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ON THE MINIMUM MODULUS OF ENTIRE FUNCTIONS OF LOWER ORDER LESS THAN ONE

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1. Introduction.

Much work has been performed to find the connection between the minimum and the maximum modulus of an entire function. Many problems concerning these moduli still remain unsolved. A survey of this field has been given by W. K. Hayman [1].

Let $f(z)$ be an entire function. We denote $\max|f(z)|$ and $\min|f(z|$ on $|z|=r$ by $M(r)$ and $m(r)$ respectively. The order ϱ and lower order λ are defined as lim sup and lim inf of $\log\log M(r)/\log r$ as $r \to \infty$.

Many years ago it was proved that

$$(1) \qquad \limsup_{r\to\infty} \frac{\log m(r)}{\log M(r)} \geqq \cos\pi\varrho$$

if $0<\varrho<1$, a result later on sharpened by Beurling (cf. [2, pp. 14–16], the result was Theorem II below with ϱ instead of λ, $0<\varrho\leqq\frac{1}{2}$).

The author has earlier proved (cf. [3]) that $\cos\pi\varrho$ in (1) can be replaced by $\cos\pi\lambda$ if $0<\lambda<\varrho<1$. Our purpose now is to remove the condition $\varrho<1$.

It is the fact that $\log|f(z)|$ is subharmonic which is used in the proof of theorems such as (1). Also in the following the theorems stated for entire functions could be given as theorems for subharmonic functions $u(z)$, replacing $\log|f(z)|$.

2. Three theorems.

THEOREM I. *Let $f(z)$ be an entire function of lower order λ, $0<\lambda<1$. Then*

$$(2) \qquad \limsup_{r\to\infty} \frac{\log m(r)}{\log M(r)} \geqq \cos\pi\lambda\,.$$

THEOREM II. *Let $f(z)$ be an entire function of positive or infinite order. Suppose that*

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$$\liminf_{r\to\infty} \frac{\log M(r)}{r^\lambda} = 0\,, \tag{3}$$

λ being a number in $0<\lambda<1$. Then

$$\log m(r) > \cos\pi\lambda \log M(r) \tag{4}$$

holds true for certain arbitrarily large values of r.

THEOREM III. *Let $f(z)$ be an entire function of positive or infinite order. Suppose that*

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^\lambda} < \infty, \quad but \quad \limsup_{r\to\infty} \frac{\log M(r)}{r^\lambda} = \infty\,, \tag{5}$$

λ being a number in $0<\lambda<1$. Then again

$$\log m(r) > \cos\pi\lambda \log M(r) \tag{4}$$

holds true for certain arbitrarily large values of r.

Perhaps it should be mentioned here that it is impossible in Theorem I to replace (2) by (4). At the beginning of this century, Wiman [5] constructed functions having $\lambda=\varrho$ for which (4) does not hold.

Theorem I follows from Theorem II so we have only to prove the last two theorems.

3. Decomposition of $f(z)$.

For the sake of simplicity we first suppose $f(0)=1$, a restriction which will be removed later on.

Let us fix a number R, large enough for zeros of $f(z)$ to exist within $|z|<R$, these zeros being denoted by $a_1, a_2, \ldots, a_N$.

Our function $f(z)$ must have an infinity of zeros. Suppose there were only a finite number, say $a_1, a_2, \ldots, a_p$. Then

$$g(z) = \frac{f(z)}{(z-a_1)(z-a_2)\ldots(z-a_p)}$$

would be an entire function without zeros. Furthermore

$$|g(z)| < e^{|z|^{\lambda+\varepsilon}}, \qquad 0<\lambda<1, \qquad \varepsilon>0\,,$$

for certain arbitrarily large values of $|z|$. By a standard argument (cf. Titchmarsh [4, Theorem 8.24]) we then conclude that $\log g(z)$ must be a polynomial of degree zero and thus $f(z)$ a polynomial of degree p. This is contrary to the supposition in Theorems I–III, hence there must be an infinity of zeros of $f(z)$.

We now define

$$f_1(z) \equiv \prod_{n=1}^{N} (1-z/a_n) , \tag{6}$$

and

$$f_2(z) \equiv \prod_{n=1}^{N} (1+z/|a_n|) . \tag{7}$$

A function $f_3(z)$ is defined by the identity

$$f(z) \equiv f_1(z) f_3(z) . \tag{8}$$

We observe that $f_3(z) \neq 0$ for $|z| < R$.

