MATH. SCAND. 8 (1960), 171–180

COMPARISON BETWEEN PLANE SYMMETRIC CONVEX BODIES AND PARALLELOGRAMS

EDGAR ASPLUND

Introduction.

Every plane symmetric compact convex body contains at least one parallelogram which expanded by $\frac{3}{2}$ will cover the original convex body. This is the main result of this paper, which thus gives a quantitative estimate of the "parallelogramness" of a general plane symmetric convex set. The writer is indebted to Dr. B. Grünbaum for the statement of the problem as well as for much help and encouragement.

In order to fix our ideas in a more general frame of reference, we make the following definitions. Let Γ be the set of all compact convex bodies in the Euclidean plane which are symmetric with respect to the origin and which have a non-empty interior. Let γ be the set of classes of affinely equivalent elements of Γ and let $i:\Gamma \to \gamma$ be the canonical map. By e, p and $h \in \gamma$ we denote respectively the class of all ellipses, parallelograms and affinely regular hexagons. The set γ becomes a metric space with the following definition of a distance.

$$d(a,b) = \log \inf\{h > 0 \mid \exists A \in i^{-1}(a),\ B \in i^{-1}(b):\ A \subset B \subset hA\}\,.$$

It is easy to verify the postulates of a distance function on the expression above. We can now state our result in the following manner.

THEOREM 1. *For any* $c \in \gamma$, $d(c,p) \leqq \log \frac{3}{2}$ *with equality only for* $c = h$.

Another result is

THEOREM 2. *For any* $c \in \gamma$, $d(c,h) \leqq \log \frac{3}{2}$ *with equality only for* $c = p$.

It is part of a result of F. Behrend [1], that for any $C \in \Gamma$ there is a unique inscribed ellipse E of maximal area and that for this ellipse $C \subset 2^{\frac{1}{2}}E$. Also if and only if C is a parallelogram, $E \subset C \subset kE$ is impossible for any ellipse E and any number $k < 2^{\frac{1}{2}}$. With the notations that we have introduced this would be stated as follows.

THEOREM 3 (Behrend). *For any* $c \in \gamma$, $d(c,e) \leqq \frac{1}{2} \log 2$ *with equality only for* $c = p$.

Received November 14, 1959.

It might also be of interest to know the diameter of the space γ. It follows immediately from Theorem 3 that the diameter is not larger than $\log 2$. This is not a particularly good estimate and it is easy to find lower estimates with the help of Theorems 1 to 3. The author believes the following conjecture to be true.

CONJECTURE. *For any* $a,\ c \in \gamma$, $d(a,c) \leq \log \frac{3}{2}$ *with equality only for* $a = p$ *and* $c = h$ *or* $a = h$ *and* $c = p$.

F. John [2] has an extension of Theorem 3 to higher dimensions.

1. Two lemmas on affinely regular hexagons.

The proof of Theorem 1 uses properties of circumscribed affinely regular hexagons. We introduce the following coordinates to describe the points on the perimeter of an affinely regular hexagon. Beginning with one of the vertices, the points on the side next to this vertex in the positive direction are measured from it so that the next vertex will have coordinate 1. Coordinates referring to this side will be indexed with the index $0: x_0, y_0$, etc. The next side in the positive direction is coordinatized in the same way with coordinates indexed with 1 and the third side with coordinates indexed with 2. This is illustrated by Figure 1. Because of the symmetry in the situations this coordinatization is sufficient for our purposes. We will also regard the indices $0, 1, 2$, as elements of the field J_3 of integers modulo 3. We make the convention that $\sum x_i$, $\prod y_i$ mean respectively $x_0 + x_1 + x_2$ and $y_0 y_1 y_2$ and that the expression "for all i" means "for all $i \in J_3$". Finally, we index the sides and the vertices of the hexagon in the same way, as indicated in Figure 1.

