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DISTRIBUTIONS INVARIANT UNDER AN ORTHOGONAL GROUP OF ARBITRARY SIGNATURE

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1. Introduction.

Let $B = B(u,v)$ be a symmetric real bilinear form on $R^n \times R^n$ and let $\mathscr{L}$ be the group of all linear transformations leaving B invariant. A distribution $T(u)$ is said to be invariant under $\mathscr{L}$ if

$$T(\Lambda u) = T(u)$$

for every Λ in $\mathscr{L}$. It is very easy to describe e.g. all invariant T with supports at $u=0$. They are of the form $P(\square)\delta(u)$ where $\square$ is Laplace's operator $B^{-1}(D,D)$ $\big(D = (\partial/\partial u_1, \ldots, \partial/\partial u_n)\big)$ and P is a polynomial. They span a linear space which we shall call $\mathscr{L}_0'$. Let $\mathscr{L}'$ be the space of all invariant distributions. A rather complete description of $\mathscr{L}'$ has been given by Methée [3] [4], when B has Lorentz signature and by de Rham [6] [7], for general indefinite signature. They show in particular that outside $u=0$ every $T \in \mathscr{L}'$ has the form

$$\langle T,f\rangle = \langle F,Nf\rangle , \tag{1}$$

where f is any function in $\mathscr{D}(R^n)$ which vanishes in a neighbourhood of $u=0$, F is a unique distribution on the real line and Nf is the mean value

$$(Nf)(\tau) = \int \delta\big(\tau - B(u,u)\big) f(u)\, du ,$$

which belongs to $\mathscr{D}(R)$. (We use Schwartz's notations. $\mathscr{D}$ is the set of infinitely differentiable functions with compact supports.) When f does not vanish at the origin, Nf becomes singular for $\tau = 0$, but has an expansion around $\tau = 0$ in powers of τ and a suitable additional set of singular functions. The singular expansion coefficients are linear invariant functionals $\langle S,f\rangle$ of f with support at $u=0$, that is, every S belongs to $\mathscr{L}_0'$ and it turns out that $\mathscr{L}_0'$ is spanned by the distributions S.

According to Gårding and Roos (see [2]) a more concise description of $\mathscr{L}'$ can be obtained by putting a suitable linear topology on $H = N\mathscr{D}$. Then to every $T \in \mathscr{L}'$ there is a unique element F in the dual H' of H such

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that (1) holds. More generally the adjoint mapping N' is a linear homeomorphism of H' onto $\mathscr{L}'$. In other words the space H' gives a parametrization of $\mathscr{L}'$. Gårding and Roos proved this for the Lorentz group. The main purpose of this paper is to prove the same result when B has the signature p,q with $p+q=n, p \geqq 2, q \geqq 2$. We note in passing that it holds also when B is definite. In this case the space $H=N\mathscr{D}$ is very simple. Changing if necessary B to $-B$ we can assume that B is positive definite. Then H consists of all functions $\tau^{n-\frac{1}{2}}f(\tau)$ where f is infinitely differentiable for $\tau \geqq 0$. Its dual can be identified with all distributions in τ with supports in $\tau \geqq 0$.

We have assumed that $\mathscr{L}$ is the entire group leaving B invariant. Let $\mathscr{L}_1$ be the connected component of $\mathscr{L}$ that contains the unit element and $\mathscr{L}_1' \supset \mathscr{L}'$ the corresponding space of invariant distributions. It is easy to see that $\mathscr{L}_1' = \mathscr{L}'$ except when B has Lorentz signature (see remark p. 13). Although this case does not concern us, we mention that then $\mathscr{L}_1'$ is the direct sum of $\mathscr{L}'$ and a space $\mathscr{L}_-'$ of odd invariant distributions with the property that

$$T(\Lambda u) = \varepsilon(\Lambda) T(u) ,$$

where $\varepsilon(\Lambda) = -1$ if Λ reverses time, $\varepsilon(\Lambda)=1$ otherwise. The space $\mathscr{L}_-'$ can be obtained in the same way as $\mathscr{L}'$ by replacing N by

$$(N_- f)(\tau) = \int \delta\big(\tau - B(u,u)\big) \operatorname{sgn} B(u,v) f(u)\, du$$

where v is any time-like vector. The space $N_-\mathscr{D}$ consists of all functions of τ with compact supports which vanish for $B(v,v)\,\tau < 0$ and are infinitely differentiable for $\tau B(v,v) \geqq 0$ (Gårding and Roos, see [2])

In outline, our paper runs as follows. In section 2 we introduce the infinitesimal rotations and prove a lemma which we need in section 5. In section 3 we describe and topologize some function spaces, $H_{s,m}$, where s assumes four values and m all integral values $\geqq 1$. In section 4 we prove that N is a continuous surjective mapping

$$\mathscr{D}(R^n) \to H = H_{s,m} ,$$

where s and m depend on p and q. The same is true if we replace $\mathscr{D}$ by the space $\mathscr{S}$ of all infinitely differentiable functions which together with their derivatives decrease faster than any negative power of $|u|$, and modify $\mathscr{L}'$ and $H_{s,m}$ accordingly. In section 5 we show that in both cases N' is a linear homeomorphism of $H_{s,m}'$ onto $\mathscr{L}'$. The rest of the paper is devoted to some applications of this result. Let G be a linear operator from $\mathscr{L}'$ to $\mathscr{L}'$. Transported to H' it becomes $\Gamma = N'GN'^{-1}$.

When $G=\square$ is Laplace's operator, Γ is the adjoint of the differential operator

$$D = 4\left(\tau D_\tau^2 + \tfrac{1}{2}(n-4)D_\tau\right), \qquad D_\tau = d/d\tau ,$$

which maps H into itself and has the property that $N\square f = DNf$ $(f \in \mathscr{D}(R^n))$. Using D it is easy to write down all invariant fundamental solutions of $\square$ (de Rham [7]). We do this in section 6 and in section 8 we prove that the equation $P(\square)T=S$ where P is an arbitrary polynomial has a solution T in $\mathscr{L}'$ for every S in $\mathscr{L}'$. When B has Lorentz signature this was shown by Methée [3]. Finally in section 7 we get an explicit expression for Γ when G is the Fourier transform $\mathscr{F}$. In the Lorentz case, Fourier-transforms of invariant distributions were studied by Methée [4].

Since the use of the homeomorphism N' makes the subject very clear and simple, I have chosen to make the paper self-contained although this leads to considerable overlappings with the papers by de Rham and Methée.

The subject of this paper was suggested to me by professor Lars Gårding. I wish to express my gratitude to him for his interest and valuable advice.

