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ON INEQUALITIES OF POINCARÉ'S TYPE

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1. Introduction.

Let Ω be an open connected domain in real n-dimensional Euclidean space E^n. We shall assume that Ω has finite Lebesgue measure $m(\Omega)$. Denote by $\mathscr{C}^1(\Omega)$ the class of all infinitely differentiable functions on Ω for which

$$\|u\|_1^2 = \int\limits_\Omega \left(\sum_1^n \left|\frac{\partial u}{\partial x_i}\right|^2 + |u|^2\right) dx \;<\; +\infty\,.$$

If we close $\mathscr{C}^1(\Omega)$ in the norm $\|u\|_1$, we get a Hilbert space $\mathscr{E}^1(\Omega)$ with the inner product

$$(u,v)_1 = \int\limits_\Omega \left(\sum_1^n \frac{\partial u}{\partial x_i}\frac{\partial \bar v}{\partial x_i} + u\bar v\right) dx\,.$$

A well-known fact is that if Ω is a cube [1, p. 488] or a Lipschitz image of a cube [7] then there is a constant σ such that Poincaré's inequality

$$\int\limits_\Omega |u|^2\,dx \;\leqq\; \sigma\int\limits_\Omega \left(\sum_1^n \left|\frac{\partial u}{\partial x_i}\right|^2\right) dx + \frac{1}{m(\Omega)}\left|\int\limits_\Omega u\,dx\right|^2 \tag{1.1}$$

is satisfied for all $u \in \mathscr{E}^1(\Omega)$. A domain Ω with this property is usually called a domain of Nikodym type.

The aim of this paper is to show that the integral in the last term of (1.1) can be replaced by other linear functionals. We obtain in this way inequalities, which are equivalent to (1.1), i.e. are valid if and only if Ω is of Nikodym type. We also study the corresponding question for inequalities similar to (1.1) but containing derivatives of higher order. Finally we study the behaviour of unbounded domains of Nikodym type at infinity.

2. Preliminaries about Beppo-Levi functions.

Concerning the details we refer to Deny and Lions [3], and Nikodym [6].

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DEFINITION. *A distribution u on Ω $(u\in\mathscr{D}'(\Omega))$ is said to be a Beppo–Levi function on Ω, if the distribution derivatives $\partial u/\partial x_i$, $i=1,\ldots,n$, belong to $L^2(\Omega)$, the class of all square integrable functions on Ω.*

Let $BL(\Omega)$ be the topological vector space of all Beppo–Levi functions on Ω with the topology defined by the seminorm

$$|u|_1 = \left(\int_\Omega \sum_1^n \left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}.$$

Denote by Λ the constant functions on Ω. Then $|u|_1$ is a norm in the quotient space $BL(\Omega)/\Lambda$, which is complete in this norm (cf. [6], [3]). The functions in the space $\mathscr{E}^1(\Omega)$ are Beppo–Levi functions, $\mathscr{E}^1(\Omega)=BL(\Omega)\cap L^2(\Omega)$.

It is known [3] that a domain Ω is of Nikodym type if and only if it satisfies the condition

$$\text{A:}\quad \text{If } \partial u/\partial x_i\in L^2(\Omega) \text{ for } i=1,\ldots,n, \text{ then } u\in L^2(\Omega)\,.$$

3. The inequality.

We are now going to show an inequality, which is equivalent to (1.1). Using the notation

$$|u|_0 = \left\{\int_\Omega |u|^2\,dx\right\}^{\frac{1}{2}},$$

THEOREM 1. *If there is a continuous linear functional L on the space $\mathscr{E}^1(\Omega)$ with $L(1)\neq 0$ and a constant σ such that*

$$|u|_0 \leqq \sigma^{\frac{1}{2}}|u|_1 + \frac{(m(\Omega))^{\frac{1}{2}}}{|L(1)|}\,|L(u)| \quad \text{for all } u\in\mathscr{E}^1(\Omega)\,,$$

then Ω is of Nikodym type,

PROOF. Put $u_0=u-\big(L(1)\big)^{-1}L(u)$, where $u_0\in\mathscr{E}^1(\Omega)$ and $L(u_0)=0$. Then

$$|u_0|_0^{\,2} \leqq \sigma|u|_1^{\,2}$$

and

$$\inf_\lambda |u+\lambda|_0^{\,2} \leqq \left|u-\frac{L(u)}{L(1)}\right|_0^2 = |u_0|_0^{\,2} \leqq \sigma|u|_1^{\,2}, \quad \lambda = \text{constant}.$$

