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ON A COLORING PROBLEM

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1. Introduction.

The present note deals with the problem of "coloring" certain families of parallelograms in the plane and with some generalizations of this problem.

We find it convenient to introduce the following terminology. Let n,m,k,r be positive integers, $k \leqq m$. A family $\mathscr{P} = \{P\}$ of polyhedra in n-dimensional Euclidean space shall be called a *family of type* (n,m,k,r) if the following conditions are fulfilled:

(i) There exist m concurrent straight lines L_i, $1 \leqq i \leqq m$, (depending on $\mathscr{P}$) such that for every $P \in \mathscr{P}$ all the edges of P are parallel to some k or less of the lines L_i.

(ii) No $r+1$ members of $\mathscr{P}$ have a common point.

Thus, e.g., a family of non-intersecting rectangles in E^3, with edges in any two of three given mutually orthogonal directions is of type $(3,3,2,1)$. The three sides of a triangle in the plane form a family of type $(2,3,1,2)$.

Let $\mathscr{P} = \{P\}$ be any family of sets. We say that $\mathscr{P}$ is q-colorable for a positive integer q if it is possible to assign to each member of $\mathscr{P}$ one of q different colors in such a way that any two members of $\mathscr{P}$, to which the same color is assigned, have an empty intersection. In other words, $\mathscr{P}$ is q-colorable if it can be decomposed into q subfamilies each of which consists of mutually disjoint sets.

We define $C(n,m,k,r)$ to be the minimum of q such that each family of type (n,m,k,r) is q-colorable, if at least one such q exists; $C(n,m,k,r) = \infty$ otherwise.

The present paper was motivated by a problem proposed by A. Bielecki [1], which in our terminology may be stated as: "Is $C(2,2,2,2) < \infty$?" Bielecki also gave an example which shows that $C(2,2,2,2) \geqq 4$, and mentions the easily proved fact that $C(1,1,1,r) = r$.

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Our main results are:

THEOREM 1. *$C(n,m,2,r) < \infty$ for all n,m,r.*

THEOREM 2. $C(2,2,2,2) = 6$.

Some reduction theorems, as well as results pertaining to the case $k = 1$, are also obtained.

2. Reduction theorems.

THEOREM 3. *$C(n,m,k,r) \leqq q$ for some positive integer q follows from the q-colorability of each finite family $\mathcal{P}$ of type (n,m,k,r) whose members have non-void k-dimensional interior and are such that no $(k-1)$-dimensional hyperplane contains more than one of the $(k-1)$-dimensional faces of the members of $\mathcal{P}$.*

PROOF. If each finite family $\mathcal{P}$ of type (n,m,k,r) is q-colorable, then $C(n,m,k,r) \leqq q$ by a theorem of de Bruijn and Erdös [2, Theorem 1] which states that a graph is (vertex) q-colorable if each of its finite subgraphs is q-colorable. Thus we have only to show that the assumptions of Theorem 3 imply that each finite family $\mathcal{P}^*$ of type (n,m,k,r) is q-colorable. But this follows immediately from the observation that, on adding (Minkowski addition) suitable k-dimensional polyhedra of sufficiently small diameter, the members of $\mathcal{P}^*$ may be "fattened up" without changing their incidence relations. This ends the proof of Theorem 3.

In view of Theorem 3 we shall from now on consider, without loss of generality, *only* finite families.

The next result is equally easy:

THEOREM 4. $C(n,m,k,r) \leqq \binom{m}{k} C(n,k,k,r)$.

PROOF. Since each family of type (n,m,k,r) may obviously be decomposed into at most

$$\binom{m}{k}$$

mutually disjoint families of types (n,k,k,r), the theorem follows from the definition of $C(n,m,k,r)$.

It seems probable that for $r > 1$ the equality sign holds in Theorem 4, though we succeeded to prove this only in a very special case (see Section 6).

If $n \geqq k$ it is immediate that $C(n,k,k,r) = C(k,k,k,r)$. Therefore, in order to prove Theorem 1, it is sufficient to establish $C(2,2,2,r) < \infty$. This will be done in the next section.