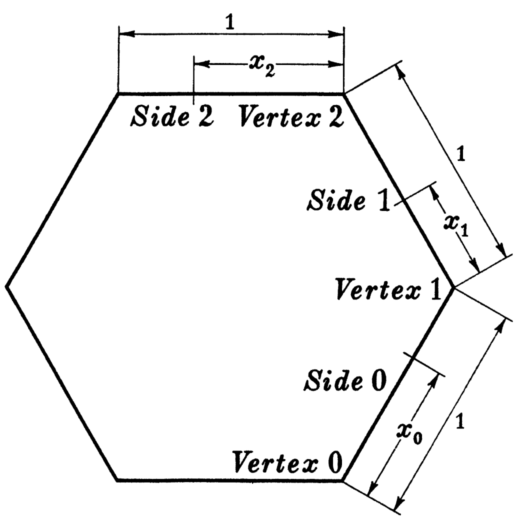

Fig. 1.

Given any convex set $C \in \Gamma$ we define a family of functions $f_\alpha(C,H)$, $0 < \alpha < \frac{2}{3}$, $H \in i^{-1}(h)$ as follows. Let $s(A)$ denote the area of the set A. Put

$$f_\alpha(C,H) = \alpha s(H - H \cap C) + (1-\alpha) s(C - H \cap C)\,.$$

These functions are continuous on $i^{-1}(h)$, provided with the evident topology. The subsets $V_{m,M} \subset i^{-1}(h)$ defined by

$$V_{m,M} = \{H \mid H \in i^{-1}(h),\ mC \subset H \subset MC,\ m \text{ and } M \text{ real and positive}\}$$

are compact in this topology. It is also clear that there exists an $m(\alpha)$ small enough and an $M(\alpha)$ large enough for the inequality

$$f_\alpha(C,H) \geqq \varrho s(C)$$

to be satisfied on the complement of $V_{m,M}$ for some positive number $\varrho = \varrho(\alpha)$. Consequently the function $f_\alpha(C,H)$ assumes its minimum in $i^{-1}(h)$ for some hexagon H.

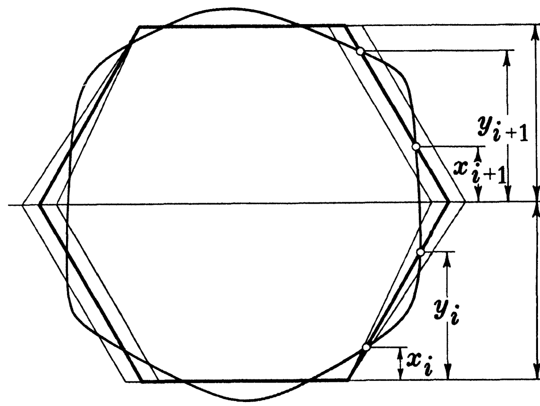

Fig. 2.

The convex set C intersects each side of H in a closed interval, which we will denote $[x_i, y_i]$ in our coordinatization. Suppose that C is strictly convex, i.e. that the boundary of C contains no nondegenerate line segment. If the function $f_\alpha(C,H)$ attains its minimum for the hexagon H, we find by studying the variation of H indicated in Figure 2, that a necessary condition on the numbers x_i, y_i is

$$-\alpha\tfrac{1}{2}x_i^2 + (1-\alpha)\tfrac{1}{2}(y_i^2 - x_i^2) - \alpha\tfrac{1}{2}(1-y_i^2) - \alpha x_{i+1} + \\ + (1-\alpha)(y_{i+1} - x_{i+1}) - \alpha(1-y_{i+1}) = 0 .$$

If we introduce the notations

$$\tfrac{1}{2}(y_i + x_i) = z_i, \qquad y_i - x_i = u_i ,$$

we get

(1) $$z_i u_i = \tfrac{3}{2}\alpha - u_{i+1} .$$

Similarly, we find

(2) $$(1 - z_i)u_i = \tfrac{3}{2}\alpha - u_{i-1} .$$

Adding these relations we get $\sum u_i = 3\alpha$. We know by the definition of z_i and u_i that

$$0 \leqq z_i \leqq 1, \qquad 0 \leqq u_i \leqq 1, \qquad \tfrac{1}{2}u_i \leqq z_i \leqq 1 - \tfrac{1}{2}u_i .$$

From equations (1) and (2) we then obtain

$$u_i \leqq \tfrac{3}{2}\alpha$$

and

$$u_i \geqq \tfrac{3}{2}\alpha - u_{i+1} + \tfrac{1}{2}u_{i+1}^2 \geqq \tfrac{9}{8}\alpha^2 .$$

Thus $u_i > 0$ for all i. Now from equations (1) and (2) we can deduce the relation

$$\prod z_i = \prod (1-z_i) .$$

We collect our principal results in a lemma.