2. Infinitesimal rotations.

We choose a coordinate system so that

$$B(u,v) = \sum_{i=1}^{p} x_i x_i' - \sum_{k=1}^{q} y_k y_k' = xx' - yy' ,$$

where $u=(x,y)$ and $v=(x',y')$, and we put

$$B(u,v) = uv .$$

Every $\Lambda \in \mathscr{L}$ can be written as

$$\Lambda = \Lambda^x \Lambda^y \Lambda_\theta ,$$

where Λ^x and Λ^y belong to $\mathscr{L}$ and leave y resp. x fixed and Λ_θ is defined by

$$\begin{aligned} x_j' &= x_j \quad \text{when} \quad j \neq i, \qquad y_l' = y_l \quad \text{when} \quad l \neq k , \\ x_i' &= x_i \cosh\theta + y_k \sinh\theta , \\ y_k' &= x_i \sinh\theta + y_k \cosh\theta . \end{aligned}$$

The group $\mathscr{L}$ consists of four connected components $\mathscr{L}_{++}$ $(=\mathscr{L}_1)$, $\mathscr{L}_{+-}$, $\mathscr{L}_{-+}$, $\mathscr{L}_{--}$, where the transformations of $\mathscr{L}_{+-}$, are characterized by $\det \Lambda^x = 1$ and $\det \Lambda^y = -1$.

Let

$$L_{ij}^x = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i},$$

$$L_{kl}^y = y_k \frac{\partial}{\partial y_l} - y_l \frac{\partial}{\partial y_k},$$

$$L_{ik} = x_i \frac{\partial}{\partial y_k} + y_k \frac{\partial}{\partial x_i}$$

be the infinitesimal rotations. We now have the following lemma:

LEMMA 2.1. *T belongs to $\mathscr{L}'_{++}$ if and only if*

$$(2) \qquad L_{ij}^x T = L_{kl}^y T = L_{ij} T = 0 \quad \text{for} \quad 1 \leqq i,j \leqq p,\ 1 \leqq k,l \leqq q\,,$$

PROOF. Define transformations Λ_θ^x by

$$\begin{aligned} x'_k &= x_k \quad \text{if} \quad k \neq i,\ k \neq j, \qquad y'_l = y_l \quad \text{for every } l\,, \\ x'_i &= x_i \cos\theta + x_j \sin\theta\,, \\ x'_j &= -x_i \sin\theta + x_j \cos\theta\,. \end{aligned}$$

T is invariant under all these transformations Λ_θ^x, that is,

$$\langle T, f(u)\rangle = \langle T, f(\Lambda_\theta^x u)\rangle \qquad \text{for every } f \in \mathscr{D}(R^n)$$

if and only if

$$\frac{d}{d\theta}\langle T, f(\Lambda_\theta^x u)\rangle = \left\langle T, \frac{\partial}{\partial\theta} f(\Lambda_\theta^x u)\right\rangle = 0$$

for every $f \in \mathscr{D}(R^n)$. But as

$$\frac{\partial}{\partial\theta} f(\Lambda_\theta^x u) = (L_{ij}^x f)(\Lambda_\theta^x u)$$

this is equivalent to $L_{ij}^x T = 0$.

In the same way we define transformations Λ_θ^y and prove that T is invariant under these transformations if and only if $L_{kl}^y T = 0$ for every k and l.

Similarly we can prove that T is invariant under all the transformations Λ_θ defined above if and only if $L_{ik} T = 0$ for every i and k. Now if $T \in \mathscr{L}'_{++}$, T is invariant under all the transformations Λ_θ^x, Λ_θ^y and Λ_θ and consequently (2) holds. As an arbitrary transformation Λ in $\mathscr{L}_{++}$ can be written as a product of Λ_θ^x, Λ_θ^y and Λ_θ we see that (2) implies that $T \in \mathscr{L}'_{++}$.

REMARK. Since $\mathscr{L}' \subset \mathscr{L}'_{++}$ it is clear $T \in \mathscr{L}'$ implies (2). Later we shall see that in fact $\mathscr{L}' = \mathscr{L}'_{++}$.

3. Some function spaces.

We are going to define spaces of functions $\varphi, \psi, \ldots$ of one variable τ which are regular for $\tau \neq 0$ and have a singularity for $\tau = 0$ defined in terms of one of four functions $\gamma = \gamma_s(\tau)$ labelled by an index s and given by

$$\theta(\tau), \quad \theta(\pm\tau)(\pm\tau)^{\frac{1}{2}}, \quad \log|\tau|^{-1}$$

according as $s = 1, \ \pm\frac{1}{2}$ and l respectively. Here θ is Heaviside's function, $\theta(\tau) = 1$ if $\tau \geqq 0$ and $\theta(\tau) = 0$ if $\tau < 0$. Let m be a fixed integer $\geqq 1$ and denote by $P_v(\tau)$ polynomials of degree $\leqq v$ divisible by τ^m. In particular $P_v = 0$ unless $v \geqq m$. It is obvious that

$$\gamma(\tau) P_v(\tau) \in C^v \quad \text{if and only if} \quad P_v = 0 . \tag{3}$$

We shall consider functions of class C^v outside the origin with the property that

$$\varphi - \gamma P_v \in C^v \tag{4}$$

at the origin for at least one polynomial P_v. It follows from (3) that P_v is uniquely determined by φ. Further, if $v < m$, $P_v = 0$ so that φ is itself of class C^m at the origin. It is clear that the coefficients $A_j(\varphi)$ of P_v,

$$P_v(\tau) = \sum_m^v A_j(\varphi)\, \tau^j ,$$

are linear functions of φ and that $A_j(\varphi) = 0$ if $\varphi \in C^j$ at the origin. In particular $A_j(\varphi) = 0$ if φ vanishes in a neighbourhood of the origin. We shall find it convenient to write P_v as

$$P_v(\tau) = \sum_0^v A_j(\varphi)\, \tau^j ,$$

where, by definition, $A_j(\varphi) = 0$ when $j < m$. Expanding (4) in a Taylor series around $\tau = 0$,

$$\varphi(\tau) - \gamma(\tau) P_v(\tau) = \sum_0^v B_j(\varphi)\, \tau^j + o(\tau^v)$$

we obtain another set B_j of linear functionals of φ with supports at the origin. Thus every φ with the property (4) has a unique expansion of the form

$$\varphi(\tau) = \sum_0^v B_j(\varphi)\, \tau^j + \gamma(\tau) \sum_0^v A_j(\varphi)\, \tau^j + o(\tau^v) .$$

Now let $a > 0$ and let C_a^v and H_a^v be the space of all $\varphi \in C^v$ with support in $|\tau| \leqq a$ and the space of all φ with the property (4) and support in $|\tau| \leqq a$ respectively. It is clear that both spaces decrease when v increases and that $H_a^v \supset C_a^v$. More precisely any $\varphi \in H_v^a$ has a unique decomposition

(5) $$\varphi = (\varphi - \gamma P_v) + \gamma P_v ,$$

where the first term belongs to C_a^v. Hence H_a^v is the direct sum of C_a^v and a space of dimension $\max(v-m, 0)$. With the norms