The inequality

$$\inf_\lambda |u+\lambda|_0^{\,2} \leqq \sigma|u|_1^{\,2} \quad \text{for all} \quad u\in\mathscr{E}^1(\Omega)$$

is easily seen to be identical with (1.1).

THEOREM 2. *If Ω is of Nikodym type, and if L is a continuous linear functional on the space $\mathscr{E}^1(\Omega)$ with $L(1)\neq 0$, then there is a constant σ such that*

$$|u|_0 \leqq \sigma^{\frac{1}{2}}|u|_1 + \frac{(m(\Omega))^{\frac{1}{2}}}{|L(1)|}|L(u)| \qquad \textit{for all} \quad u \in \mathscr{E}^1(\Omega)\,.$$

PROOF. Let N be the class of all functions u in $\mathscr{E}^1(\Omega)$ for which $L(u)=0$. As L is continuous, N is a closed hyperplane in $\mathscr{E}^1(\Omega)$. N is a Banach space with the norm

$$\|u\|_1 = \left(|u|_1^2+|u|_0^2\right)^{\frac{1}{2}}\,.$$

Let M be the class of all functions in $BL(\Omega)$ for which $L(u)=0$. Then $|u|_1$ is a norm in M, since $L(1) \neq 0$. Furthermore, M is a closed subspace of $BL(\Omega)$. For let $\{u_n\}$ be a Cauchy sequence in M.

$$|u_n-u_m|_1 \to 0 \quad \text{as} \quad n, m \to \infty\,.$$

Denote by $\dot{u}$ the element in $BL(\Omega)/\Lambda$ that contains $u \in BL(\Omega)$. With this notation

$$u_n \in \dot{u}_n \in BL/\Lambda\,.$$

Clearly $\{\dot{u}_n\}$ is a Cauchy sequence in BL/Λ. As BL/Λ is complete $\dot{u}_n$ has a limit $\dot{u} \in BL/\Lambda$ in the norm $|\ |_1$, when n tends to infinity. Take $u_0 \in \dot{u}$ and put

$$u = u_0 - \frac{L(u_0)}{L(1)}\,.$$

Then u belongs to M and

$$|u_n-u|_1 \to 0 \qquad \text{when} \quad n \to \infty\,.$$

As Ω is of Nikodym type and therefore satisfies the condition (A), N and M are algebraically isomorphic. Let φ be the identical mapping of N on M. If u_n tends to zero in the norm $|\ |_1$ and to g in $\|\ \|_1$, then as a consequence of the inequality $|u|_1 \leqq \|u\|_1$ g is zero. The theorem of the closed graph [2] then shows that φ is continuous, e.g. there is a constant σ' such that

$$\|u\|_1 \leqq \sigma'|u|_1$$

or

$$|u|_0 \leqq \sigma^{\frac{1}{2}}|u|_1 \qquad \text{for all} \quad u \in N\,.$$

For $u \in \mathscr{E}^1$ we write

$$u = u_0 + \frac{L(u)}{L(1)},$$

where $u_0 \in N$.

$$\begin{aligned} |u|_0 \leqq \left| u_0 + \frac{L(u)}{L(1)} \right|_0 &\leqq |u_0|_0 + \frac{(m(\Omega))^{\frac{1}{2}}}{|L(1)|}|L(u)| \\ &\leqq \sigma^{\frac{1}{2}}|u_0|_1 + \frac{(m(\Omega))^{\frac{1}{2}}}{|L(1)|}|L(u)| \\ &= \sigma^{\frac{1}{2}}|u|_1 + \frac{(m(\Omega))^{\frac{1}{2}}}{|L(1)|}|L(u)|\,. \end{aligned}$$

REMARK. The inequality

$$|u|_0 \leqq \sigma^{\frac{1}{2}}|u|_1 + \frac{m^{\frac{1}{2}}}{|L(1)|}|L(u)|$$

can be improved to

$$|u|_0^2 \leqq \sigma'|u|_1^2 + \frac{m}{|L(1)|^2}|L(u)|^2$$

if and only if $L(u) = \text{constant}\cdot\int_\Omega u\,dx$.

