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ON GENERALIZED POTENTIALS OF FUNCTIONS IN THE LEBESGUE CLASSES

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1. Introduction.

The generalized potentials considered in the present paper are those of order α, $0<\alpha<n$, in R^n, with the logarithmic potential as a kind of limit case corresponding to the order $\alpha=n$. These potentials refer to the kernels

$$|x-y|^{\alpha-n}, \qquad 0 < \alpha < n ,$$

and

$$-\log|x-y| ,$$

respectively. Here x and y range over the n-dimensional Euclidean space R^n. The first systematic investigations of these potentials are those of M. Riesz [13] and O. Frostman [7]. Various applications of L^p-norms to potentials of order α were made by Frostman [8].

Our principal aim is to establish the following result referred to in an earlier paper [10, p. 198]:

Let $p\geqq 1$, $0<\alpha p<n$, and $E\subset R^n$. *In order that there exist a function* $f\geqq 0$, $f\in \mathscr{L}^p(R^n)$, *whose potential of order* α *is infinite at every point of* E (without being identically infinite) *it is*[1]

$$1°)\ \textit{necessary that}\begin{cases} \operatorname{cap}^*_{\alpha p}E = 0, & \text{provided } p \leqq 2\ . \\ \operatorname{cap}^*_{\alpha p-\varepsilon}E = 0 & \text{for every } \varepsilon > 0, \text{ provided } p > 2\ . \end{cases}$$

$$2°)\ \textit{sufficient that}\begin{cases} \operatorname{cap}^*_{\alpha p}E = 0, & \text{provided } p \geqq 2\ . \\ \operatorname{cap}^*_{\alpha p-\varepsilon}E = 0 & \text{for some } \varepsilon > 0, \text{ provided } p < 2\ . \end{cases}$$

The same conclusions subsist in the case $\alpha p=n$ *provided potentials and capacities of order* n *are taken with reference to the logarithmic kernel.*

Part 1° was established in the case $\alpha p<n=1$ by N. du Plessis [13], who also showed that the ε cannot be dispensed with when $p>2$. Using the same method, J. Deny [5] treated Part 1° for general n in the case $p\leqq 2$, $\alpha p<n$. (In view of the intended applications, only the value

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[1] Here $\operatorname{cap}^*_\beta E$ denotes the exterior capacity "of order β", that is, associated with the above kernel of order β, $0<\beta<n$. For $\beta>n$, $\operatorname{cap}^*_\beta E=0$ is meant to imply that E is empty.

$\alpha = 1$ was considered specifically.) Part 2° is related to a result due to H. Cartan [2, théorème 3bis, p. 96]. (Cf. Theorem 3.2 of the present paper.) A complete discussion of the case $\alpha p < n$ is given in §§ 3, 4 of the present paper, based on the methods of du Plessis (Part 1°) and Cartan (Part 2°). Cartan's method depends on the existence of an interior capacitary distribution to be associated with any given, say open, set of finite capacity. Such a measure λ was constructed by Cartan [2] in the Newtonian case $\alpha = 2$. Extensions to arbitrary order $\alpha < n$ were made by Aronszajn and Smith [1] and later by the present author [11]. A brief survey over the relevant material concerning the potentials of order $\alpha < n$ is given in § 2 of the present paper.

The logarithmic case $\alpha p = n$ is settled in § 6[2] and § 7 with reference to the author's study [12] of the logarithmic kernel in higher dimensions.

2. Potentials of order α.

The potentials of a given *order* α, $0 < \alpha < n$, in Euclidean n-dimensional space R^n refer to the following convolution kernel

$$|x-y|^{\alpha-n}, \qquad x \in R^n, \quad y \in R^n ,$$

the value of which is interpreted as $+\infty$ for $x = y$. The corresponding *kernel function* $|x|^{\alpha-n}$ is lower semi-continuous and locally Lebesgue integrable in R^n. A fundamental property of these kernel functions is expressed in the formula of composition due to M. Riesz [14]:

$$|x|^{\alpha-n} * |x|^{\beta-n} = c_{\alpha,\beta} |x|^{\alpha+\beta-n} , \tag{1}$$

valid for $\alpha > 0$, $\beta > 0$, $\alpha + \beta < n$. The constant $c_{\alpha,\beta}$ is given by

$$c_{\alpha,\beta} = \pi^{n/2} \frac{\Gamma(\alpha/2)\, \Gamma(\beta/2)\, \Gamma(n/2-\alpha/2-\beta/2)}{\Gamma(n/2-\alpha/2)\, \Gamma(n/2-\beta/2)\, \Gamma(\alpha/2+\beta/2)} .$$

The *potential* of order α of a measure μ on R^n is defined as the convolution $U^\mu_\alpha = |x|^{\alpha-n} * \mu$, that is,

$$U^\mu_\alpha(x) = \int |x-y|^{\alpha-n}\, d\mu(y) , \tag{2}$$

provided the integral on the right exists, that is, $U^{\mu^+}_\alpha$ and $U^{\mu^-}_\alpha$ are not both infinite. If $\mu \geqq 0$, U^μ_α is everywhere defined ($\leqq +\infty$) and lower semi-continuous. If μ has a density f (that is, $d\mu = f dx$, where f is locally Lebesgue integrable), we may write U^f_α instead of U^μ_α.