LEMMA 1. *Let $C \in \Gamma$ be strictly convex. If $f_\alpha(C,H)$, $0<\alpha<\frac{2}{3}$, attains its minimum for the hexagon $H \in i^{-1}(h)$, then the lengths u_i of the intervals in which C cuts the sides of H satisfy the inequalities*

$$\tfrac{9}{8}\alpha^2 \leqq u_i \leqq \tfrac{3}{2}\alpha \tag{3}$$

and the coordinates z_i of the midpoints of these intervals the equation

$$\prod z_i = \prod (1-z_i) . \tag{4}$$

Our aim is now to preserve equation (4) while α tends to zero and while we drop the requirement that C be strictly convex. To do this we choose a sequence $\{\alpha_n\}$ of numbers satisfying $0<\alpha_n<\frac{2}{3}$ and $\lim_{n\to\infty}\alpha_n = 0$ and a sequence $\{C_n\}$ of strictly convex sets in Γ tending to the arbitrary set $C \in \Gamma$. We construct a corresponding sequence $\{H_n\}$ of solutions in $i^{-1}(h)$ to the respective problems of minimizing $f_{\alpha_n}(C_n,H)$. It follows from the properties of the hexagons H_n and the convergence of the sequence $\{C_n\}$ to a convex body C with interior points that the H_n are contained in some compact subset of $i^{-1}(h)$. We can therefore, by passing to a subsequence if necessary, think of the sequence $\{H_n\}$ as having a limit $H \in i^{-1}(h)$. By further selection of subsequences we may assume that the sequences $\{z_{ni}\}$ of midpoints of the intervals which are cut out from the sides of H_n by C_n also converge to points z_i which then by the closedness of the sets must belong both to C and to the appropriate sides of H. The relation (4) is of course preserved. Also we must have $H \supset C$, since otherwise the relation (3) would be contradicted for small enough α_n in the sequence. However, it is no longer certain that the points on the sides of H with coordinates z_i are midpoints of those intervals on which C touches the sides of H. We have proved the following lemma.

LEMMA 2. *For any $C \in \Gamma$ there exists an $H \in i^{-1}(h)$ such that $H \supset C$ and such that for each side of H there is a point on this side with coordinate x_i which also belongs to C (i.e., H is circumscribed to C), and the numbers x_i satisfy the relation*

$$\prod x_i = \prod (1-x_i) .$$

This proof of lemma 2 is based on an idea of professor A. Beurling. The writer first used the fact that those circumscribed H which have extremal area do satisfy the required relation, but this leads to much less perspicuous computations.

2. Proof of Theorem 1.

As we now go on to prove the inequality $d(c,p) \leq \log \frac{3}{2}$ we need a couple of lemmas to single out the freak case $c=h$.

LEMMA 3. $d(p,h) = \log \frac{3}{2}$.

This lemma is proved by elementary computation.

LEMMA 4. *If for a given $C \in \Gamma$ one can find a parallelogram $P \in i^{-1}(p)$ such that $P \subset C \subset \frac{3}{2}P$ but such that at least one pair of opposite vertices of P lie in the interior of C, then $d(i(C),p) < \log \frac{3}{2}$.*

Project the two vertices of P that lie in the interior of C by a line through the origin onto the boundary of C. Then take the two projections together with the other pair of vertices of P as the vertices of another parallelogram $P' \in i^{-1}(p)$. It is obvious that

$$P' \subset C \subset kP'$$

for some $k < \frac{3}{2}$. This proves Lemma 4.