Let $u \in \mathscr{E}^1$ be realvalued and k a real constant. Then

$$|u+k|_0^2 \leqq \sigma'|u+k|_1^2 + \frac{m}{|L(1)|^2}|L(u+k)|^2$$

or

$$|u|_0^2 + 2k\int_\Omega u\,dx \leqq \sigma'|u|_1^2 + \frac{m}{|L(1)|^2}|L(u)|^2 + \frac{2kmL(u}{|L(1)|}$$

for all k gives

$$L(u) = m^{-1}L(1)\int_\Omega u\,dx\,.$$

4. A generalization of the preceding result.

Let

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u\,,$$

where

$$\alpha = (\alpha_1,\ldots,\alpha_n), \qquad |\alpha| = \sum_{i=1}^n \alpha_i\,,$$

and

$$x^\alpha = x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}\,.$$

Denote by $BL_m(\Omega)$ the class of all Beppo–Levi functions of order m on Ω, that is, all $u \in \mathscr{D}'(\Omega)$ such that

$$D^\alpha u \in L^2(\Omega) \quad \text{for all } \alpha \quad \text{with} \quad |\alpha| = m\,.$$

$|u|_m$ defined by

$$|u|_m^2 = \sum_{|\alpha|=m}\int_\Omega |Du|^2$$

is a seminorm in $BL_m(\Omega)$. Let Z be the class of all u such that $|u|_m = 0$, that is, the class of all polynomials of degree less than or equal to $m-1$. The quotient space $BL_m(\Omega)/Z$ is a complete [3], normed space with the norm $|u|_m$.

We denote by $\mathrm{A_m}$ the following condition on the domain Ω.

$\mathrm{A_m}$: If $u \in BL_m(\Omega)$ then $D^\alpha\, u \in L^2(\Omega)$ for all α, $|\alpha| \leqq m-1$.

LEMMA. *If a domain Ω satisfies* (A), *it satisfies* ($\mathrm{A_m}$).

PROOF. $u \in BL_m(\Omega)$ and (A) implies that $D^\beta u$, $|\beta| = m-1$, belongs to $L^2(\Omega)$. Also $D^\beta u \in L^2(\Omega)$ for all β, $|\beta| = m-1$, and (A) then gives us $D^\gamma u \in L^2(\Omega)$ for $|\gamma| = m-2$ etc.

$\mathscr{E}^m(\Omega)$ is the class of all $u \in \mathscr{D}'(\Omega)$ such that $D^\alpha u \in L^2$ for all $|\alpha| \leqq m$. The space $\mathscr{E}^m(\Omega)$ is a Banach space with the norm

$$\|u\|_m = \Big(\sum_{|\alpha| \leqq m} \int_\Omega |D^\alpha u|^2 \Big)^{\frac{1}{2}} .$$

THEOREM 3. *Suppose that Ω satisfies* (A_m) *and let L_α, $|\alpha| \leqq m-1$, be a collection of continuous linear functionals on $\mathscr{E}^m(\Omega)$ such that* $\det\big(L_\alpha(x^\beta)\big) \neq 0$, $|\beta| \leqq m-1$. *Then there are constants σ and τ_α such that*

$$\|u\|_{m-1} \leqq \sigma |u|_m + \sum_{|\alpha| \leqq m-1} \tau_\alpha |L_\alpha(u)| .$$

PROOF. Let N_α be the hyperplane in $\mathscr{E}^m(\Omega)$ defined by $L_\alpha(u) = 0$. Then

$$N = \bigcap_{|\alpha| \leqq m-1} N_\alpha$$

is a closed subspace of $\mathscr{E}^m(\Omega)$. The space M of all $u \in BL_m(\Omega)$, which satisfies $L_\alpha(u) = 0$ for all $|\alpha| \leqq m-1$, is a normed space with the norm

$$|u|_m = \Big(\sum_{|\alpha| = m} \int_\Omega |D^\alpha u|^2 \Big)^{\frac{1}{2}} .$$

This is a consequence of the assumption that $\det\big(L_\alpha(x^\beta)\big) \neq 0$.

