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# A NOTE ON STEINER TRIPLE SYSTEMS

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## 1.

Given a set $E$ of $e$ elements, we denote by Steiner triple system [5] an arrangement of the elements of $E$ in triples in such a way that each pair of elements is contained in exactly one triple. It is known [2, 1], that a necessary and sufficient condition for the existence of a Steiner triple system is that $e \equiv 1$ or $3 \pmod 6$. The number of the triples in such a system is $\frac{1}{6}e(e-1)$.

In this note a new method of construction of Steiner triple systems in the case $e=6t+1$ will be given. This method has been employed by Skolem [4] for $t \equiv 0$ or $1 \pmod 4$, but it seems that for $t \equiv 2$ and $3 \pmod 4$ its use has been unknown so far.

## 2.

It has been proved by Skolem [3] that it is possible to distribute the integers $1,2,\ldots,2n$ in $n$ pairs $(a_r,b_r)$ such that $b_r-a_r=r$, $r=1,2,\ldots,n$, if and only if $n \equiv 0$ or $1 \pmod 4$.

For $n=4m$ denote the pairs by $(a_r^0,b_r^0)$, and the distribution is made as follows (see [3]):

| $r$ | $a_r^0$ | $b_r^0$ | |
|---|---|---|---|
| $4m-2\alpha$ | $4m+\alpha$ | $8m-\alpha$ | $\alpha = 0,1,\ldots,2m-1;$ |
| $1$ | $m$ | $m+1$ | |
| $2m-3-2\alpha$ | $m+2+\alpha$ | $3m-1-\alpha$ | $\alpha = 0,1,\ldots,m-3;$ |
| $2m-1$ | $2m$ | $4m-1$ | $m \geqq 2;$ |
| $4m-1-2\alpha$ | $\alpha$ | $4m-1-\alpha$ | $\alpha = 1,2,\ldots,m-1;$ |
| $4m-1$ | $2m+1$ | 6m | |

For $n=4m+1$ denote the pairs by $(a_r^1,b_r^1)$, and the distribution is

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| $r$ | $a_r^1$ | $b_r^1$ | |
|---|---|---|---|
| $2\alpha$ | $2m+1-\alpha$ | $2m+1+\alpha$ | $\alpha = 1,2,\ldots,2m-1;$ |
| $4m$ | $2m+1$ | $6m+1$ | |
| $1$ | $7m+2$ | $7m+3$ | $m \geqq 1;$ |
| $1+2\alpha$ | $6m+1-\alpha$ | $6m+2+\alpha$ | $\alpha = 1,2,\ldots,m-1;$ |
| $2m+1$ | $4m+1$ | $6m+2$ | |
| $2m+1+2\alpha$ | $5m+2-\alpha$ | $7m+3+\alpha$ | $\alpha = 1,2,\ldots,m-1;$ |
| $4m+1$ | $1$ | $4m+2$ | |

which is slightly different from that of Skolem [3] but has the advantage of being valid for $m=1$.

For $n \equiv 2$ or $3 \pmod 4$ such distribution is impossible. For $n=4m+3$, however, we can distribute the integers $1,2,\ldots,4m+3,4m+5,\ldots,8m+7$ (the first $8m+7$ positive integers, $4m+4$ excepted) in $n$ pairs $(a_r^3, b_r^3)$ such that $b_r^3-a_r^3=r$, $r=1,2,\ldots,n$. This is done in the following way:

| $r$ | $a_r^3$ | $b_r^3$ | |
|---|---|---|---|
| $2\alpha$ | $2m+2-\alpha$ | $2m+2+\alpha$ | $\alpha = 1,2,\ldots,2m+1;$ |
| $1$ | $7m+6$ | $7m+7$ | |
| $1+2\alpha$ | $6m+4-\alpha$ | $6m+5+\alpha$ | $\alpha = 1,2,\ldots,m;$ |
| $2m+3$ | $6m+4$ | $8m+7$ | $m \geqq 1;$ |
| $2m+3+2\alpha$ | $5m+4-\alpha$ | $7m+7+\alpha$ | $\alpha = 1,2,\ldots,m-1;$ |
| $4m+3$ | $2m+2$ | $6m+5$ | |