In order to exclude the possibility that the distant masses force the

[2] In the case $p = 1$, $\alpha = n$, we have, for brevity, confined the attention to bounded parts of R^n. The extension to the global case is easy.

potential U_α^μ to become infinite (or undefined) at some point (and hence actually everywhere), we shall restrict the attention mostly to measures μ which are *finite of order* α in the sense that

$$\int\limits_{|x|\geqq 1} |x|^{\alpha-n}\,|d\mu(x)| < +\infty\,. \tag{3}$$

Simple estimates show that this condition is equivalent to the following more general condition[3]

$$\int\limits_{\complement B_\varrho(x)} |x-y|^{\alpha-n}\,|d\mu(y)| < +\infty \tag{3'}$$

for any given point $x\in R^n$ and radius $\varrho>0$. Here $B_\varrho(x)$ denotes the (closed) ball of radius ϱ about x, and hence $\complement B_\varrho(x)=\{y\in R^n\colon |x-y|>\varrho\}$. For $x=0$ we may write simply B_ϱ instead of $B_\varrho(0)$.—The class of all measures which are finite of order α will be denoted by $\mathscr{M}_\alpha$. It is easily proved that, for any $\mu\in\mathscr{M}_\alpha$, $U_\alpha^\mu(x)$ is an analytic function of x in the complement of the support $S(\mu)$ of μ. This fact permits us to *extend to the case* $\mu\in\mathscr{M}_\alpha$ *any local regularity property of the potential* U_α^μ *which is known in the case where* μ *has compact support.* In particular, U_α^μ is defined almost everywhere and represents a locally Lebesgue integrable function, provided $\mu\in\mathscr{M}_\alpha$.

From the formula of composition (1) follows that

$$U_\alpha^f = c_{\alpha,\beta}\,U_{\alpha+\beta}^\mu \qquad \text{almost everywhere}\,, \tag{4}$$

when $f=U_\beta^\mu$ and $\mu\in\mathscr{M}_{\alpha+\beta}$. (Cf. M. Riesz [14, no. 4]. The assumption $\alpha+\beta<n$ remains in force.) In fact, the relation

$$|x|^{\alpha-n}*(|x|^{\beta-n}*\mu) = (|x|^{\alpha-n}*|x|^{\beta-n})*\mu$$

holds whenever the right hand side exists as a Lebesgue–Stieltjes integral, thus in particular almost everywhere in R^n. Note that (4) holds *everywhere* in R^n if $\mu\geqq 0$.

The mutual energy $\langle\mu,\nu\rangle_\alpha$ of order α of two measures μ,ν is defined by

$$\langle\mu,\nu\rangle_\alpha = \iint |x-y|^{\alpha-n}\,d\mu(x)\,d\nu(y)\,,$$

provided the double integral on the right exists (that is, $\langle\mu^+,\nu^+\rangle_\alpha+\langle\mu^-,\nu^-\rangle_\alpha$ and $\langle\mu^+,\nu^-\rangle_\alpha+\langle\mu^-,\nu^+\rangle_\alpha$ are not both infinite). Under the same assumption we have the *formula of reciprocity*

[3] The condition (3′) is evidently independent of ϱ. The equivalence of (3) and (3′) follows, therefore, from the fact that the ratio $|x-y|^{\alpha-n}/|y|^{\alpha-n}$ is bounded away from 0 and ∞ as $|y|\to+\infty$, x being fixed.

$$(5)\qquad \int U_\alpha^\mu d\nu = \int U_\alpha^\nu d\mu \quad (= \langle\mu,\nu\rangle_\alpha)\,.$$

In view of the formula of composition (1), the kernels of order α are *consistent* (cf. [11, § 3.3 and Theorem 7.4]). Since they are strictly definite (cf. M. Riesz [14, nos. 4 and 10]), we conclude that they are *perfect.* The *energy* of order α of a measure μ will be denoted often by $\|\mu\|_\alpha^2$ instead of $\langle\mu,\mu\rangle_\alpha$. The class $\mathscr{E}_\alpha$ of all measures μ for which the energy $\langle\mu,\mu\rangle_\alpha$ is defined and *finite* is a pre-Hilbert space with the scalar product $\langle\mu,\nu\rangle_\alpha$ and the norm

$$\|\mu\|_\alpha = \langle\mu,\mu\rangle_\alpha^{1/2}\,.$$

The identity

$$(6)\qquad \|\mu\|_\alpha^2 = \frac{1}{w_\alpha}\int [U_{\alpha/2}^\mu]^2 dx, \qquad \mu\in\mathscr{E}_\alpha\,,$$

where $w_\alpha = c_{\alpha/2,\,\alpha/2}$, is a consequence of (1). Note also that $\mathscr{E}_\alpha \subset \mathscr{M}_\alpha$.