In the same way as in theorem 2 one sees that M is complete in the norm $|\ |_m$, and that N and M are topologically equivalent, i.e. there is a constant σ' such that

$$\|u\|_m^2 \leqq \sigma' |u|_m^2$$

or

$$\|u\|_{m-1} \leqq \sigma |u|_m \quad \text{for all} \quad u \in N .$$

If $u \in \mathscr{E}^m(\Omega)$ we write $u = u_0 + p$, where $u_0 \in N$ and p is a polynomial of degree less than or equal to $m-1$, and get

$$\|u\|_{m-1} \leqq \|u_0\|_{m-1} + \|p(x)\|_{m-1} \leqq \sigma |u|_m + \sum_{|\alpha| \leqq m-1} \tau_\alpha |L_\alpha(u)| ,$$

where the constants τ_α depend only on Ω and L^β.

5. Examples.

1. Let Ω_1 be a Lebesgue-measurable subset of Ω with $\int_{\Omega_1} dx > 0$. Put $L(u) = \int_{\Omega_1} u\,dx$. L is a continuous linear functional on the space $\mathscr{E}^1(\Omega)$. According to theorem 2 the norms

$$\|u\|_1 \quad \text{and} \quad |u|_1 + |Lu|$$

are equivalent, if Ω is of Nikodym type.

2. Let Ω be of Nikodym type and let Γ be a compact sufficiently smooth part of the boundary of Ω. Then we can define a continuous mapping (cf. [5])

$$\mathscr{E}^1(\Omega) \ni u \to \gamma u \in L^2(\Gamma)$$

such that if u is continuous in $\Omega \cup \Gamma$ we have $u(x) = \gamma u(x)$ for $x \in \Gamma$. If moreover $\varrho \in L^2(\Gamma)$ and $\int_\Gamma \varrho\, ds \neq 0$, then

$$L(u) = \int_\Gamma \varrho u\, ds$$

defines a continuous linear functional on $\mathscr{E}^1(\Omega)$ such that $L(1) \neq 0$. Hence according to theorem 2 there is a constant σ wuch that

$$|u|_0 \leqq \sigma |u|_1 + \frac{m^{\frac{1}{2}}}{|\int_\Gamma \varrho\, ds|} \left| \int_\Gamma \varrho u\, ds \right| .$$

(See also Sandgren [7].)

6. A boundary problem.

Let Ω be an open connected domain in E^n such that there is a domain Ω_1 of Nikodym type containing Ω. $m(\Omega_1 - \Omega) > 0$. For $u \in \mathscr{D}(E^n)$ put

$$\|u\|_\Omega{}^2 = \int_{\complement \Omega} \left(\sum_1^n \left| \frac{\partial u}{\partial x_i} \right|^2 + \varkappa^2 |u|^2 \right) dx + a \int_\Omega \left(\sum_1^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right),$$

where $\varkappa$ and $a > 0$ are real constants. If we close $\mathscr{D}(E^n)$ in the norm $\|u\|_\Omega$, we get a Hilbert space H with the inner product

$$(u,v)_\Omega = \int_{\complement \Omega} \left(\sum_1^n \frac{\partial u}{\partial x_i} \frac{\partial \bar v}{\partial x_i} + \varkappa^2 u \bar v \right) dx + a \int_\Omega \left(\sum_1^n \frac{\partial u}{\partial x_i} \frac{\partial \bar v}{\partial x_i} \right) dx .$$