(6) $$|\varphi|_v = \max_{k \leqq v} \max_{\tau} |\varphi^{(k)}(\tau)|$$

and

(7) $$g_v(\varphi) = |\varphi - \gamma P_v|_v + \sum_0^v |A_j(\varphi)| ,$$

C_a^v and H_a^v become Banach spaces. It is clear that the A_j and B_j with $j \leqq v$ are continuous on H_a^v. Equipped with the norms (6),

$$C_a = C_a^\infty = \bigcap_{v \geqq 0} C_a^v$$

becomes a reflexive Fréchet space. The reflexivity follows from the classical fact that the injections $C_a^{v+1} \to C_a^v$ are completely continuous. Since, obviously, the injections $H_a^{v+1} \to H_a^v$ are also completely continuous, the space

$$H_a = H_a^\infty = \bigcap_{v \geqq 0} H_a^v$$

equipped with the norms (7), is also a reflexive Fréchet space. All the A_j and B_j are continuous on H_a. It is clear that H_a consists of all φ with support in $|\tau| \leqq a$ which are in C^∞ for $\tau \neq 0$ and for which there exists a formal power series

$$P(\tau) = \sum_0^\infty a_j \tau^j, \qquad a_j = 0 \quad \text{if} \quad j < m ,$$

with partial sums $P_v(\tau) = \sum_0^v a_j \tau^j$ such that

(8) $$\varphi - \gamma P_v \in C^v \qquad \text{at } \tau = 0 \quad \text{for all } v .$$

In particular, $a_j = A_j(\varphi)$ so that P is uniquely determined by φ. The direct decomposition (5) fails to hold in the infinite case. Instead we have

Lemma 3.1. *φ belongs to H_a if and only if*

(9) $$\varphi(\tau) = \varphi_1(\tau) + \gamma(\tau)\varphi_2(\tau)\tau^m, \quad \textit{where} \quad \varphi_1, \varphi_2 \in C_a .$$

Proof. It is clear that any φ of the form (9) is in H_a. Conversely, let $\varphi \in H_a$. Then, by a classical result due to Borel, we can find a $\varphi_2 \in C_a$ such that $\tau^m \varphi_2$ has the Taylor series P at the origin. Hence $\varphi - \gamma \tau^m \varphi_2 \in C_a$.

Finally, we shall define spaces of functions with arbitrarily large supports having the property (4) for all v. This can of course be done in various ways. We choose to define the analogue of L. Schwartz's space $\mathscr{D}(R)$ which is the inductive limit (Bourbaki [1, Chap. II p. 61–65]) of the spaces C_a when, for example, $a=1,2,\ldots$. We let H be the inductive limit of the spaces H_a for $a=1,2,\ldots$. It consists of all φ with compact supports, belonging to C^∞ for $\tau \neq 0$ and having the property (4). We have

LEMMA 3.2. *H is reflexive.*

PROOF. H is in fact the strict inductive limit of the H_a and hence H is reflexive (Bourbaki [1, Chap. IV, p. 95, Exc. 17b]).

A complete set of seminorms for $\mathscr{D}(R)$ can be obtained by putting

$$h(\varphi) = \sum_k \max_\tau |h_k(\tau)\varphi^{(k)}(\tau)| ,$$

where the h_k are continuous functions with the property that for every compact K there is a $\lambda(K)$, such that $h_k=0$ in K when $k>\lambda$. Correspondingly, we obtain a complete set of seminorms on H by choosing a function $\chi \in \mathscr{D}(R)$ which is 1 in a neighbourhood of $\tau=0$ and putting

$$g(\varphi) = h(\varphi-\chi\gamma P_v) + \sum_1^\mu |A_j(\varphi)| ,$$

where $v>\lambda(K)$, $(K=\operatorname{supp}\chi)$ and μ is an arbitrary integer. Changing χ, we get an equivalent set of seminorms.

Any element F in the dual H' of H can be described in terms of a distribution and the functionals A_j. We have

LEMMA 3.3. *F belongs to H' if and only if F has the form*

$$\langle F,\varphi\rangle = \langle F_0,\varphi-\chi\gamma P_v\rangle + \sum_0^v c_j A_j(\varphi) , \tag{10}$$

where F_0 belongs to $\mathscr{D}'(R)$ and the order of F_0 on the support of χ is less than v.

PROOF. Clearly $F\in H'$ if F is defined by (10). Let $F\in H'$ and let F_0 be the restriction of F to $\mathscr{D}(R)$. Then $F_0\in\mathscr{D}'(R)$, and if $v\geqq$ the order of F_0 on the support of χ, we define F_1 by

$$\langle F_1,\varphi\rangle = \langle F_0,\varphi-\chi\gamma P_v\rangle .$$

Then we have $F_1=F$ when $A_j(\varphi)=0$, $j\leqq v$, which implies

$$F = F_1+\sum_0^v c_j A_j .$$

COROLLARY. *Any $F \in H'$ with support at $\tau = 0$ has the form*

$$\sum c'_j B_j + \sum c''_j A_j ,$$

where the sums are finite.

REMARK. We can also define H as the analogue of the space $\mathcal{S}$ (Schwartz [8]), i.e. the space of all functions $\varphi \in C^\infty$ for which the norms

$$\sum_{j,\,k \leqq v} |\tau^j \varphi^{(k)}(\tau)|_0$$

are finite. The lemmas 3.1–3.3 are still true.

REMARK. When we want to exhibit that H and H' depend on s and m we write $H_{s,m}$ and $H'_{s,m}$.

4. The mapping of $\mathcal{D}(R^n)$ onto $H_{s,m}$.

Let $g \in \mathcal{D}(R)$ and $f \in \mathcal{D}(R^n)$ and consider the integral

$$\int g(uu) f(u)\, du .$$

Make a change of variables such that

$$2\sigma = uu, \quad 2\varrho = xx + yy, \quad x = (\varrho + \sigma)^{\frac{1}{2}} w_x, \quad y = (\varrho - \sigma)^{\frac{1}{2}} w_y ,$$

where w_x belongs to the $(p-1)$-dimensional sphere, S_{p-1}, and $w_y \in S_{q-1}$. Then the integral becomes

$$\int_{\varrho \geqq |\sigma|} g(\sigma) Mf(\varrho, \sigma)(\varrho + \sigma)^{\bar{p}} (\varrho - \sigma)^{\bar{q}}\, d\varrho\, d\sigma ,$$

where

$$(Mf)(\varrho, \sigma) = \int f\big((\varrho + \sigma)^{\frac{1}{2}} w_x,\ (\varrho - \sigma)^{\frac{1}{2}} w_y\big)\, dw_x\, dw_y .$$

dw_x and dw_y are the surface elements of S_{p-1} and S_{q-1} and $\bar{p} = \frac{1}{2}(p-2)$ and $\bar{q} = \frac{1}{2}(q-2)$.

Hence letting g approach $\delta(\tau - \sigma)$ we get

$$\int \delta(uu - \tau) f(u)\, du = \int_{\varrho \geqq |\tau|} Mf(\varrho, \tau)(\varrho + \tau)^{\bar{p}} (\varrho - \tau)^{\bar{q}}\, d\varrho .$$

We observe that $\delta(uu - \tau) \in \mathcal{L}'$ and that the support of $\delta(uu - \tau)$ is the hyperboloid $uu = \tau$.