Given a convex set $C \in \Gamma$, we now circumscribe a hexagon $H \in i^{-1}(h)$ about C in such a way that there are three tangent points (points where C meets the boundary of H) on successive sides of H whose coordinates x_i satisfy $\prod x_i = \prod(1-x_i)$. This is possible because of Lemma 2. We trace the tangents to C parallel to those lines which join the midpoints of opposite sides of H. These tangents together with their respective opposite sides of H, suitably prolonged, form three parallelograms circumscribed to H. We contract these three parallelograms by the factor $\frac{2}{3}$. For clarification see Figure 3, where we have

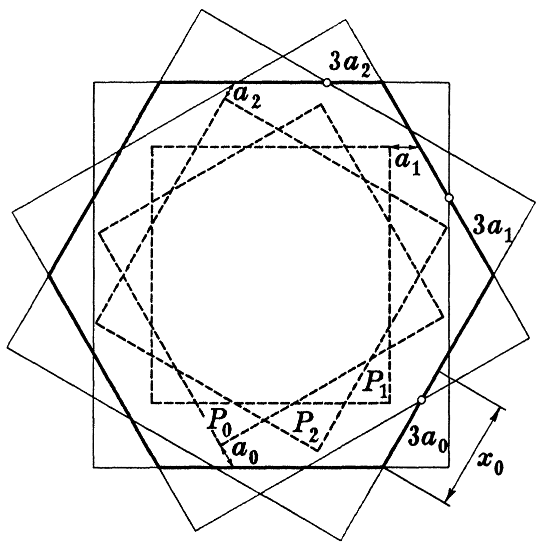

Fig. 3.

also introduced three lengths a_0, a_1 and a_2 which determine the configuration. All measures have not been written out in order not to clutter up the figure needlessly. By Lemmas 3 and 4 we have proved Theorem 1 if we can show that among the three dotted parallelograms P_i there is at least one which is included in C and that, unless C is a hexagon in $i^{-1}(h)$, at least one of the vertices of this parallelogram lies in the interior of C. We proceed to do so by splitting up the possible configurations of Figure 3 in different cases. First, if $a_i = 0$ for all i, the set C reaches out into every vertex of H and thus coincides with H. Next, suppose $a_i \neq 0$ for exactly one $i \in J_3$. We will find that the corresponding dotted parallelogram P_i satisfies the condition of Lemma 4, which thus settles this case. To do so we will need another lemma.

LEMMA 5. *If C meets a certain side of H in one of its closed outer third part intervals and the other end of the same side corresponds to an $a_i \neq 0$, then the vertex of P_i nearest to the side in question lies in the interior of C.*

Lemma 5 is proved by elementary computation from Figure 4, where $A \in C$ denotes the given point in the (closed) outer third of the side and B is constructed as indicated and must be an interior point of C since otherwise the indicated tangent would not touch C.

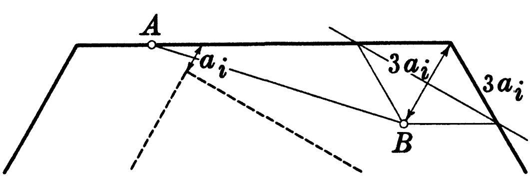

Fig. 4.

Reverting to our case $a_i \neq 0$ for exactly one i, Lemma 5 now shows that P_i will satisfy the conditions of Lemma 4, hence we are through with that case.

Now, suppose $a_i = 0$ for exactly one i. Then by Lemma 5 the parallelograms P_{i+1} and P_{i-1} have at least one vertex each in the interior of C. To arrive at the desired conclusion through Lemma 4 we have thus to show that at least one of the two vertices nearest to the $i+1$-st side are in C.

DEFINITION. *C is arrow-shaped with respect to the i-th side of H if for some real number k, $0 < k < 1$, the closed straight line segment bounded by $x_i = \frac{1}{2}$ and $x_{i+1} = k$ and the segment bounded by $x_{i-1} = 1 - k$ and $x_i = \frac{1}{2}$ both belong to the boundary of C.*

COROLLARY. *If C is arrow-shaped with respect to more than one side of H, then C is the hexagon with vertices $x_i = \frac{1}{2}$.*

LEMMA 6. *Suppose a_i, $a_{i+1} \neq 0$. The corresponding vertices enclose the i-th side of H. Then either the vertex of P_i nearest to the i-th side of H or the vertex of P_{i+1} nearest to the same side belong to the interior of C, or else C is arrow-shaped with respect to the i-th side, in which case both vertices lie on the boundary of C.*