The restriction of $u \in H$ to Ω_1 is of course in $\mathscr{E}^1(\Omega_1)$. Hence (example 1) there is a constant C' such that

$$\int_{\Omega_1} |u|^2\, dx \leqq C' \left[\int_{\Omega_1} \left(\sum_1^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right) dx + \left| \int_{\Omega_1 - \Omega} u\, dx \right|^2 \right]$$

$$\leqq C \left[\int_{\Omega_1} \left(\sum_1^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right) dx + \int_{\Omega_1 - \Omega} |u|^2\, dx \right],$$

and we get

$$\int_{\Omega_1} |u|^2 \, dx \leqq C \|u\|_\Omega^2 . \tag{6.1}$$

We denote by V the vector space of all functions $u \in H$ with the property that there is a function $w \in L^2(\Omega)$ such that

$$a \int_\Omega w\bar{v} \, dx = (u,v)_\Omega \qquad \text{for all} \quad v \in H . \tag{6.2}$$

The mapping P defined by

$$V \ni u \overset{P}{\rightarrow} w \in L^2(\Omega)$$

is linear and closed. Let $D(P)$ and $R(P)$ be the domain of definition and the range of values of P, respectively. By definition $D(P) = V$.

THEOREM 4. *P is a selfadjoint extension of the Laplace-operator $-\Delta$ defined on $\mathscr{D}(\Omega)$, and $R(P) = L^2(\Omega)$.*

PROOF. That P is selfadjoint is obvious, and that it is an extension of $-\Delta$ follows from Weyl's lemma [4].

Let g be an arbitrary function in $L^2(\Omega)$. The functional F defined by

$$F(f) = \int_\Omega g\bar{f} \, dx, \qquad f \in H$$

is a continuous linear functional on H. Indeed, using (6.1) we have

$$|F(f)|^2 \leqq \int_\Omega |f|^2 \, dx \int_\Omega |g|^2 \, dx \leqq C \int_\Omega |g|^2 \, dx \, \|f\|_\Omega^2 .$$

Then, according to the theorem of Riesz–Fréchet, there is an element $h \in H$ such that

$$F(f) = \int_\Omega g\bar{f} \, dx = (h,f)_\Omega \qquad \text{for all} \quad f \in H ,$$

where $g = Ph$. As g is arbitrary in $L^2(\Omega)$, we have proved that $R(P) = L^2(\Omega)$.

The space V. For $u \in V$ we get from (6.2)

$$0 = (u,v)_\Omega = \int_{\complement \Omega} \left(\sum_1^n \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} + \varkappa^2 u\bar{v} \right) dx$$

$$= \int_{\complement \Omega} u(-\Delta + \varkappa^2)\bar{v} \, dx \qquad \text{for all} \quad v \in \mathscr{D}(\complement \Omega) .$$

We use Weyl's lemma and get that u is equal, a.e., to an infinitely differentiable function u_0 in $\complement\Omega$, and that

$$\Delta u_0 - \varkappa^2 u_0 = 0 .$$

Because $V \subset BL(E^n)$, the functions in V are absolutely continuous over the boundary of Ω on almost every line parallel with any co-ordinate axis [8].

If the boundary of Ω is sufficiently smooth, we get from Green's formula that

$$\frac{\partial u}{\partial n_e} = a\frac{\partial u}{\partial n_i},$$

where $\partial/\partial n_i$ and $\partial/\partial n_e$ are the interior and exterior normal derivatives taken in the same direction.

7. Unbounded domains of Nikodym type.

In this section we are going to study the behaviour at infinity of unbounded domains of Nikodym type.

Let C_R be the sphere of radius R around the origin in E^n and put

$$\Omega_R = \Omega \cap \complement C_R .$$

THEOREM 5. *If Ω is of Nikodym type, there is a constant $k>0$ such that*

$$m(\Omega_R) = O(e^{-kR}) \quad \text{when} \quad R \to +\infty . \tag{7.1}$$

PROOF. Put

$$L(u) = \int_{\Omega \cap C_{R_0}} u\, dx, \qquad m(\Omega \cap C_{R_0}) > 0 .$$

According to theorem 2 we have

$$|u|_0^2 \leqq \sigma(|u|_1^2 + |Lu|^2) \quad \text{for all} \quad u \in \mathscr{E}^1(\Omega) .$$