Define the mapping N by

$$(Nf)(\tau) = \int \delta(uu - \tau) f(u)\, du \quad \text{for every} \quad f \in \mathcal{D}(R^n) .$$

Put

$$\Omega = R^n - \{0\}, \quad Q = \{(\varrho, \sigma);\ \varrho \geqq |\sigma|\} \quad \text{and} \quad Q_0 = Q - \{0\} .$$

LEMMA 4.1. *M defines linear surjective continuous mappings*

$$\mathscr{D}(R^n) \to \mathscr{D}(Q)\,, \tag{11}$$

$$\mathscr{D}(\Omega) \to \mathscr{D}(Q_0) \tag{12}$$

PROOF. (12) follows from (11). We prove (11).

The only thing which is difficult to prove is that Mf has continuous derivatives of any order in $(0,0)$ which evidently is equivalent to proving the same property for $M_1 f$, where

$$M_1 f(\xi,\eta) = \int f(\xi^{\frac{1}{2}} w_x, \eta^{\frac{1}{2}} w_y)\, dw_x\, dw_y = \psi(\mu,v)$$

with $\mu^2=\xi$, $v^2=\eta$. We observe that $\psi \in C^\infty(R^2)$ and that ψ is even in μ and v. We have

$$(\partial/\partial\xi)(M_1 f) = (2\mu)^{-1}(\partial/\partial\mu)\psi = \psi_1(\mu,v)\,,$$

where ψ_1 is continuous and even in μ for $(\partial/\partial\mu)\psi$ is odd in μ. Now it follows by induction that an arbitrary derivative of $M_1 f$ is continuous. Furthermore we have

$$(\partial/\partial\xi)(M_1 f)(0,0) = \tfrac{1}{2}(\partial/\partial\mu)\psi(0,0)\,.$$

REMARK. From the above proof it follows that

$$D_\varrho^\beta D_\sigma^\gamma (Mf)(0,0) = \sum_{|\alpha| \leqq 2(\beta+\gamma)} c_\alpha (D_\alpha f)(0)\,,$$

where some $c_\alpha \neq 0$ with $|\alpha| = 2(\beta+\gamma)$.

LEMMA 4.2. *N is a linear continuous surjective mapping $\mathscr{D}(\Omega) \to \mathscr{D}(R)$.*

PROOF. According to Lemma 4.1 it is sufficient to prove that the mapping $L\colon \mathscr{D}(Q_0) \to \mathscr{D}(R)$ defined by

$$(Lg)(\tau) = \int_{\varrho \geqq |\tau|} g(\varrho,\tau)(\varrho+\tau)^{\bar{p}}(\varrho-\tau)^{\bar{q}}\, d\varrho$$

is a linear continuous surjective mapping. As $g(\varrho,\sigma)=0$ in a neighbourhood of $(0,0)$, it is easily seen that $Lg \in \mathscr{D}(R)$. Clearly L is linear and continuous.

Let $\varphi \in \mathscr{D}(R)$ and put $K = \operatorname{supp}\varphi$. Let $I \subset R$ be a compact interval so that $I \times K$ is contained in the interior of Q_0 and let $\psi \in \mathscr{D}(I)$ so that

$$\int \psi(\varrho)\, d\varrho = 1\,.$$

If

$$h(\varrho,\sigma) = \psi(\sigma)\varphi(\varrho)(\varrho+\sigma)^{-\bar{p}}(\varrho-\sigma)^{-\bar{q}}$$

we have $Lh(\tau) = \varphi(\tau)$ and hence N is surjective.

Now we can prove the following basic lemma.

LEMMA 4.3. *N is a linear, continuous, surjective mapping $\mathscr{D}(R^n) \to H_{s,m}$ where $m=[\frac{1}{2}(n-2)]$ and*

I) $s = 1$ *if* *p and q both are even,*
II) $s = +\frac{1}{2}$ *if* *p is odd and q is even,*
II′) $s = -\frac{1}{2}$ *if* *p is even and q is odd,*
III) $s = l$ *if* *p and q both are odd.*

REMARK. We observe that II′ follows from II by interchanging x and y.

PROOF. At first we show that $N(\mathscr{D}(R^n)) \subset H_{s,m}$. Clearly Nf has compact support and is of class C^∞ for $\tau \neq 0$. Put $(Mf)(\varrho,\sigma)=h(\varrho+\sigma,\varrho-\sigma)$ with $h \in \mathscr{D}(R^+ \times R^+)$. We examine Nf in $\mathcal{O}=\{\tau;\ |\tau| \leqq \frac{1}{2}\}$. It is easily seen that

$$\int_1^\infty h(\varrho+\tau,\varrho-\tau)(\varrho+\tau)^{\overline{p}}(\varrho-\tau)^{\overline{q}}\,d\varrho \in C^\infty(\mathcal{O})\,.$$

I. *p and q both even.* It is easily seen that

$$\int_{|\tau|}^1 h(\varrho+\tau,\varrho-\tau)(\varrho+\tau)^i(\varrho-\tau)^j\,d\varrho \in C^{i+j}(\mathcal{O})\,.$$

If we put $h_{\beta,\gamma}=D_1^\beta D_2^\gamma h$ we get by developing in a Taylor series and integrating by parts

$$\begin{aligned} Nf(\tau) &= \sum_{\beta+\gamma\leqq v} \binom{\beta+\gamma}{\beta} h_{\beta,\gamma}(0,0) \int_{|\tau|}^1 (\varrho+\tau)^{\overline{p}+\beta}(\varrho-\tau)^{\overline{q}+\gamma}\,d\varrho + w \\ &= \theta(\tau)\sum_{\beta+\gamma\leqq v} a_{\beta,\gamma}\tau^{\overline{p}+\overline{q}+\beta+\gamma+1}h_{\beta,\gamma}(0,0) + \sum_{\beta+\gamma\leqq v} h_{\beta,\gamma}(0,0)w_{\beta,\gamma} + w\,, \end{aligned}$$

where $w \in C^{v+1}(\mathcal{O})$ and $w_{\beta,\gamma} \in C^\infty(\mathcal{O})$ is independent of f, and

$$a_{\beta,\gamma} = \frac{\Gamma(\overline{p}+\beta+1)\,\Gamma(\overline{q}+\gamma+1)}{\Gamma(\overline{p}+\overline{q}+\beta+\gamma+2)}\cdot\binom{\beta+\gamma}{\beta}\,2^{\overline{p}+\overline{q}+\beta+\gamma+1}\,.$$

Hence $N(\mathscr{D}(R^n)) \subset H_{1,m}$ and the expression for Nf shows that

$$A_{m+k}(Nf) = \sum_{|\alpha|\leqq 2k} c_\alpha (D_\alpha f)(0)\,, \tag{13}$$

where $c_\alpha \neq 0$ for some α, $|\alpha|=2k$.