## 3.

Consider now the set $T=\{1,2,\ldots,6t\}$ of the first $6t$ positive integers. It is possible to form $t$ mutually disjoint triples $(s,h_s,k_s)$, $s=1,2,\ldots,t$, in such a way that each integer of $T$ appears exactly once in some triple either as an element or as the sum of two elements and that $s+h_s+k_s=6t+1$, $s=1,2,\ldots,t$.

The construction of such triples is given below and it may be easily checked that they have the required properties.

| For | $s$ | $h_s$ | $k_s$ | |
|---|---|---|---|---|
| $t = 4m$ | $r$ | $t+a_r^0$ | $5t+1-b_r^0$ | $r = 1,2,\ldots,4m;$ |
| $t = 4m+1$ | $r$ | $t+a_r^1$ | $5t+1-b_r^1$ | $r = 1,2,\ldots,4m+1;$ |
| $t = 8m+2$ or | $2r$ | $2t+2a_r^1$ | $4t+1-2b_r^1$ | $r = 1,2,\ldots,4m+1;$ |
| $8m+3$ | $2r-1$ | $2t-4m-r$ | $4t+4m+2-r$ | $r = 1,2,\ldots,t-4m-1;$ |
| | $2r$ | $2t+2a_r^3$ | $4t+1-2b_r^3$ | $r = 1,2,\ldots,4m+3;$ |
| $t = 8m+6$ or | $2r-1$ | $2t-4m-3-r$ | $4t+4m+5-r$ | $r = 1,2,\ldots,t-4m-4;$ |
| $8m+7$ | $t-1$ | $2t$ | $3t+2$ | for $t = 8m+6;$ |
| | $t$ | $2t$ | $3t+1$ | for $t = 8m+7.$ |

## 4.

Given a set $E=\{0,1,\ldots,6t\}$ of $e=6t+1$ elements it can now be easily shown that the triples

$$\text{(i)} \qquad (x,x+s,x+s+h_s), \quad x = 0,1,\ldots,6t; \qquad s = 1,2,\ldots,t\,,$$

(all numbers taken modulo $6t+1$) form a Steiner triple system.

Note that the number of triples in our system $(6t+1)t=\frac{1}{6}e(e-1)$ is the correct number of triples in Steiner triple system; accordingly it will suffice to show that every pair of elements of $E$ appears at least once in some triple (i). Now the differences between the elements of a triple (i) are modulo $6t+1$:

$$s, \quad h_s, \quad s+h_s, \quad 6t+1-s = h_s+k_s, \quad 6t+1-h_s = s+k_s,$$
$$6t+1-s-h_s = k_s\,,$$

and according to § 3 each of the numbers $1,2,\ldots,6t$ appears as the difference of two elements of a triple (i) for some $s$. For any pair of elements of $E$ we can therefore find the triple which contains it by choosing a suitable $s$ according to the difference between the elements of the pair and a suitable $x$.

### REFERENCES


1. E. Netto, *Lehrbuch der Combinatorik*, Zweite Auflage, Berlin, 1927.
2. M. Reiss, *Über eine Steinersche combinatorische Aufgabe*, J. Reine Angew. Math. 56 (1859), 326–344.
3. Th. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. 5 (1957), 57–68.
4. Th. Skolem, *Some remarks on the triple systems of Steiner*, Math. Scand. 6 (1958), 273–280.
5. J. Steiner, *Combinatorische Aufgabe*, J. Reine Angew. Math. 45 (1853), 181–182 (= *Gesammelte Werke* II, Berlin, 1884, 435–436).



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