4. Estimation of $|f_1(z)|$ and $|f_2(z)|$.

The maximum and the minimum of $|f_\nu(z)|$, $\nu = 1,2,3$, on $|z| = r$ are denoted by $M_\nu(r)$ and $m_\nu(r)$. Obviously

$$m_2(r) \leqq m_1(r) < M_1(r) \leqq M_2(r) . \tag{9}$$

Let $n(t)$ be the number of zeros of $f(z)$ for $|z| < t$. Jensen's theorem gives

$$\int_0^{2R} \frac{n(t)}{t}\, dt \leqq \log M(2R) \quad \text{thus} \quad n(R) \log 2 \leqq \log M(2R) . \tag{10}$$

We then estimate $\log M_2(R)$:

$$\begin{aligned} \log M_2(R) = \sum_1^N \log(1+R/|a_n|) &= \int_0^R \log(1+R/t)\, dn(t) \\ &= n(R) \log 2 + \int_0^R \frac{R}{R+t} \frac{n(t)}{t}\, dt \\ &< n(R) \log 2 + \int_0^R \frac{n(t)}{t}\, dt \\ &\leqq n(R) \log 2 + \log M(R) < 2 \log M(2R) . \end{aligned}$$

Hence

$$\log M_1(R) \leqq \log M_2(R) < 2 \log M(2R) . \tag{11}$$

5. Estimation of $|f_3(z)|$.

We obtain from (8)

$$\log M_3(r) \leqq \log M(r) - \log m_1(r) . \tag{12}$$

Because $|a_n| < R$, that is $\left|1 - 2R/|a_n|\right| > 1$, we get

$$m_1(2R) \geqq m_2(2R) = \prod_{1}^{N} \left|1-2R/|a_n|\right| > 1 , \tag{13}$$

and so from (12)

$$\log M_3(2R) < \log M(2R) .$$

Thus

$$\log M_3(R) < \log M(2R) . \tag{14}$$

Since $f_3(z) \neq 0$ for $|z| < R$ we can define

$$\psi(z) = \log f_3(z) , \tag{15}$$

where $\psi(z)$ is regular for $|z| < R$ and $\psi(0) = 0$. By a well-known theorem of Carathéodory (cf. [4, Theorem 5.5]) we get for $|z| < R$

$$|\psi(z)| \leqq \frac{2A(R)\,|z|}{R-|z|} \tag{16}$$

where $A(R) = \max_{|z|=R} \mathrm{Re}\{\psi(z)\} = \log M_3(R)$. Hence

$$\left|\log|f_3(z)|\right| = |\mathrm{Re}\{\psi(z)\}| \leqq |\psi(z)| \leqq \frac{2\log M_3(R)}{R-|z|} \cdot |z| . \tag{17}$$

For $|z| \leqq \frac{1}{2}R$, we get the formula

$$\left|\log|f_3(z)|\right| \leqq \frac{4\log M_3(R)}{R}\,|z| . \tag{18}$$

From (14) we at last obtain

$$\left|\log|f_3(z)|\right| \leqq \frac{4\log M(2R)}{R}\,|z| , \tag{19}$$

valid for $|z| \leqq \frac{1}{2}R$.

We may observe that, given a function $f(z)$ of lower order λ, $0 < \lambda < 1$, it is possible to choose a sequence $R_1, R_2, \ldots, R_\nu, \ldots \to \infty$ such that the right-hand side of (19) tends to zero uniformly in every circle $|z| \leqq R_0$. Thus $f(z)$ can be uniformly approximated in $|z| \leqq R_0$ by means of a sequence of Weierstrassian products of the simplest kind.

6. A formula for $|f_2(z)|$.

Let $0 < R_1 < R_2$. By taking the integral

$$\int \frac{\log(1+z/|a_n|)}{z^{1+\lambda}}\,dz , \tag{20}$$

where $0 < \lambda < 1$, around the upper half of the annulus $R_1 < |z| < R_2$, one can deduce (cf. [2, p. 16], and [3, p. 136]) that

$$\int_{R_1}^{R_2} \frac{\log|1-r/|a_n||-\cos\pi\lambda \log(1+r/|a_n|)}{r^{1+\lambda}}\,dr > k(\lambda)\frac{\log(1+R_1/|a_n|)}{R_1^{\lambda}} - K(\lambda)\frac{\log(1+R_2/|a_n|)}{R_2^{\lambda}}. \tag{21}$$

The value of $k(\lambda)$ is (cf. [2, p. 18])

$$k(\lambda) = \frac{1-\sin\pi|\frac{1}{2}-\lambda|}{\frac{1}{2}-|\frac{1}{2}-\lambda|}. \tag{22}$$

The best value of $K(\lambda)$ seems to be more difficult to obtain, but a rough estimation gives $K(\lambda) < 10$.

Summation with respect to n from 1 to N gives from (21)

$$\int_{R_1}^{R_2} \frac{\log m_2(r)-\cos\pi\lambda \log M_2(r)}{r^{1+\lambda}}\,dr > k(\lambda)\frac{\log M_2(R_1)}{R_1^{\lambda}} - K(\lambda)\frac{\log M_2(R_2)}{R_2^{\lambda}}, \tag{23}$$

valid for $0 < \lambda < 1$, $0 < R_1 < R_2$.