Choosing $u = u_n$,

$$u_n(x) = \begin{cases} 0 & \text{when} \quad |x| \leq R \\ (r-R)^n & \text{when} \quad |x| > R \end{cases} \qquad R > R_0, \quad r = (\sum x_i^2)^{\frac{1}{2}} ,$$

we get

$$\int_{\Omega_R} |u_n|^2\, dx \leqq \sigma n^2 \int_{\Omega_R} |u_{n-1}|^2\, dx .$$

$$\int_{\Omega_R} |u_n|^2\, dx \leqq \sigma^n (n!)^2 m(\Omega_R) . \tag{7.2}$$

On the other hand we have

$$(7.3) \qquad \int_{\Omega_R} |u_n|^2\, dx = \int_{\Omega_R} |r-R|^{2n}\, dx \geqq \int_{\Omega_{2R}} |r-R|^{2n}\, dx \geqq R^{2n} m(\Omega_{2R}) \,.$$

From (7.2) and (7.3) we get

$$\frac{m(\Omega_{2R})}{m(\Omega_R)} \leqq \sigma^n (n!)^2 R^{-2n} \leqq C \sigma^n n^{2n+1} e^{-2n} R^{-2n} \,,$$

in the last step using Stirling's inequality, where C is a constant independent of σ, n and R. Now we choose $n = [R\sigma^{-\frac{1}{2}}]$ and get

$$\frac{m(\Omega_{2R})}{m(\Omega_R)} \leqq C' [R\sigma^{-\frac{1}{2}}] e^{-2R\sigma^{-\frac{1}{2}}}$$

$$m(\Omega_R) \leqq C' e^{-R(\sigma^{-\frac{1}{2}}-\varepsilon)} m(\Omega_{\frac{1}{2}R}), \qquad \varepsilon > 0 \,.$$

Iterating this inequality once we have

$$m(\Omega_R) \leqq C'' e^{-R\sigma^{-\frac{1}{2}}} \,.$$

The estimation (7.1) is best possible in the sense that there are domains with the property

$$\lim_{R \to \infty} e^R m(\Omega_R) = 1 \,,$$

which satisfy Poincaré's inequality. We will show that the domain D in E^2 defined by

$$x_1 > 0, \quad 0 < x_2 < e^{-x_1}$$

is such a domain. We first prove the following lemma.

LEMMA. *All functions of one variable* $v \in \mathscr{E}^1(0,\infty)$ *satisfy*

$$(7.4) \qquad \int_0^\infty |v|^2\, dx \leqq 4 \int_0^\infty \left| \frac{dv}{dx} + \tfrac{1}{2} v \right|^2 dx + \left| \int_0^\infty v e^{-\frac{1}{2}x}\, dx \right|^2 .$$

PROOF. We observe that $\mathscr{E}^1(0,\infty)$ is the closure in the norm

$$\left(\int_0^\infty \left(\left| \frac{dv}{dx} \right|^2 + |v|^2 \right) dx \right)^{\frac{1}{2}}$$

of the linear closure of $\mathscr{D}(0,\infty) \cup e^{-\frac{1}{2}x}$. Putting $v = v_0 + \lambda e^{-\frac{1}{2}x}$, $v_0 \in \mathscr{D}(0,\infty)$, $\lambda =$ constant, we obtain

$$(7.5) \qquad \int_0^\infty |v|^2\, dx - \left| \int_0^\infty v e^{-\frac{1}{2}x}\, dx \right|^2 \leqq \int_0^\infty |v_0|^2\, dx \,,$$

and

$$(7.6)\qquad \int_0^\infty |v_0|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{v}_0|^2\,d\xi \leqq \frac{4}{2\pi}\int_{-\infty}^{+\infty} |\hat{v}_0|^2|\tfrac{1}{2}+i\xi|^2\,d\xi$$

$$= 4\int_0^\infty \left|\frac{dv_0}{dx}+\tfrac{1}{2}v_0\right|^2 dx = 4\int_0^\infty \left|\frac{dv}{dx}+\tfrac{1}{2}v\right|^2 dx\,,$$

where

$$\hat{v}_0 = \int_0^\infty e^{-ix\xi}v_0(x)\,dx\,.$$

Combining (7.5) and (7.6) we get (7.4).