To prove Lemma 6 we first note that by Lemma 5 it is possible to assume that C meets the i-th side in a point with coordinate $x_i \in (\frac{1}{3},\frac{2}{3})$. We now assume that none of the two vertices lies in the interior of C. Join the point $x_{i+1} = 3a_i$ and the vertex of P_i with a straight line. It is clear that the continuation of this line beyond the vertex contains no interior points of C. It follows that $3a_{i+1} > 2a_i$ and

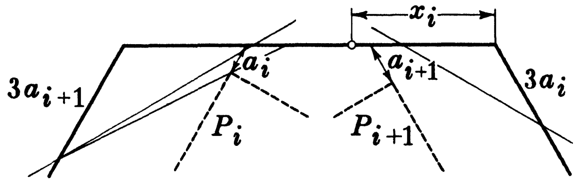

Fig. 5.

$$\frac{a_{i+1}}{3a_{i+1}-a_i} \leqq 1-x_i .$$

By the same construction from the other side we have $2a_{i+1} < 3a_i$ and

$$\frac{a_i}{3a_i-a_{i+1}} \leqq x_i .$$

We add these two relations and get

$$1 \geqq \frac{a_i}{3a_i-a_{i+1}} + \frac{a_{i+1}}{3a_{i+1}-a_i} = \frac{(a_i+a_{i+1})^2-2(a_i-a_{i+1})^2}{(a_i+a_{i+1})^2-4(a_i-a_{i+1})^2} .$$

Under the conditions $3a_i > 2a_{i+1}$, $3a_{i+1} > 2a_i$ this relation is possible only if $a_i = a_{i+1}$ and $x_i = \frac{1}{2}$, and by the construction made, one finds that this possibility exists only when C is arrow-shaped with respect to the i-th side.

Now Lemma 6 is proved. Substituting $i+1$ for i in Lemma 6 settles the case $a_i = 0$ for exactly one i.

Finally, we suppose $a_i \neq 0$ for all i so that Lemma 6 is applicable to all sides of H. If C is arrow-shaped with respect to more than one side of H, then C is a hexagon. If C is arrow-shaped with respect to exactly one side of H, then it is easy to see on Figure 3 that by Lemma 6 at least one of the parallelograms P_i satisfies the requirements of Lemma 4. We may thus suppose that C is not arrow-shaped with respect to any side. Then, by Lemma 6, at most one of the two parallelogram vertices pointing towards any given side is not in the interior of C. Furthermore, unless it be immediately possible to pick out a wholly interior parallelogram among the P_i, the non-interior vertices must be arranged in the pattern of Figure 6 (or in the opposite pattern, which can be treated in exactly the same way). Indeed, these vertices must even be exterior to C lest the conditions of Lemma 4 be apparently fulfilled for some P_i. Similarly, by Lemma 5, we may suppose that $x_i > \frac{1}{3}$ for all i. As in the proof of Lemma 6, we now find

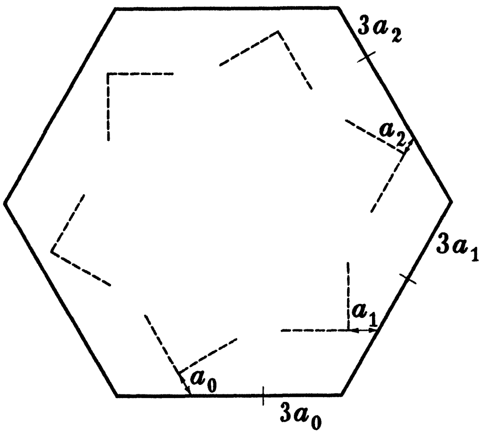

Fig. 6.

$$x_i > \frac{a_i}{3a_i - a_{i+1}}$$

and, moreover, $2a_i > a_{i+1}$ for all i. Hence we get

$$\frac{\prod a_i}{\prod (3a_i - a_{i+1})} < \prod x_i, \qquad \prod (1 - x_i) < \frac{\prod (2a_i - a_{i+1})}{\prod (3a_i - a_{i+1})}.$$

Because of the equation $\prod x_i = \prod (1 - x_i)$ these two relations imply

$$\prod a_i^2 = \prod a_i \prod a_{i+1} < \prod a_{i+1}(2a_i - a_{i+1}) \leqq \prod a_i^2$$

the last step consisting of the familiar inequality between arithmetic and geometric means, which is applicable since $2a_i - a_{i+1} > 0$ for all i. But this last relation is impossible and we thus have the final contradiction needed in the proof of Theorem 1.