7. Proof of the theorems in the case of $\lambda \leq \frac{1}{2}$.

Let us add the integral

$$I(R_1,R_2) = \int_{R_1}^{R_2} \frac{\log m_3(r)-\cos\pi\lambda \log M_3(r)}{r^{1+\lambda}}\,dr \tag{24}$$

to both sides of (23). Denoting

$$A(R_1,R_2) = \int_{R_1}^{R_2} \frac{\log m_2(r)m_3(r)-\cos\pi\lambda \log M_2(r)M_3(r)}{r^{1+\lambda}}\,dr \tag{25}$$

we then get from (23)

$$A(R_1,R_2) > k(\lambda)\frac{\log M_2(R_1)}{R_1^{\lambda}} - K(\lambda)\frac{\log M_2(R_2)}{R_2^{\lambda}} + I(R_1,R_2)\,. \tag{26}$$

We shall now prove that it is possible to choose arbitrarily large R_1 and R_2 such that the right-hand side of (26) is positive. Let us set

$$\alpha = \liminf_{r\to\infty}\frac{\log M(r)}{r^{\lambda}}, \qquad \beta = \limsup_{r\to\infty}\frac{\log M(r)}{r^{\lambda}}. \tag{27}$$

We have supposed in Theorem II that $\alpha = 0$, and, in Theorem III, that $\alpha < \infty$.

We choose a number $\varepsilon > 0$. For certain arbitrarily large values of R it then holds that

$$\log M(2R) < (\alpha+\varepsilon)(2R)^{\lambda}\,. \tag{28}$$

Let (28) be true for a value R. We then choose $R_2 = \frac{1}{2}R$ and estimate the terms in the right-hand side of (26). To begin with (the value of R_1 will be chosen later)

$$(29) \qquad \log M_2(R_1) = \log M_2(R_1) M_3(R_1) - \log M_3(R_1) \geqq \log M(R_1) - \log M_3(R_1) .$$

From (19) and (28) we get

$$(30) \quad \log M_3(R_1) \leqq \frac{4 \log M(2R)}{R} R_1 < 4(\alpha+\varepsilon) 2^{\lambda} R^{\lambda-1} R_1 = (\alpha+\varepsilon) 2^{\lambda+2} (R_1/R)^{1-\lambda} R_1^{\lambda} .$$

Hence

$$(31) \qquad \log M_2(R_1) > \log M(R_1) - (\alpha+\varepsilon) 2^{\lambda+2} (R_1/R)^{1-\lambda} R_1^{\lambda} .$$

Furthermore, from (11) and (28).

$$(32) \quad -\log M_2(R_2) = -\log M_2(\tfrac{1}{2}R) > -\log M_2(R) > -2 \log M(2R) > -(\alpha+\varepsilon) 2^{1+\lambda} R^{\lambda} .$$

By (19), for $r \leqq \frac{1}{2}R$,

$$\log m_3(r) \geqq -\frac{4 \log M(2R)}{R} r, \qquad -\log M_3(r) \geqq -\frac{4 \log M(2R)}{R} r .$$

Thus

$$(33) \quad \log m_3(r) - \cos \pi\lambda \log M_3(r) > -\frac{8 \log M(2R)}{R} r > -8(\alpha+\varepsilon) 2^{\lambda} R^{\lambda-1} r .$$

Inserting this in (24) we obtain

$$I(R_1, \tfrac{1}{2}R) > -(\alpha+\varepsilon) 2^{\lambda+3} R^{\lambda-1} \int_{R_1}^{\frac{1}{2}R} \frac{dr}{r^{\lambda}} = -\frac{\alpha+\varepsilon}{1-\lambda} 2^{\lambda+3} R^{\lambda-1} \{(\tfrac{1}{2}R)^{1-\lambda} - R_1^{1-\lambda}\}$$

and then

$$(34) \qquad I(R_1, \tfrac{1}{2}R) > -\frac{\alpha+\varepsilon}{1-\lambda} 2^{2\lambda+2} .$$

Combining (26), (31), (32) and (34) we get

$$(35) \quad A(R_1, \tfrac{1}{2}R) > k(\lambda) \frac{\log M(R_1)}{R_1^{\lambda}} - k(\lambda)(\alpha+\varepsilon) 2^{\lambda+2} (R_1/R)^{1-\lambda} - K(\lambda)(\alpha+\varepsilon) 2^{1+2\lambda} - \frac{\alpha+\varepsilon}{1-\lambda} 2^{2\lambda+2} .$$

Let us now consider the case $\alpha = 0$. For an arbitrarily chosen R_1 the first term on the right-hand side of (35) is some positive number. We then choose ε small enough (that is R large) to make the right-hand side of (35) positive.