3. Proof of $C(2,2,2,r) < \infty$.

By Theorem 3 it is sufficient to consider finite families $\mathcal{P}$ of parallelograms in the plane, no $r+1$ of which have common points; for convenience of expression we may assume that $\mathcal{P}$ consists of rectangles with "horizontal" and "vertical" sides. A rectangle R_1 is said to be *vertex-intruding* into another rectangle R_2 if R_2 contains at least one of the vertices of R_1. Two rectangles are *vertex-incident* if at least one of them is vertex-intruding into the other. We use these concepts in order to establish the following proposition:

$\mathcal{P}$ may be decomposed into $8r$ *subfamilies, each of type* $(2,2,2,r)$, *in such a way that no vertex-incidence occurs among members of the same subfamily.*

Indeed, otherwise there would exist a family $\mathcal{P}$ with a minimal number of elements for which such a decomposition is impossible. The minimality of $\mathcal{P}$ implies that each member of $\mathcal{P}$ is vertex-incident to more than $8r$ other members of $\mathcal{P}$. Now, each member of $\mathcal{P}$ is vertex-intruding into at most $4r$ other members of $\mathcal{P}$. Therefore, if N is the number of rectangles in $\mathcal{P}$, there are at most $4rN$ vertex-incidences among the members of $\mathcal{P}$ and thus there exists at least one rectangle in $\mathcal{P}$ into which not more than $4r$ other rectangles are vertex-intruding. This rectangle, therefore, takes part in not more than $8r$ vertex-incidences, in contradiction to the above assumption.

We remark parenthetically that by using a slightly more careful procedure it is possible to show that the above proposition holds also with "into less than $4r$" instead of "into $8r$". On the other hand, it is obvious that a similar "vertex-incidence elimination" is possible for (finite) families of any (n,m,k,r) type.

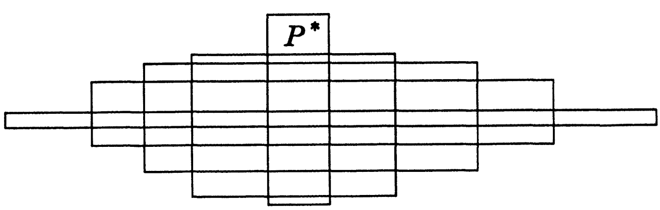

Fig. 1.

Now we shall complete the proof of $C(2,2,2,r) < \infty$ by showing that any family $\mathcal{P}$ of type $(2,2,2,r)$ among whose members no vertex-incidence occurs is r-colorable. Since no two members of $\mathcal{P}$ are vertex-incident, it is obvious that if the intersection of any r members of $\mathcal{P}$ is not empty, they must intersect in a pattern represented schematically in Fig. 1 (for $r=5$).

Let $\mathscr{P}^*$ be the subfamily of $\mathscr{P}$ consisting of those (and only those) members P^* of $\mathscr{P}$ which satisfy the condition: There exists an r-membered subfamily of $\mathscr{P}$ which includes P^*, all of whose members have a non-void intersection, and P^* is the tallest and narrowest of the r rectangles (corresponding to P^* in Fig. 1). Then, obviously, the members of $\mathscr{P}^*$ are disjoint, and therefore 1-colorable, while the members of $\mathscr{P}$ which do not belong to $\mathscr{P}^*$ form a family of type $(2,2,2,r-1)$. An inductive argument now establishes the assertion, and with it Theorem 1. Moreover, we proved that $C(2,2,2,r) \leqq 8r^2$; the more precise estimate $C(2,2,2,r) < 4r^2$ may be derived from the strengthened form of the first part of the present proof, but even this bound seems to be quite crude.

4. Proof of Theorem 2.

By using arguments similar to those in the previous section, we shall first establish $C(2,2,2,2) \leqq 6$.

Let $\mathscr{P}$ be a finite family of rectangles (with horizontal and vertical sides) of type $(2,2,2,2)$; without loss of generality we assume (see Section 2) that no vertical or horizontal straight line contains more than one of the edges of the members of $\mathscr{P}$.

We may, moreover, assume that no member of $\mathscr{P}$ completely contains another member of $\mathscr{P}$ (the same argument would apply to all families of types $(n,m,k,2)$). Indeed, if the subfamily $\mathscr{P}^*$ of $\mathscr{P}$, obtained by omitting from $\mathscr{P}$ those rectangles contained in another rectangle of $\mathscr{P}$, is q-colorable with $q \geqq 2$, the family $\mathscr{P}$ is also q-colorable since each of the omitted sets meets precisely one other member of $\mathscr{P}$ (as a matter of fact, of $\mathscr{P}^*$).

Only families $\mathscr{P}$ satisfying these additional conditions shall be considered in the remaining part of this section.

A family $\mathscr{P}$ shall be called *horizontally simple* if it has the following property: *No member of the family intersects both vertical edges of another member of the family.*

In other words, intersections of the types represented in Fig. 2 do not occur among members of a horizontally simple family.

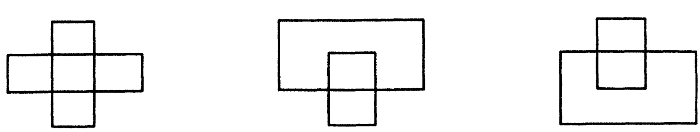

Fig. 2.

Vertically simple families are defined similarly, interchanging "horizontal" and "vertical" in the obvious way.