By the transformation

$$x_1' = x_1,\quad x_2' = x_2e^{x_1},\quad v = ue^{-\frac{1}{2}x_1}\,,$$

the inequality

$$(7.7)\qquad \int_D |u|^2\,dx \leqq \sigma\left[\int_D \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial u_2}\right|^2\right)dx + \left|\int_D u\,dx\right|^2\right]$$

is transformed into

$$(7.8)\quad \int_{D'} |v|^2\,dx \leqq \sigma\left[\int_{D'}\left(\left|\frac{\partial v}{\partial x_1}+\tfrac{1}{2}v+x_2\frac{\partial v}{\partial x_2}\right|^2 + \left|\frac{\partial v}{\partial x_2}\right|^2 e^{2x_1}\right)dx + \right.$$

$$\left. + \left|\int_{D'} ve^{-\frac{1}{2}x_1}\,dx\right|^2\right],$$

where D' is the stripe $x_1>0$, $0<x_2<1$.

Instead of (7.8) we prove a somewhat stronger inequality

$$(7.9)\qquad \int_{D'} |v|^2\,dx \leqq \sigma\left[\int_{D'}\left(\left|\frac{\partial v}{\partial x_1}+\tfrac{1}{2}v\right|^2 + \left|\frac{\partial v}{\partial x_2}\right|^2\right)dx + \left|\int_{D'} ve^{-\frac{1}{2}x_1}\,dx\right|^2\right].$$

If $v\in\mathscr{E}^1(D')$, then $v(\cdot,x_2)\in\mathscr{E}^1(0,\infty)$ for almost every x_2, $0<x_2<1$. Hence $v(\cdot,x_2)$ satisfies (7.4). Integrating (7.4) from zero to one we get

$$(7.10)\qquad \int_{D'} |v|^2\,dx \leqq 4\int_{D'}\left|\frac{\partial v}{\partial x_1}+\tfrac{1}{2}v\right|^2 dx + \int_0^1\left|\int_0^\infty ve^{-\frac{1}{2}x_1}\,dx_1\right|^2 dx_2\,.$$

For $u(x_2)\in\mathscr{E}^1(0,1)$ we have

$$(7.11)\qquad \int_0^1 |u|^2\,dx_2 \leqq \int_0^1\left|\frac{du}{dx_2}\right|^2 dx_2 + \left|\int_0^1 u\,dx_2\right|^2.$$

Putting

$$u(x_2) = \int_0^\infty v(x_1, x_2) e^{-\frac{1}{2}x_1} \, dx_1$$

we obtain

$$\frac{du}{dx_2} = \int_0^\infty \frac{\partial v}{\partial x_2} e^{-\frac{1}{2}x_1} \, dx_1 \, .$$

Then (7.11) gives us the inequality

$$\int_0^1 \left| \int_0^\infty v e^{-\frac{1}{2}x_1} \, dx_1 \right|^2 dx_2 \leqq \int_{D'} \left| \frac{\partial v}{\partial x_2} \right|^2 dx + \left| \int_{D'} v e^{-\frac{1}{2}x_1} \, dx \right|^2 . \tag{7.12}$$

Combining (7.10) and (7.12) we have the desired inequality (7.9) with $\sigma = 4$.

REFERENCES

1. R. Courant und D. Hilbert, *Methoden der mathematischen Physik* II, Berlin, 1937.
2. N. Bourbaki, *Espaces vectoriels topologiques*, Ch. I (Act. Sci. Ind. 1189), Paris, 1953.
3. J. Deny and J. L. Lions, *Les espace du type de Beppo Levi*, Ann. Inst. Fourier Grenoble 5 (1955), 305–370.
4. L. Gårding, *On a lemma by H. Weyl*, Kungl. Fysiogr. Sällsk. i Lund Förh. 20 (1950), 250–253.
5. J. L. Lions, *Problèmes aux limites en théorie des distributions*, Acta Math. 94 (1955), 13–153.
6. O. Nikodym, *Sur une classe de fonctions considérée dans l'étude du problème de Dirichlet*, Fund. Math. 21 (1933), 129–150.
7. L. Sandgren, *A vibration problem*, Medd. Lunds Univ. Mat. Sem. 13 (1955).
8. L. Schwartz, *Théorie des distributions* I (Act. Sci. Ind. 1091), Paris, 1950.

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