such a way that $f_\lambda(x) = g_\lambda^{-1}(x)$ whenever x is a member of $\mathfrak{C}_\lambda$. It readily follows that (5)–(7) hold with λ replaced by $\lambda+1$. We conclude that we can choose $\mathfrak{A}_\xi$, $\mathfrak{B}_\xi$ and f_ξ for all $\xi < \omega_\beta$ in such a way that (5)–(7) hold with $\lambda = \omega_\beta$. In view of (6) we have

$$\mathfrak{A} = \bigcup_{\xi<\omega_\beta} \mathfrak{A}_\xi' \qquad \text{and} \qquad \mathfrak{B} = \bigcup_{\xi<\omega_\beta} \mathfrak{B}_\xi ,$$

and because of (7) there eixts a function f mapping $\mathfrak{A}$ isomorphically onto $\mathfrak{B}$ in such a way that $f(x) = f_\xi(x)$ whenever $\xi < \omega_\beta$ and x is a member of $\mathfrak{A}_\xi'$.

In conclusion three remarks are in order.

Remark 1. Under the assumption that the Generalized Continuum Hypothesis holds, it was shown in [5] that if $\boldsymbol{K}$ satisfies the conditions I–V and VI_1, then there exists an $(\aleph_\alpha, \boldsymbol{K})$ universal system for every ordinal $\alpha > 0$, and of course the same is true if $\boldsymbol{K}$ satisfies I′, II, III, IV′, V and VI_1. It is interesting to observe that neither set of conditions implies the existence of an $(\aleph_\alpha, \boldsymbol{K})$ universal, $\boldsymbol{K}$ homogeneous system. In fact, these conditions are clearly satisfied if $\boldsymbol{K}$ is the class of all simply ordered systems, and it is easy to show that the existence of an $\aleph_\alpha$ universal, homogeneous, simply ordered system $\mathfrak{A} = \langle A, \leqq \rangle$ implies that α satisfies the condition (i) in Theorem A. To show this, assume that (i) fails. Then $\mathfrak{A}$ must be cofinal with a well-ordered subsystem $\mathfrak{B}$ of type ω_β where $\beta < \alpha$. Now take a well-ordered subsystem $\mathfrak{C}$ of $\mathfrak{A}$ of the type $\omega_\beta + 1$, and let $\mathfrak{D}$ be the initial segment of $\mathfrak{C}$ of type ω_β. Then $\mathfrak{B}$ and $\mathfrak{D}$ are isomorphic, but this isomorphism cannot be extended to an automorphism of $\mathfrak{A}$ because $\mathfrak{D}$ is not cofinal with $\mathfrak{A}$. Clearly, similar remarks apply to any class of partially ordered sets which includes the class of all well-ordered sets, e.g. to the class of all partially ordered sets, the class of all lattices, the class of all modular lattices, and the class of all distributive lattices.

Remark 2. In [5] it was observed that when the results of that paper are applied to various familiar classes of algebraic systems, condition IV is usually the only one whose verification causes difficulties. While IV′ is a weaker and in some respects a more natural condition than IV, it appears to be equally difficult to apply as IV. Thus we do not know whether IV′ holds in the class of all demi-groups (systems with an associative binary operation and an identity element), the class of all distributive lattices, or the class of all modular lattices. As was shown in [5], the first two of these classes do not satisfy IV.

REMARK 3. R. Vought has announced in [6] certain far-reaching extensions of the principal results of [5]. Roughly speaking, under the assumption of the Generalized Continuum Hypothesis these results are extended to all classes K which can be characterized by first order sentences, and for which the conditions I′ and III hold. However, it is not known whether similar generalizations can be found for the stronger results given here. For instance, it follows from Vought's theorem that if the Generalized Continuum Hypothesis holds, then there exists for each ordinal α an $\aleph_\alpha$ universal distributive lattice, but we do not know whether there exists (even when α is not a limit ordinal) an $\aleph_\alpha$ universal, homogeneous, distributive lattice.

ADDED IN PROOFS: Professor Vought has informed me of certain unpublished results by Mr. Michael Morley concerning the existence of homogeneous universal systems. Like Vought, Morley is concerned with the class K of all models of a consistent set of well-formed formulas in a countable first order language. Except for this, the conditions which he imposes upon K are similar to the ones used here. His principal result is therefore closely related to Theorem A above.

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