If $\alpha > 0$ we have supposed that $\beta = \infty$. Then, by appropriate choice of R_1, the first term on the right-hand side of (35) can be as large as we wish. Thus there exist also in this case arbitrarily large R_1 and R such that $A(R_1, \frac{1}{2}R) > 0$.

Remembering (25) we conclude that

$$\log m_2(r) m_3(r) - \cos\pi\lambda \log M_2(r) M_3(r) > 0 \tag{36}$$

for certain values of $r \to \infty$. Since $f(z) \equiv f_1(z) f_3(z)$ we have

$$m(r) \geqq m_1(r) m_3(r) \geqq m_2(r) m_3(r) \tag{37}$$

and

$$M(r) \leqq M_1(r) M_3(r) \leqq M_2(r) M_3(r) . \tag{38}$$

Because, in this section, we have $\lambda \leqq \frac{1}{2}$, that is $\cos\pi\lambda \geqq 0$, (36), (37) and (38) yield what is to be proved in Theorems II and III:

$$\log m(r) - \cos\pi\lambda \log M(r) > 0 \tag{4}$$

for certain values of $r \to \infty$.

8. Proof of the theorems in the case of $\lambda > \frac{1}{2}$.

This time we add the integral

$$J(R_1, R_2) = \int_{R_1}^{R_2} \frac{(1 - \cos\pi\lambda) \log m_3(r)}{r^{1+\lambda}} dr \tag{39}$$

to both sides of the inequality (23). We perform all the estimations in exactly the same manner as before and finally we conclude that

$$\log m_2(r) m_3(r) - \cos\pi\lambda \log M_2(r) m_3(r) > 0 \tag{40}$$

for certain values of $r \to \infty$. Because $\cos\pi\lambda < 0$, the inequality $M_2(r) \geqq M_1(r)$ now goes in the wrong direction. We avoid this difficulty in the usual manner (cf. [4, Theorem 8.74]): Let z_0 be a point on $|z| = r$ where $m_1(r) = |f_1(z_0)|$. Then

$$\begin{aligned} m_2(r) M_2(r) &= \prod_1^N \left|1 - r^2/|a_n|^2\right| \leqq \prod_1^N |1 - z_0^2/a_n^2| \\ &= |f_1(z_0)| \cdot |f_1(-z_0)| \leqq m_1(r) M_1(r) . \end{aligned} \tag{41}$$

Hence, by means of (40),

$$\begin{aligned} m_1(r) M_1(r) \geqq m_2(r) M_2(r) &> M_2(r)^{1+\cos\pi\lambda} m_3(r)^{\cos\pi\lambda - 1} \\ &\geqq M_1(r)^{1+\cos\pi\lambda} m_3(r)^{\cos\pi\lambda - 1} , \end{aligned} \tag{42}$$

which gives

$$m_1(r)m_3(r) > M_1(r)^{\cos\pi\lambda}m_3(r)^{\cos\pi\lambda}$$

or

$$\log m_1(r)m_3(r) - \cos\pi\lambda \log M_1(r)m_3(r) > 0 . \tag{43}$$

Since $m(r) \geqq m_1(r)m_3(r)$ and $M(r) \geqq M_1(r)m_3(r)$, we obtain, once again, in the case $\frac{1}{2} < \lambda < 1$ that

$$\log m(r) - \cos\pi\lambda \log M(r) > 0 \tag{4}$$

for certain arbitrarily large values of r.

9. Zeros at the origin.

For simplicity we have up to now supposed $f(0) = 1$. If there are p zeros at the origin, the entire function can be written

$$f(z) \equiv Az^p f_1(z) f_3(z) . \tag{44}$$

There $f_1(z)$ is defined as in (6) and $f_3(z)$ by (44) and by $f_3(0) = 1$. To complete the proof in order to cover also this case, we must in addition add to (23) the integral

$$\int_{R_1}^{R_2} \frac{(1-\cos\pi\lambda)(\log|A| + p \log r)}{r^{1+\lambda}} dr . \tag{45}$$

The small changes in the proof which are necessary do not affect the conclusions about the positive sign of the right-hand sides of the formulae corresponding to (35). Thus (4) still holds, and Theorems I–III are proved without any restriction on $f(0)$.

In Theorem III, the condition $\beta = \infty$ is stronger than necessary. As we can see from (35), β should be larger than a certain multiple of α:

$$\beta > \frac{2^{1+2\lambda}}{k(\lambda)} \left(K(\lambda) + \frac{2}{1-\lambda} \right) \alpha , \tag{46}$$

but, of course, the value of the factor is far from being the best. But if $\beta < \infty$, the function is of order λ, $0 < \lambda < 1$, a case not of much interest here.

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