We have

LEMMA. *Every horizontally [vertically] simple family $\mathscr{P}$ is* 3-*colorable.*

PROOF. We shall prove the Lemma only for horizontally simple families, the proof for vertically simple ones requiring only obvious changes.

Let $n(\mathscr{P})$ be the number of rectangles in $\mathscr{P}$. By induction, we may suppose that the Lemma is proved for $n(\mathscr{P}) < N$ (it is obvious for $n(\mathscr{P}) < 4$). Let $P_0 \in \mathscr{P}$ be such that its lower edge is higher than the lower edge of any other member of $\mathscr{P}$. By hypothesis, $\mathscr{P} - \{P_0\} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$, where the members of $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$ are mutually disjoint. But P_0 meets at most two members of $\mathscr{P} - \{P_0\}$, since if $P \in \mathscr{P} - \{P_0\}$ meets P_0, it must contain at least one of the lower vertices of P_0. Hence we can adjoin P_0 to one of the families $\mathscr{P}_1$, $\mathscr{P}_2$ or $\mathscr{P}_3$ while preserving the mutual disjointness property of this family. This proves the assertion for $n(\mathscr{P}) = N$, and thus completes the proof of the Lemma.

The proof of $C(2,2,2,2,) \leqq 6$ follows now easily. Let any family $\mathscr{P}$ of rectangles be given. We decompose it into two subfamilies $\mathscr{P}_1$ and $\mathscr{P}_2$ in the following way: A member of $\mathscr{P}$ shall belong to $\mathscr{P}_1$ if and only if its horizontal edges may be joined by a vertical segment which is completely contained in another member of $\mathscr{P}$. The subfamily $\mathscr{P}_2$ is formed by all the members of $\mathscr{P}$ that do not belong to $\mathscr{P}_1$. It is immediate that $\mathscr{P}_1$ is a horizontally simple family, while $\mathscr{P}_2$ is a vertically simple one. By the Lemma, each of them is 3-colorable, and therefore $\mathscr{P}$ itself is 6-colorable, i.e. $C(2,2,2,2,) \leqq 6$ as claimed.

5. Proof of Theorem 2 (end).

We shall now complete the proof of Theorem 2 by describing a family of type (2,2,2,2) which is not 5-colorable. For simplicity of description we shall use also "degenerated rectangles", i.e. horizontal (and vertical) segments (their endpoints shall be indicated by arrows).

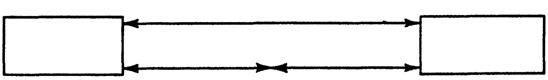

Fig. 3.

We start with the obvious remark that the five-membered family represented in Fig. 3 is not 2-colorable.

It follows then easily that in any coloring of the twelve-membered family represented in Fig. 4 at least one of the regions bounded by a dotted line meets rectangles colored by (at least) three different colors. We shall call such a family a horizontal "filter bed", each of the eight dotted regions representing a "filter".

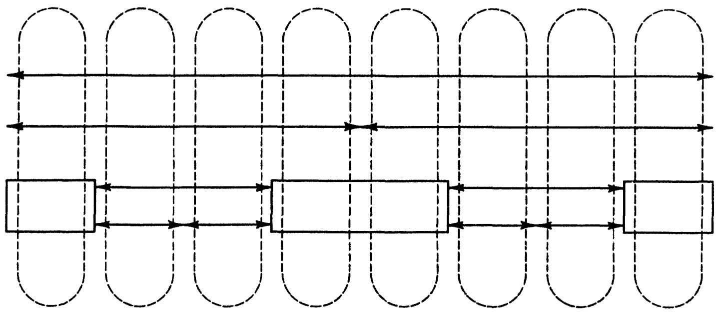

Fig. 4.

We now consider the following system of filter beds. Take 8^8 congruent horizontal filter beds and arrange them vertically one above another at equal distances (Fig. 5). Over the leftmost filters of these filter beds put a vertical filter bed such that each of its filters intersects 8^7 of the filters in the horizontal beds. Put a chain of 8 vertical filter beds over the next column of filters so that each of their filters intersects 8^6 horizontal filters, and so on, until the rightmost column is intersected by a chain