II. *p odd, q even.* It is easily seen that

$$\int_{|\tau|}^{1} h(\varrho+\tau,\varrho-\tau)(\varrho+\tau)^{i+\frac{1}{2}}(\varrho-\tau)^{j}d\varrho \in C^{i+j+1}(\mathcal{O}) .$$

By developing in a Taylor series and integrations by parts we get

$$\begin{aligned} Nf(\tau) &= \sum_{\beta+\gamma\leq v}\binom{\beta+\gamma}{\beta}h_{\beta,\gamma}(0,0)\int_{|\tau|}^{1}(\varrho+\tau)^{\bar{p}+\beta}(\varrho-\tau)^{\bar{q}+\gamma}d\varrho + w \\ &= \theta(\tau)(2\tau)^{\frac{1}{2}}\sum_{\beta+\gamma\leq v}a_{\beta,\gamma}\tau^{\bar{p}+\bar{q}+\beta+\gamma+\frac{1}{2}}h_{\beta,\gamma}(0,0)+\sum_{\beta+\gamma\leq v}h_{\beta,\gamma}(0,0)w_{\beta,\gamma}+w , \end{aligned}$$

where $w\in C^{v+1}(\mathcal{O})$ and $w_{\beta,\gamma}\in C^{\infty}(\mathcal{O})$ is independent of f. Hence $N(\mathcal{D}(R^n))\subset H_{+\frac{1}{2},m}$ and the expression for Nf gives again (13).

III. *p and q both odd.* It is easily seen

$$\int_{|\tau|}^{1} h(\varrho+\tau,\varrho-\tau)(\varrho+\tau)^{i+\frac{1}{2}}(\varrho-\tau)^{j+\frac{1}{2}}d\varrho \in C^{i+j+1}(\mathcal{O}) .$$

We get by developing in a Taylor series and integrating by parts

$$\begin{aligned} Nf(\tau) &= \sum_{\beta+\gamma\leq v}\binom{\beta+\gamma}{\beta}h_{\beta,\gamma}(0,0)\int_{|\tau|}^{1}(\varrho+\tau)^{\bar{p}+\beta}(\varrho-\tau)^{\bar{q}+\gamma}d\varrho + w \\ &= 2\pi^{-1}\log|\tau|^{-1}\sum_{\beta+\gamma\leq v}a_{\beta,\gamma}\tau^{\bar{p}+\bar{q}+\beta+\gamma+1}h_{\beta,\gamma}(0,0)+ \\ &\quad+\sum_{\beta+\gamma\leq v}h_{\beta,\gamma}(0,0)w_{\beta,\gamma}+w , \end{aligned}$$

where $w\in C^{v+1}(\mathcal{O})$ and $w_{\beta,\gamma}\in C^{\infty}(\mathcal{O})$ is independent of f. Hence $N(\mathcal{D}(R^n))\subset H_{lm}$ and the expression for Nf gives again (13).

We shall now show that N is surjective. Let $\varphi\in H_{s,m}$. From the calculations above it follows that there is an $h\in\mathcal{D}(R^+\times R^+)$ so that

$$\int_{\varrho\leq|\tau|} h(\varrho+\tau,\varrho-\tau)(\varrho+\tau)^{\bar{p}}(\varrho-\tau)^{\bar{q}}d\varrho$$

has the same singular part as φ. Take an $f_1\in\mathcal{D}(R^n)$ so that $Mf_1=h(\varrho+\tau,\varrho-\tau)$. We have $Nf_1-\varphi\in\mathcal{D}(R)$. By Lemma 4.2 there is an f_2 in $\mathcal{D}(\Omega)$ so that $Nf_2=Nf_1-\varphi$. We now have $Nf=\varphi$ with $f=f_1-f_2\in\mathcal{D}(R^n)$.

Proof of the continuity of N: Let $f_j\to 0$ in $\mathcal{D}(R^n)$. It is easily seen that $\operatorname{supp}(Nf_j)$ is contained in a fixed compact set $K\subset R$. From (13) it follows that $A_k(Nf_j)\to 0$. Furthermore we have with $0<\vartheta<1$

$$|D_\tau^v(Nf_j) - \gamma_s(\tau)\chi(\tau)P_v(\tau)|$$

$$\leqq \left| D_\tau^v \left(\int_{|\tau|}^{1} \sum_{\beta+\gamma\leqq v} c_{\beta,\gamma} h_{j,\beta,\gamma}\big(\vartheta(\varrho+\tau), \vartheta(\varrho-\tau)\big)\cdot(\varrho+\tau)^{\bar{p}+\beta}(\varrho-\tau)^{\bar{q}+\gamma}\,d\varrho + \right.\right.$$

$$\left.\left. + \sum_{\beta+\gamma\leqq v} h_{j,\beta,\gamma}(0,0)\,w_{\beta,\gamma} + \int_1^\infty h_j(\varrho+\tau,\varrho-\tau)(\varrho+\tau)^{\bar{p}}(\varrho-\tau)^{\bar{q}}\,d\varrho \right)\right| +$$

$$+ C\sum_{k=0}^{v} |A_k(Nf_j)|\,,$$

which tends to zero uniformly in K and hence N is continuous.

5. Parametrization of $\mathscr{L}'$.

Lemma 5.1. *If $T \in \mathscr{L}'$ there is one and only one distribution $F \in \mathscr{D}'(R)$ such that*

$$\langle T,f\rangle = \langle F,Nf\rangle \qquad \text{for every } f \in \mathscr{D}(\Omega)\,.$$

Proof. After having in a convenient manner introduced new coordinates $\eta(x,y) = (\varrho, \sigma, \theta_x, \theta_y)$ where $2\varrho = xx + yy$ and $2\sigma = uu$, using Lemma 2.1 we can prove that $T\circ\eta$ is independent of $(\varrho, \theta_x, \theta_y)$. Hence we easily get the existence of F. (We have here defined $T\circ\eta$ by $\langle T\circ\eta, f\rangle = \langle T, f\circ\eta^{-1}|J(\eta^{-1})|\rangle$, where J denotes the Jacobi determinant; we assume that $\inf|J(\eta)| > 0$ in the set). By use of lemma 4.2, the uniqueness of F follows from the fact that $N\big(\mathscr{D}(\Omega)\big) = \mathscr{D}(R)$. (Cf. de Rham [6] [7] and for the case p = 1 Methée [3].)

Lemma 5.2. *$T \in \mathscr{L}'$ and* $\operatorname{supp} T \subset \{0\}$ *if and only if $T = P(\Box)\delta$, where P is a polynomial.*

Proof. In fact, every such T has the form $Q(D)\delta(u)$ where Q is an invariant polynomial and hence of the form $P(\Box)\delta$.

We observe that if $F \in H'$ then F defines a distribution $T \in \mathscr{L}'$ by

$$\langle T,f\rangle = \langle F,Nf\rangle \qquad \text{for every} \quad f \in \mathscr{D}(R^n)\,.$$

Put $T = N'F$ where N' is the adjoint mapping to N.