The interior Wiener *capacity* of order α of an arbitrary set $E\subset R^n$ is defined as the supremum of $\mu(R^n)$ when μ ranges over all positive measures concentrated on E of potential

$$(7)\qquad U_\alpha^\mu(x) \leqq 1 \qquad \text{for every } x\in S(\mu)\,.$$

Observe that the kernels of order $\alpha\leqq 2$ fulfill Frostman's *maximum principle*, and hence (7) implies: $U_\alpha^\mu(x)\leqq 1$ everywhere (under the assumption $\mu\geqq 0$). As a substitute we have, for any order $\alpha<n$, the following consequence of (7), again provided $\mu\geqq 0$,[4]

$$(8)\qquad U_\alpha^\mu(x) \leqq 2^{n-\alpha} \quad \text{everywhere}\,.$$

For the interior capacity of order α of a set E we use the notation $\mathrm{cap}_{*\alpha}E$. It can be shown that

$$(9)\qquad \mathrm{cap}_{*\alpha}E = \sup \mathrm{cap}_{*\alpha}K$$

as K ranges over the class of all compact subsets of E.

The *exterior capacity* of order α of a set E is defined by

$$(10)\qquad \mathrm{cap}_\alpha^* E = \inf \mathrm{cap}_{*\alpha}G$$

as G ranges over the class of all open sets containing E. The set function

[4] In fact, if ϱ denotes the shortest distance between $S(\mu)$ and a given point $x\in R^n$, there is a point $x'\in S(\mu)$ with $|x-x'|=\varrho$. For any other point $y\in S(\mu)$, the inequality

$$2|x-y| \geqq |x-y|+|x-x'| \geqq |x'-y|$$

shows that

$$U_\alpha^\mu(x) = \int_{S(\mu)} |x-y|^{\alpha-n}\,d\mu(y) \leqq 2^{n-\alpha}U_\alpha^\mu(x') \leqq 2^{n-\alpha}\,.$$

cap_α^* is countably subadditive. A set E is called *capacitable* of order α if $\mathrm{cap}_{*\alpha}E = \mathrm{cap}_\alpha^* E$. For such a set we shall often write $\mathrm{cap}_\alpha E$ to signify the capacity of E $(=\mathrm{cap}_{*\alpha}E = \mathrm{cap}_\alpha^* E)$. Since the kernels of order α are perfect, it follows by application of Choquet's theory that *all analytic sets, in particular all Borel sets in R^n are capacitable* (cf. [11, Theorem 4.4] and Choquet [3, § 30.2]).

A property P is said to subsist *nearly everywhere* of order α if P fails to subsist at most in some set N with $\mathrm{cap}_{*\alpha}N = 0$. Similarly, P is said to subsist *quasi-everywhere* of order α if $\mathrm{cap}_\alpha^* N = 0$.

THEOREM 2.1. *With any set $E \subset R^n$ of finite interior capacity of order α can be associated a unique positive measure λ such that*

$$\lambda(R^n) = \|\lambda\|_\alpha{}^2 = \mathrm{cap}_{*\alpha}E \tag{11}$$

and

$$U_\alpha^\lambda(x) \geqq 1 \quad \textit{nearly everywhere of order } \alpha \textit{ in } E, \tag{12}$$

$$U_\alpha^\lambda(x) \leqq 1 \quad \textit{everywhere in the support of } \lambda\,. \tag{13}$$

(Cf. [11], Theorem 4.1.) This measure λ is called the *interior capacitary distribution* of order α associated with E. As shown by Frostman [8, p. 37] it follows from (12) that

$$U_\alpha^\lambda(x) \geqq 1 \quad \text{everywhere in the interior of } E\,. \tag{14}$$

As pointed out earlier, (13) implies

$$U_\alpha^\lambda(x) \leqq 2^{n-\alpha} \quad \text{everywhere in } R^n\,. \tag{15}$$

There is an analogous theorem concerning the *exterior capacitary distribution* of order α associated with an arbitrary set E such that $\mathrm{cap}_\alpha^* E < +\infty$ (cf. [11], Theorem 4.3).