3. Proof of Theorem 2.

To prove Theorem 2 we will first show that it is possible to regard the set of all hexagons $H \in i^{-1}(h)$ circumscribed to C as the set of values of a function defined on a circle. Take the ellipse of maximal area that is inscribed to C and make it a circle S by an affine transformation, carrying C into C'. Take a point $t \in S$ and trace the tangents to C' parallel to the line joining t to the origin. This line cuts the frontier of C' in a point A. We may trace the two half-tangents to C' at A and con-

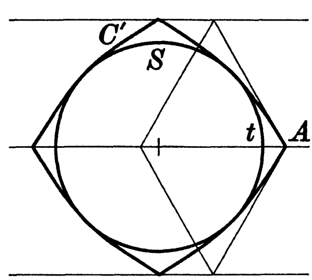

Fig. 7.

struct the (possibly degenerate) parallelogram which has for sides these two half-tangents between A and their intersections with their corresponding tangents to C', see Figure 7. Unless this parallelogram contains the origin in its interior there will be a unique hexagon $H(t)$ in $i^{-1}(h)$ circumscribed to C', which has one pair of sides parallel to the given direction. In case the parallelogram has the origin in its interior, there is a family of hexagons $H(t,\theta) \in i^{-1}(h)$ with vertex A that all fulfill the above conditions, and it may be parametrized e.g. with the angle θ that the side of H issuing from A in the positive direction makes with the direction from the origin to A. However, this case can happen only for a finite number of points $t \in S$, and it is easy to arrive at a definite estimate of this number by means of the inclusion $S \subset C' \subset 2^{1/2}S$. Thus, we can cut S at these points t_n and piece in the corresponding functions $H(t_n,\theta)$ so that we get the desired function, and it is easy to see that this may be done so that the function becomes continuous. We rename the resulting function $H(t)$, $t \in S$.

Now we consider three consecutive vertices of $H(t)$ and follow their variation as t runs through the circle S. To each vertex i there is a contraction number k_i, which is the factor by which H has to be contracted to bring that particular vertex to the boundary of C'. Thus we have three continuous functions $k_i(t)$ on S. Moreover, all these functions have the same range of values. Accordingly, there must be some t for which two of the functions $k_i(t)$ are equal and the third function larger than or equal to this common value of the two others.

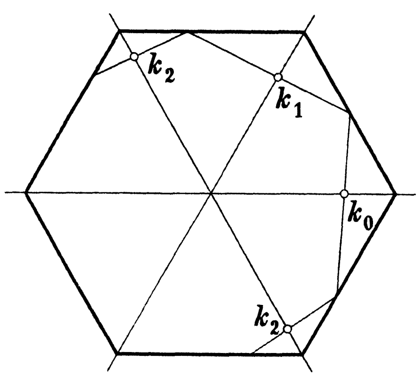

Fig. 8.

We will now show that for the corresponding $H(t)$, $C' \supset \frac{2}{3}H(t)$ and C' will contain $kH(t)$ for some $k > \frac{2}{3}$ unless it is a parallelogram. This proves Theorem 2, since C' represents the same class in Γ as does C. The proof is immediate from Figure 8. Let $k_0(t) = k_1(t) < k_2(t)$. Then C cannot meet every side of H unless $k_0 = k_1 > \frac{2}{3}$, and in case $k_0 = k_1 = \frac{2}{3}$ this can only be done by letting C be the parallelogram with vertices at the midpoint of side 0 of H and at vertex 2 of H.

REFERENCES

1. F. Behrend, *Über einige Affininvarianten konvexer Bereiche*, Math. Ann. 113 (1937), 713–747.
2. F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays presented to R. Courant, New York, 1948, 187–204.

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N.J.. U.S.A.