Lemma 5.3. *$T \in \mathscr{L}'$ and* $\operatorname{supp} T \subset \{0\}$ *if and only if $T = \sum c_j N'A_j$, where the sum is finite.*

Proof. Let G_v be all distributions in question whose orders are $\leqq 2v$. Lemma 5.2 shows that $\dim G_v = v+1$. From (13) it follows that $N'A_{m+k} \in G_v$ for every $k \leqq v$ and that the $N'A_{m+k}$ are linearly independent, and hence we have the lemma.

Now we can easily get the parametrization of $\mathscr{L}'$.

THEOREM 5.1. *N' is a linear homeomorphism $H'_{s,m} \to \mathscr{L}'$, where s and m depend on p and q as in Lemma 4.3.*

PROOF. At first we prove that $N'H'_{s,m} = \mathscr{L}'$. Clearly $N'H'_{s,m} \subset \mathscr{L}'$. Let $T \in \mathscr{L}'$. From Lemma 5.1 it follows that there is a unique $F_0 \in \mathscr{D}'(R)$ such that

$$\langle T, f\rangle = \langle F_0, Nf\rangle \qquad \text{for every} \quad f \in \mathscr{D}(\Omega) .$$

If $v \geqq$ the order of F_0 in supp χ, we define an extension F_1 of F_0 by

$$\langle F_1, \varphi\rangle = \langle F_0, \varphi - \gamma_s(\tau)\chi(\tau) \sum_{j=0}^{v} A_j(\varphi)\tau^j\rangle \qquad \text{for every} \quad \varphi \in H_{s,m} .$$

But as supp $(T - N'F_1) \subset \{0\}$ we have $T - N'F_1 = \sum c_j A_j$ (Lemma 5.3) and consequently $T = N'F$ with $F = F_1 + \sum c_j A_j$.

Now the theorem follows from general theorems if we observe that $\mathscr{L}'$ (see Bourbaki [1, Chap. IV, p. 80, Exc. 9a] and Schwartz [8, I, p. 72]), that the spaces $H'_{s,m}$ (inductive limits of Fréchet spaces) are barrelspaces, and that N is continuous and surjective $\mathscr{D}(R^n) \to H_{s,m}$. For the general theorems see Bourbaki [1, Chap. IV, p. 70, 102–104].

REMARK. Lemma 5.1 holds even if we only suppose $T \in \mathscr{L}'_{++}$ which depends on the fact that $\mathscr{L}'_{++}$ as well as $\mathscr{L}'$ acts transitively on Ω (which is not true when $p = 1$ or $q = 1$), which implies that the distribution F is independent of the set in which we introduced new variables. As also Lemmas 5.2 and 5.3 hold if we change $\mathscr{L}'$ to $\mathscr{L}'_{++}$, Theorem 5.1 holds if we change $\mathscr{L}'$ to $\mathscr{L}'_{++}$ and consequently $\mathscr{L}' = \mathscr{L}'_{++}$.

REMARK. We put $\bar{n} = \frac{1}{2}(n-2)$ and $m = [\bar{n}]$. From the proof of Lemma 4.3 we get

(14′) $$N'A_m = (2\pi)^{\bar{n}+1}\delta/\Gamma(\bar{n}+1) \qquad \text{when } pq \text{ is even,}$$

and

(14″) $$N'A_m = (2\pi)^{\bar{n}+1}\delta/\big(\pi\Gamma(\bar{n}+1)\big) \qquad \text{when } pq \text{ is odd.}$$

6. Fundamental solutions of $\square$.

By direct calculation we get

LEMMA 6.1. *$N\square f = DNf$ for every $f \in \mathscr{D}(R^n)$, where*

$$D = 4\big(\tau D_\tau^{\,2} + \tfrac{1}{2}(n-4)D_\tau\big) .$$

We make the following

DEFINITION. *Define* Pf. τ^{-v}, $v \leqq m+1$, *by*

$$\langle \text{Pf.}\,\tau^{-v}, f\rangle = \text{p.v.} \int_{-\infty}^{+\infty} \tau^{-v}\Big(f(\tau) - \sum_{0}^{[v]-2} \big(f^{(j)}(0)\tau^j\big)(j!)^{-1}\, d\tau$$

for every $f \in H_{s,m}$, where p.v. *denotes the principal value of Cauchy.*

Now we get the following theorems from the parametrization of $\mathscr{L}'$.

THEOREM 6.1 *If we put*

$$E = \tfrac{1}{8}(2\pi)^{-\bar{n}-1}\Gamma(\bar{n}+1)(n-2)^{-1}\,\mathrm{Pf.}\,\tau^{-\bar{n}}$$

when at least one of p and q is even, and

$$E = \tfrac{1}{16}(2\pi)^{-\bar{n}}\Gamma(\bar{n}+1)\,B_{\bar{n}}$$

when both p and q are odd, we have $\Box N'E=\delta$.

PROOF. It is easily seen that if $F\in H'_{s,m}$ and $D'F=A_m$ then $\Box(N'F)=k\delta$, where $m=[\bar{n}]$ and $k=k(p,q)$ is given by (14) and D' is the formal adjoint of D. When at least one of p and q is even it is immediately verified that $D'E=A_m/k$. When both p and q are odd the equality $D'E=A_m/k$ follows from the asymptotic development of $D\varphi$ at the origin

$$\tfrac{1}{4}D\varphi \sim \sum \big(B_j j(j-\bar{n})+A_j(2j-\bar{n})\big)\tau^{j-1}+\log|\tau|^{-1}\sum A_j(j-\bar{n})\tau^j$$

if

$$\varphi \sim \sum B_j\tau^j+\log|\tau|^{-1}\sum A_j\tau^j\,.$$

THEOREM 6.2. *Put $G=B_{\bar{n}}$ when at least one of p and q is even, and $G=\mathrm{Pf.}\,\tau^{\bar{n}}$ when both p and q are odd. Every solution $T\in\mathscr{L}'$ of $\Box T=0$ can be written in the form $aN'G+b$.*

PROOF. It is easily seen that $D'F=0$ if and only if $\Box N'F=0$. When at least one of p and q is even, the equality $D'G=0$ follows from the asymptotic development of $D\varphi$ at the origin where $\varphi\sim\sum B_j\tau^j+\gamma_s\sum A_j\tau^j$.

I. *p and q both even.*

$$\tfrac{1}{4}D\varphi \sim \sum B_j j(j-\bar{n})\tau^j + \theta(\tau)\sum A_j(j+1)(j-\bar{n}+1)\tau^j\,.$$

II. *p odd and q even.*

$$\tfrac{1}{4}D\varphi \sim \sum B_j j(j-\bar{n})\tau^{j-1} + \theta(\tau)\tau^{\frac{1}{2}}\sum A_j(j+\tfrac{1}{2})(j-\bar{n}+\tfrac{1}{2})\tau^{j-1}\,.$$

When both p and q are odd it is directly verified that $D'G=0$. As 1 is a solution of $D'F=0$ and 1 and G are linearly independent, the theorem follows.

