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FJELDSTAD'S SERIES AND THE q-ANALOG OF A SERIES OF DOUGALL

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1.

Fjeldstad [3] proved the identity

$$(1)\quad \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{2n}{n-m+k}\binom{2p}{p-m+k} = (-1)^m\frac{(m+n+p)!\,(2m)!\,(2n)!\,(2p)!}{(m+n)!\,(m+p)!\,(n+p)!\,m!\,n!\,p!},$$

and later Carlitz [1] gave its q-analog

$$(2)\quad \sum_{k=0}^{2m}(-1)^k\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}2n\\n-m+k\end{bmatrix}\begin{bmatrix}2p\\p-m+k\end{bmatrix}q^{\frac{1}{2}\{3(k-m)^2+(k-m)\}} = (-1)^m\frac{[m+n+p]!\,[2m]!\,[2n]!\,[2p]!}{[m+n]!\,[m+p]!\,[n+p]!\,[m]!\,[n]!\,[p]!},$$

where

$$\begin{bmatrix}n\\k\end{bmatrix} = \prod_{j=1}^{k}\frac{q^{n-j+1}-1}{q^j-1} = \frac{[n]!}{[k]!\,[n-k]!},$$

$$[n]! = \prod_{j=1}^{n}(q^j-1), \quad [0]! = 1\,.$$

It may be of interest to show how (1) follows from a general formula of Dougall [2] and how (2) leads to the q-analog of Dougall's formula.

2.

As a special case of his more general theorem, Dougall [2, formula (10)] gave the relation

$$(3)\quad 1-\frac{2xyz}{(x+1)(y+1)(z+1)}+\frac{2x(x-1)y(y-1)z(z-1)}{(x+1)(x+2)(y+1)(y+2)(z+1)(z+2)}-\ldots = \frac{x!\,y!\,z!\,(x+y+z)!}{(x+y)!\,(y+z)!\,(z+x)!}, \quad R(x+y+z) > -1\,,$$

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which may be rewritten in the form

$$(4)\qquad \sum_{k=0}^{\infty}(-1)^k\frac{\binom{x}{k}\binom{y}{k}\binom{z}{k}}{\binom{x+k}{k}\binom{y+k}{k}\binom{z+k}{k}} = \tfrac{1}{2}\left\{\frac{x!\,y!\,z!\,(x+y+z)!}{(x+y)!\,(y+z)!\,(z+x)!}+1\right\}.$$

We note the binomial coefficient identity

$$(5)\qquad \frac{\binom{x}{k}\binom{y}{k}\binom{z}{k}}{\binom{x+k}{k}\binom{y+k}{k}\binom{z+k}{k}} = \frac{\binom{2x}{x-k}\binom{2y}{y-k}\binom{2z}{z-k}}{\binom{2x}{x}\binom{2y}{y}\binom{2z}{z}},$$

and let $x=n$ be a non-negative integer. Then the relation (4) implies that

$$\tfrac{1}{2}\frac{(n+y+z)!\,(2n)!\,(2y)!\,(2z)!}{(n+y)!\,(n+z)!\,(y+z)!\,n!\,y!\,z!}+\tfrac{1}{2}\binom{2n}{n}\binom{2y}{y}\binom{2z}{z}$$
$$=\sum_{k=0}^{n}(-1)^k\binom{2n}{n-k}\binom{2y}{y-k}\binom{2z}{z-k}$$
$$=\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k}\binom{2y}{y-n+k}\binom{2z}{z-n+k}$$
$$=(-1)^n\tfrac{1}{2}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2y}{y-n+k}\binom{2z}{z-n+k}+\tfrac{1}{2}\binom{2n}{n}\binom{2y}{y}\binom{2z}{z},$$

and consequently we have

$$(6)\qquad \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2y}{y-n+k}\binom{2z}{z-n+k} = (-1)^n\frac{(n+y+z)!\,(2n)!\,(2y)!\,(2z)!}{(n+y)!\,(n+z)!\,(y+z)!\,n!\,y!\,z!}.$$

When y and z are non-negative integers this is equivalent with (1).

3.

To obtain the q-analog of (4) in the case where $x=n$ is a non-negative integer we proceed as follows. From (2) we have

$$\sum_{k=0}^{2m}(-1)^k\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}2n\\n-m+k\end{bmatrix}\begin{bmatrix}2p\\p-m+k\end{bmatrix}q^{\frac{1}{2}\{3(k-m)^2+(k-m)\}}$$
$$=\sum_{k=0}^{m}+\sum_{k=m}^{2m}-(-1)^m\begin{bmatrix}2m\\m\end{bmatrix}\begin{bmatrix}2n\\n\end{bmatrix}\begin{bmatrix}2p\\p\end{bmatrix}$$
$$=\sum_{k=0}^{m}(-1)^{m-k}\begin{bmatrix}2m\\m-k\end{bmatrix}\begin{bmatrix}2n\\n-k\end{bmatrix}\begin{bmatrix}2p\\p-k\end{bmatrix}q^{\frac{3}{2}k^2}(q^{\frac{1}{2}k}+q^{-\frac{1}{2}k})-(-1)^m\begin{bmatrix}2m\\m\end{bmatrix}\begin{bmatrix}2n\\n\end{bmatrix}\begin{bmatrix}2p\\p\end{bmatrix},$$

Now, obviously the identity (5) applies to the q-binomial coefficients also, and applying this to our equations we find that we have

$$(7)\qquad \sum_{k=0}^{m}(-1)^k \frac{\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} p \\ k \end{bmatrix}}{\begin{bmatrix} m+k \\ k \end{bmatrix}\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} p+k \\ k \end{bmatrix}} q^{\frac{3}{2}k^2}(q^{\frac{1}{2}k}+q^{-\frac{1}{2}k}) = 1 + \frac{[m+n+p]!\,[m]!\,[n]!\,[p]!}{[m+n]!\,[m+p]!\,[n+p]!},$$

which is the desired q-analog of (5) when $x=m=$ non-negative integer.

4.

An interesting special case of (7) might be noted. Since

$$\begin{bmatrix} -m-1 \\ k \end{bmatrix} = \frac{(-1)^k}{q^{mk}q^{\frac{1}{2}k(k+1)}}\begin{bmatrix} m+k \\ k \end{bmatrix}$$

it follows that

$$\frac{\begin{bmatrix} -m-1 \\ k \end{bmatrix}}{\begin{bmatrix} -m+k-1 \\ k \end{bmatrix}} = \frac{\begin{bmatrix} m+k \\ k \end{bmatrix}}{\begin{bmatrix} m \\ k \end{bmatrix}} q^{-k^2}.$$

Therefore, setting $p=-m-1$ in (7) we have

$$(8)\qquad \sum_{k=0}^{m}(-1)^k \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{\begin{bmatrix} n+k \\ k \end{bmatrix}}(q^k+1)q^{\frac{1}{2}k(k-1)} = 1 + (-1)^m \frac{\begin{bmatrix} n-1 \\ m \end{bmatrix}}{\begin{bmatrix} n+m \\ m \end{bmatrix}} q^{\frac{1}{2}m(m+1)}.$$

REFERENCES

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