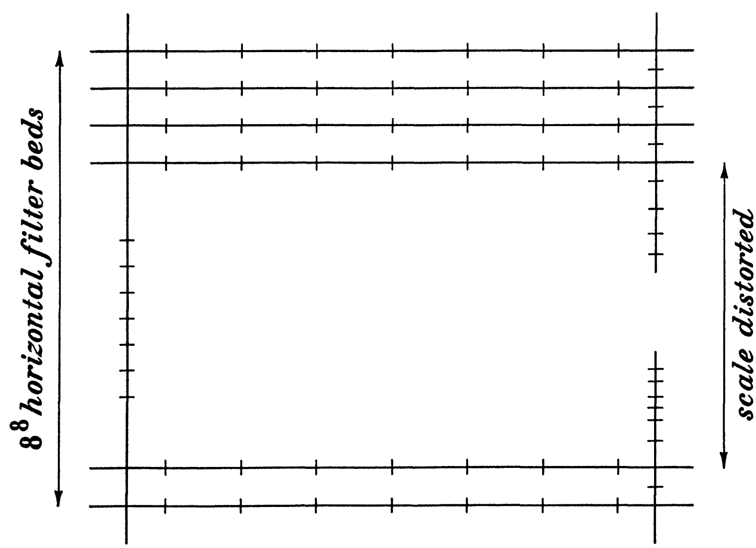

Fig. 5. System of filter beds. A line with seven crossmarks represents a filter bed.

of 8^7 vertical filter beds such that each filter in this chain intersects one horizontal filter.

Let a coloring of this system be given. Looking at the leftmost vertical filter bed, we find at least one bunch of 8^7 horizontal filter beds such that their leftmost filters are intersected by a 3-color vertical filter. Next, out of these horizontal filter beds at least 8^6 adjacent ones intersect the next vertical chain in a 3-color vertical filter. Proceeding in this manner, we finally find a horizontal filter bed all of whose filters are intersected by vertical 3-color filters. But this horizontal bed contains at least one 3-color horizontal filter, hence in the whole system there is at least one intersection between a 3-color vertical and a 3-color horizontal filter, which proves that the system cannot be colored by less than six colors. This ends the proof of Theorem 2.

We remark that the above example of a family of type (2,2,2,2) which is not 5-colorable contains more than 10^8 rectangles; we found another example of the same kind containing "only" about 50000, but its construction is quite complicated. Since it is easy to find families of type (2,2,2,2) which are not 4-colorable and which contain less than 200 sets, it would be interesting to know whether the size of non-5-colorable families may be substantially reduced.

6. Some additional results.

The easily established relation $C(1,1,1,r)=r$ implies, using the reduction theorems of Section 2, that $C(n,m,1,r)\leqq mr$. We shall show that, for $n\geqq 2$, we have $C(n,m,1,2)=C(2,m,1,2)=2m$.

Following an idea similar to that used in Section 5, we remark first that in any coloring of the "filter-bed" represented in Fig. 6 at least one of the two "filters" (regions indicated by a dotted line) contains segments colored by two different colors. Then, superimposing a sufficient number of such filter beds in m directions and of suitable sizes and positions, families of type $(2,m,1,2)$ not colorable by $2m-1$ colors, may be obtained. For $m=2$ an example is given in Fig. 7.

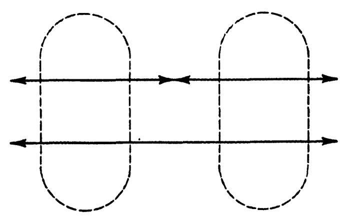

Fig. 6.

This family consists of 21 sets; the corresponding example for $m=3$ consists of 765 sets, etc. By slight changes, a small "saving" may be achieved, reducing the number of sets to 19 for $m=2$, to 649 for $m=3$, etc.

It is easy to find families of segments showing that $C(2,2,1,3)\geqq 5$,

$C(2,2,1,4) \geqq 6$, etc. But we were unable to settle the question whether $C(2,m,1,r) = mr$ for $r > 2$.

By methods similar to those used in Section 4 it is possible to show, e.g. that $C(2,3,3,2)$ is finite; we were unable to prove the finiteness in the general case $C(2,k,k,r)$.

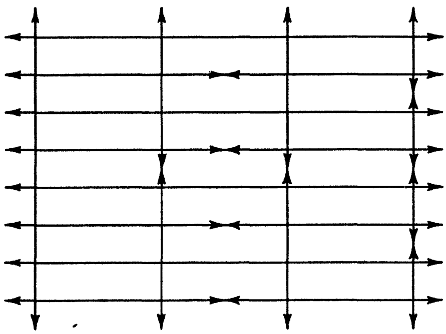

Fig. 7.

For $n = 3$ even the simplest problem (not reducible to lower dimensions), viz. the finiteness of $C(3,3,3,2)$, is still unsolved.

Another interesting problem is the following. We proved that $C(3,3,1,2) > 5$ by indicating the construction of a family of type (2,3,1,2) which is not 5-colorable. Among families of type (3,3,1,2) which are *not* reducible to families of type (2,3,1,2) situated in parallel planes, we were unable to find examples showing $C(3,3,1,2) > 5$ (although we constructed such a family which is not 4-colorable). Are all families of type (3,3,1,2) containing segments in 3 independent directions 5-colorable?

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