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PYTHAGOREAN INEQUALITIES FOR CONVEX BODIES

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Consider a convex body in Euclidean three dimensional space and let $\sigma(u)$ be the area of the projection of this body upon a plane with normal direction u. It was conjectured by C. Carathéodory and proved by W. Blaschke [1, p. 148] that if u_1, u_2, u_3 are three mutually perpendicular directions, then

$$\sigma^2(u) \leqq \sum \sigma^2(u_i) .$$

In this note a class of such Pythagorean inequalities for convex bodies, (which includes that of Carathéodory), is shown to follow from one particular inequality. Also, in a certain sense, the cases of equality are determined.

Let K be a convex body in Euclidean n-dimensional space ($n \geqq 2$), $B(u)$ the breadth of K in the direction u. That is, if $h(u)$ is the support function of K, then

$$B(u) = h(u) + h(-u) .$$

Suppose $\{u_i\}$, $i = 1, 2, \ldots, n$, are a set of n mutually perpendicular directions and consider the circumscribing rectangular parallelotope of K which is bounded by the $2n$ supporting hyperplanes of K having outer normal directions $\pm u_i$. Call this circumscribing body C. Finally, let $P(u)$ be some point common to K and its supporting hyperplane with outer normal direction u. For any u, the points $P(u)$ and $P(-u)$ belong to the body C and so the distance d between them is less than or equal to the length of a diagonal of C, that is $d^2 \leqq \sum B^2(u_i)$. On the other hand, since $B(u)$ is the minimum distance between points of the hyperplanes containing $P(u)$ and $P(-u)$, we have

$$B^2(u) \leqq \sum B^2(u_i) . \tag{1}$$

The class of Pythagorean inequalities which we have in mind follow from (1). If $V(K_1, \ldots, K_{n-1}, u)$ denotes the mixed volume of convex bodies $K_1, \ldots, K_{n-1}$ and the unit segment in the direction u, then $V(K_1, \ldots, K_{n-1}, u)$ as a function of u, is known to be the support function of a convex body having a centre of symmetry [2, p. 44]. The breadth

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of this body in the direction u is then $2V(K_1,\ldots,K_{n-1},u)$ and so, with the preceding meaning for u_i, we have

$$V^2(K_1,\ldots,K_{n-1},u) \leqq \sum V^2(K_1,\ldots,K_{n-1},u_i)\,. \tag{2}$$

In particular, if we let $K_1=K_2=\ldots=K_p=K$ and $K_{p+1}=\ldots=K_{n-1}=S$ (where S denotes the unit spherical body), then

$$V(K_1,\ldots,K_{n-1},u) = W'_{n-p-1}(K,u)$$

which is the $(n-p-1)$th cross-section integral of the projection of K upon a hyperplane with normal direction u [2, p. 49]. In particular, $W'_0(K,u)=\sigma(u)$ for which (2) is Carathéodory's inequality.

Returning to (1), we introduce a Pythagorean defect $\delta(K)$ of K by

$$\begin{aligned}\delta(K) &= \max_{\{u_i\}}\left(\min_u \frac{\sum B^2(u_i)-B^2(u)}{B^2(u)}\right)\\ &= \max_{\{u_i\}}\left(\frac{\sum B^2(u_i)}{\max_u B^2(u)}-1\right).\end{aligned} \tag{3}$$

We have

THEOREM. $0\leqq\delta(K)\leqq n-1$. *Both bounds are attained, the lower being attained if and only if K is a segment.*

Inequality (1) asserts that δ is non-negative. By Pythagoras' theorem, $\delta=0$ if K is a segment. On the other hand, if $\delta=0$, from (3)

$$\sum B^2(u_i) = \max_u B^2(u)$$

for all $\{u_i\}$. Let $\bar{u}_1$ be a direction in which $B(u)$ attains its maximum and let $\bar{u}_2,\ldots,\bar{u}_n$ be such that $\{\bar{u}_i\}$ form a mutually perpendicular set of n directions. Then $\sum_{i\neq 1}B^2(\bar{u}_i)=0$, whence $B^2(\bar{u}_j)=0$ for $j=2,3,\ldots,n$ and so K is a one dimensional convex body, that is, a segment.

To show $\delta\leqq n-1$ we need only remark that $B^2(u_i)/\max_u B^2(u)\leqq 1$ with equality if and only if $B^2(u_i)=\max_u B^2(u)$. Therefore, if and only if K admits a circumscribing hypercube of edge length equal to the maximum breadth of K, we have $\delta=n-1$. This is the case, for example, if K is of constant breadth, or if K is an n-dimensional hypercube.

We finally note that the cases of equality in (2) can be found directly by suitably interpreting the statement of the theorem. Thus, for each $\{u_i\}$ there is a u for which we have equality in Carathéodory's inequality if and only if K lies in a hyperplane. On the other hand, the ratio $\sum\sigma^2(u_i)/\sigma^2(u)$ attains its maximum of n for a body of constant brightness or for a hypercube.

REFERENCES

1. W. Blaschke, *Kreis und Kugel*, Leipzig, 1916, Reprint New York 1949, Second edition, Berlin, 1956.
2. T. Bonnesen und W. Fenchel, *Konvexe Körper*, Berlin, 1934, Reprint, New York 1948.

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