7. Fouriertransforms in $\mathscr{L}'$.

Everywhere in the above sections we can (with obvious modifications) change $\mathscr{D}$ to $\mathscr{S}$ and $\mathscr{D}'$ to $\mathscr{S}'$. In the following the spaces $H_{s,m}$ and $\mathscr{L}'$ refer to $\mathscr{S}$. The elements of $H_{s,m}$ then have other properties at infinity.

Define the Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ by

$$(\mathscr{F}f)(\mu) = (2\pi)^{-\frac{1}{2}n}\int f(u)e^{-iu\mu}\,du$$

and

$$(\mathscr{F}^{-1}f)(\mu) = (2\pi)^{-\frac{1}{2}n}\int f(u)e^{iu\mu}\,du$$

for every $f \in \mathscr{S}$, where $u\mu$ is defined on p. 203 above. Define $\mathscr{F}T$ where $T \in \mathscr{S}'$ by

$$\langle \mathscr{F}T, f\rangle = \langle T, \mathscr{F}^{-1}f\rangle \qquad \text{for every} \quad f \in \mathscr{S}\,.$$

Put $\mathscr{N}' = N'^{-1}\mathscr{F}N'$, where $\mathscr{N}'$ is a linear homeomorphism of $H'_{s,m}$ onto $H'_{s,m}$. As $H_{s,m}$ is reflexive, it follows that $\mathscr{N}$ is a linear homeomorphism of $H_{s,m}$ onto $H_{s,m}$. Let $H_{s,m} \ni \varphi = Ng$. We have

$$\langle F, \mathscr{N}\varphi\rangle = \langle N'\mathscr{N}'F, g\rangle = \langle \mathscr{F}N'F, g\rangle = \langle F, N\mathscr{F}^{-1}g\rangle$$

for every $F \in H'_{s,m}$. This implies that $\mathscr{N}Ng = N\mathscr{F}^{-1}g$, and we have the following theorem

THEOREM 7.1. *$\mathscr{F}$ induces a linear homeomorphism $\mathscr{N}' = N'^{-1}\mathscr{F}N'$ of $H'_{s,m}$ onto $H'_{s,m}$. Its adjoint $\mathscr{N}$ is a linear homeomorphism of $H_{s,m}$ onto $H_{s,m}$ defined by*

$$\mathscr{N} = \{[Ng, N\mathscr{F}^{-1}g]\}\,.$$

Furthermore $\mathscr{N}^2 = 1$.

The last statement follows from $\mathscr{N}^2Ng = N\mathscr{F}^{-1}\mathscr{F}^{-1}g = N\check{g} = Ng$.

We shall now give an explicit expression of the kernel of $\mathscr{N}$.

Let $\varphi = Ng$. We have

$$\mathscr{N}\varphi(\tau) = N\mathscr{F}^{-1}g(\tau) = \lim_{R\to\infty}\int_{|u|\le R}\delta(uu-\tau)\,du\int e^{iu\mu}g(\mu)\,d\mu$$

and

$$e^{iu(\Lambda\mu)} = \int e^{iu\eta}\delta(\mu\mu-\sigma)\,d\sigma\,,$$

where $\eta(\sigma)$ is a function so that $\eta\eta = \sigma$, and where $\Lambda_\mu = \Lambda \in \mathscr{L}$ and $\Lambda\mu = \eta(\mu\mu)$. Now

$$\begin{aligned}\mathscr{N}\varphi(\tau) &= \lim_{R\to+\infty}\int_{|u|\le R}\delta(uu-\tau)\left(\int e^{iu(\Lambda^{-1}\eta)}\delta(\mu\mu-\sigma)\,d\sigma\right)g(\mu)\,d\mu du\\ &= \lim_{R\to+\infty}\int\left(\int_{|u|\le R}\delta(uu-\tau)e^{iu\eta}\,duNg(\sigma)\right)d\sigma\\ &= \lim_{R\to+\infty}\int \Delta_R(\sigma,\tau)Ng(\sigma)\,d\sigma\,.\end{aligned}$$

Clearly $\Delta_R(\sigma,\tau)$ tends to an element of $H'_{s,m}(\sigma)$, the element $\Delta(\sigma,\tau)$ say, as $R \to \infty$.

(i) If $\sigma = \eta\eta > 0$ we have

$$\Delta_R(\sigma,\tau) = (2\pi)^{-\frac{1}{2}n} \int\limits_{|u|\leq R} \delta(uu-\tau)e^{i\sigma^{\frac{1}{2}}x_1}du$$

$$= 2^{-\bar{q}}\Gamma(\bar{q}+1)^{-1}\int\limits_{|\tau|}^{R^2} (\varrho+\tau)^{\bar{p}}(\varrho-\tau)^{\bar{q}}d\varrho \int\limits_0^{\pi} e^{i\sigma(\varrho+\tau)^{\frac{1}{2}}\cos\vartheta}(\sin\vartheta)^{\bar{p}}\,d\vartheta$$

$$= 2^{-\bar{q}}\Gamma(\bar{q}+1)^{-1}\int\limits_{|\tau|}^{R^2} (\varrho+\tau)^{\bar{p}}(\varrho-\tau)^{\bar{q}}J_{\bar{p}}\big((\sigma(\varrho+\tau))^{\frac{1}{2}}\big)\,d\varrho\,.$$

Here J_v denotes the Bessel function of order v. If we put $t=(\sigma(\varrho+\tau))^{\frac{1}{2}}$ and $a(\tau)=(\max(0,\tau))^{\frac{1}{2}}$ we get

$$(15') \qquad \Delta_R(\sigma,\tau) = 2^{1-\bar{q}}\Gamma(\bar{q}+1)^{-1}\sigma^{-\bar{n}} \int\limits_{a(2\tau\sigma)}^{(\sigma(R^2+\tau))^{\frac{1}{2}}} t^{\bar{p}+1}(t^2-2\tau\sigma)^{\bar{q}}J_{\bar{p}}(t)\,dt\,.$$

(ii) If we have $\sigma=\eta\eta<0$ we get in the same way

$$(15'') \quad \Delta_R(\sigma,\tau) = 2^{1-\bar{p}}\Gamma(\bar{p}+1)^{-1}\sigma^{-\bar{n}} \int\limits_{a(2\sigma\tau)}^{(-\sigma(R^2+\tau))^{\frac{1}{2}}} t^{\bar{q}+1}(t^2-2\tau\sigma)^{\bar{p}}J_{\bar{q}}(t)\,dt\,.$$

Introduce the notations

$$c_{p,q}(\sigma) = 2^{1-\bar{q}}\Gamma(\bar{q}+1)^{-1}\sigma^{-\bar{n}}, \qquad \mathscr{H}_{\alpha,\beta}(k) = \int\limits_{a(k)}^{\infty} t^{\alpha+1}(t^2+k)^{-\beta}J_{\alpha}(t)\,dt\,.$$

In order to give $\mathscr{H}_{\alpha,\beta}(k)$ a meaning when $\alpha=\bar{p}$ and $\beta=\bar{q}$, we prove the following lemma

Lemma 7.1. *Let k be fixed.*

For fixed β, $\mathscr{H}_{\alpha,\beta}(k)$ can be continued to a function of α which is analytic in the whole complex plane.

For fixed α, $\mathscr{H}_{\alpha,\beta}(k)$ can be continued to a function which is analytic in the whole complex plane if $k<0$ and analytic in the whole plane except when $\beta=2,3,\ldots$, where it has simple poles if $k>0$.

Proof. Let $k>0$. When the integral converges we have

$$(16) \quad \mathscr{H}_{\alpha,\beta}(k) = \int\limits_{k}^{\infty} t^{\alpha+1}(t^2-k)^{-\beta}J_{\alpha}(t)dt$$

$$= \Gamma(1-\beta)k^{\frac{1}{2}(\alpha+\beta-1)}\big(e^{\alpha\pi i}J_{-\alpha-\beta-1}(k^{\frac{1}{2}})-i\sin\alpha\pi\, H^1_{-\alpha-\beta-1}(k^{\frac{1}{2}})\big)\,.$$

Here H^1_v is a Hankel cylinder function which is analytic in v. Hence the lemma follows for $k>0$.

Let $k<0$. When the integral converges we have

$$(17)\quad \mathscr{H}_{\alpha,\beta}(k) = \int_0^\infty t^{\alpha+1}(t^2-k)^{-\beta}J_\alpha(t)\,dt$$
$$= i(-k)^{(\alpha-\beta)/2}e^{(\alpha-\beta)\pi i/2}2^{-\beta-1}\Gamma(\beta+1)^{-1}H^1_{\alpha-\beta}\big((-k)^{\frac{1}{2}}\big)\,.$$

Hence the lemma follows.—For the formulas (16) and (17) see Nielsen [5. pp. 222–224].

By analytic continuation and from (15′) and (15″) follows

THEOREM 7.2. *The kernel $\Delta(\sigma,\tau)$ to $\mathscr{N}$ is given by*

$$\mathscr{H}_{\bar p,\,-\bar q}(2\sigma\tau)\ \mathrm{Pf.}\,c_{p,q}(\sigma) \quad \textit{when} \quad \sigma>0$$

and

$$\mathscr{H}_{\bar q,\,-\bar p}(2\sigma\tau)\ \mathrm{Pf.}\,c_{q,p}(\sigma) \quad \textit{when} \quad \sigma<0\,.$$

8. Solutions in $\mathscr{S}'$ of $P(\square)S=T$.

LEMMA 8.1. *If P is a polynomial, the mapping $H_{s,m}\ni\varphi \to P(\tau)\varphi\in H_{s,m}$ has a continuous inverse.*

PROOF. As $P(\tau)=a_0\prod(\tau-\lambda_j)$ it is sufficient to prove the lemma when $P(\tau)=\tau-\lambda$.

If $\operatorname{Im}\lambda\neq 0$ the lemma is trivial.

Let λ be real $\neq 0$. Let $\vartheta_\lambda(\tau)\in\mathscr{D}(R)$ so that $\vartheta_\lambda(\tau)=1$ for $|\tau|\leq|\frac{1}{3}\lambda|$, and $\vartheta_\lambda=0$ for $|\tau|>|\frac{1}{2}\lambda|$. Let $\varphi_j\in(\tau-\lambda)H_{s,m}$ and let $\varphi_j\to 0$ in $H_{s,m}$. Clearly

$$\varphi_j(\tau)\,\vartheta_\lambda(\tau)(\tau-\lambda)^{-1}\to 0 \quad \text{in} \quad H_{s,m}\,,$$

and as in Schwartz [8, p. 123] it is proved that

$$\varphi_j(\tau)\big(1-\vartheta_\lambda(\tau)\big)(\tau-\lambda)^{-1}\to 0 \quad \text{in} \quad \mathscr{D}(R)\,.$$

Now we prove the lemma for $\lambda=0$. Let $\varphi_j\in\tau H_{s,m}$. It is easily seen that $\operatorname{supp}(\varphi_j/\tau)$ is contained in a fixed compact set $K\subset R$ and that

$$A_k(\varphi_j/\tau)\to 0\,.$$

We have

$$\sup_{\tau\in K}|D_\tau^v\psi_j|\to 0, \quad \text{where} \quad \psi_j(\tau) = \varphi_j(\tau)-\gamma_s(\tau)\chi(\tau)P_v(\tau)\,,$$

and

$$(\varphi_j/\tau)-\sum_{k=0}^{v}A_k(\varphi_j/\tau)\tau^k\gamma_s(\tau)\chi(\tau) = \psi_j(\tau)/\tau = \int_0^\infty \psi_j(\tau\sigma)\,d\sigma\,.$$

Now the lemma follows from the inequality

$$\sup_{\tau\in K}\big|D_\tau^{v-1}\big(\psi_j(\tau)/\tau\big)\big| = \sup_{\tau\in K}\left|\int_0^1 \psi_j^{(v)}(\sigma\tau)\,\sigma^{v-1}\,d\sigma\right| \leq v^{-1}\sup_{\tau\in K}|D_\tau^v\psi_j|\,.$$

COROLLARY. $P(\tau)H'_{s,m}=H'_{s,m}$.

THEOREM 8.1. *The equation* $P(\square)S=T$ *with* $T\in\mathscr{L}'$ *has a solution in* $\mathscr{L}'$.

PROOF. We shall prove that $P(\square)\mathscr{L}'=\mathscr{L}'$ which clearly is equivalent to proving that $P(D')H'_{s,m}=H'_{s,m}$. It is well known that $\mathscr{F}\square\mathscr{F}^{-1}$ is multiplication by uu. Hence $\mathscr{N}'^{-1}D'\mathscr{N}'$ is multiplication by τ; for if $H_{s,m}\ni\varphi=Ng$ where $g\in\mathscr{D}(R^n)$, then

$$\begin{aligned}\langle\mathscr{N}'^{-1}D'\mathscr{N}'F,\varphi\rangle &= \langle N'\mathscr{N}'^{-1}D'\mathscr{N}'F,g\rangle\\ &= \langle\mathscr{F}N'D'\mathscr{N}'F,g\rangle\\ &= \langle F,\mathscr{N}N\square\mathscr{F}^{-1}g\rangle\\ &= \langle F,N\mathscr{F}\square\mathscr{F}^{-1}g\rangle = \langle\tau F,\varphi\rangle\,.\end{aligned}$$

Clearly $P(D')H'_{s,m}=H'_{s,m}$, if and only if $\mathscr{N}'^{-1}P(D')\mathscr{N}'H'_{s,m}=H'_{s,m}$, and as

$$\mathscr{N}'^{-1}P(D')\mathscr{N}' = P(\mathscr{N}'^{-1}D'\mathscr{N}') = P(\tau)$$

this is true by the corollary of Lemma 8.1.

REMARK. This result is also true relative to $\mathscr{D}$.

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