3. On the set of points of infinite potential.

THEOREM 3.1. *If $\mu \in \mathscr{M}_\alpha$, $\mu \geqq 0$, and $0<\alpha<n$, then*

$$\mathrm{cap}_\alpha^*\{x \in R^n\colon U_\alpha^\mu(x) = +\infty\} = 0\,.$$

This result is due to H. Cartan [2, théorème 3, p. 95]. Previously it was known that the corresponding interior capacity is zero (cf. G. C. Evans [7, Theorem I], Frostman [8, p. 81], or De la Vallée-Poussin [17, p. 21]). Actually, the set $\{x \in R^n\colon U_\alpha^\mu(x) = +\infty\}$ is of class G_δ and hence capacitable, as mentioned above. It suffices, in the proof of Theorem 3.1, to consider the case where $S(\mu)$ is compact. For any number t, the open set

$$G_t = \{x \in R^n\colon U_\alpha^\mu(x) > t\}$$

contains the set in which U^{μ}_{α} is infinite, and hence it suffices to prove that $\text{cap}_{*\alpha} G_t \to 0$ as $t \to +\infty$. Denoting by λ the capacitary distribution on an arbitrary compact subset $K \subset G_t$, we obtain from (5) and (15), § 2,

$$t \, \text{cap}_{\alpha} K \leqq \int U^{\mu}_{\alpha} d\lambda = \int U^{\lambda}_{\alpha} d\mu \leqq 2^{n-\alpha} \mu(R^n) .$$

This implies on account of (9), § 2,

$$\text{cap}_{\alpha} G_t \leqq t^{-1} 2^{n-\alpha} \mu(R^n) .$$

It follows from Theorem 3.1 that U^{μ}_{α} is defined and finite quasi-everywhere of order α provided $\mu \in \mathcal{M}_{\alpha}$.

In order that a set $E \subset R^n$ be of the form

$$E = \{x \in R^n : U^{\mu}_{\alpha}(x) = +\infty\}$$

for some positive $\mu \in \mathcal{M}_{\alpha}$, it is necessary that E be of class G_{δ} and that $\text{cap}^*_{\alpha} E = 0$ (Theorem 3.1). The question whether this condition is sufficient was answered in the affirmative by J. Deny [6] in the Newtonian case $\alpha = 2$. (The first result in this direction is due to Evans [7, Theorem II], who treated the case where E is closed and $\alpha = 2$.) We shall, however, restrict the attention to the simpler problem of characterizing the sets *contained in* some set of the above type. The answer to this question was given by H. Cartan [2, théorème 3bis, p. 96], likewise for $\alpha = 2$. The same method applies, however, in the general case $0 < \alpha < n$, the crucial point being the existence of an interior capacitary distribution on any open set of finite capacity.

THEOREM 3.2. *Let* $0 < \alpha < n$, $E \subset R^n$, *and suppose* $\text{cap}^*_{\alpha} E = 0$. *Then there exists a positive measure* $\mu \in \mathcal{M}_{\alpha}$ *such that* $U^{\mu}_{\alpha}(x) = +\infty$ *for every* $x \in E$; *that is,*

$$E \subset \{x \in R^n : U^{\mu}_{\alpha}(x) = +\infty\} .$$

PROOF. Choose open sets $G_k \supset E$ so that $\text{cap}_{\alpha} G_k < 2^{-k}$, $k = 1, 2, \ldots$; and denote by λ_k the interior capacitary distribution associated with G_k (Theorem 2.1). Since

$$\sum_k \lambda_k(R^n) = \sum_k \text{cap}_{\alpha} G_k < \infty ,$$

the sum $\sum_k \lambda_k$ defines a bounded measure μ. It follows from (14) that $U^{\mu}_{\alpha}(x) = +\infty$ for $x \in E$. Moreover, $\mu \in \mathcal{E}_{\alpha} \subset \mathcal{M}_{\alpha}$ because

$$\sum_k \|\lambda_k\|_{\alpha} = \sum_k (\text{cap}_{\alpha} G_k)^{1/2} < \infty .$$

LEMMA 3.1. $\text{cap}^*_{\alpha} E = 0$ *implies* $\text{cap}^*_{\beta} E = 0$ *provided* $0 < \beta < \alpha < n$.

Although a more direct proof could easily be given, we shall derive this well-known result from the two preceding theorems: If $\mu \geqq 0$, $\mu \in \mathscr{M}_\alpha$, and $U_\alpha^\mu = +\infty$ everywhere in E, then $\mu \in \mathscr{M}_\beta$, and a simple estimate shows that $U_\beta^\mu(x) = +\infty$ for every $x \in E$, the masses outside $B_1(x)$ being of no influence upon the infiniteness of $U_\alpha^\mu(x)$ (or of $U_\beta^\mu(x)$).—The following simple lemma will be used in § 4:

Lemma 3.2. *Let $0 \leqq \theta \leqq 1$, $0 < \beta < n$, $0 < \gamma < n$, and write $\alpha = \theta\beta + (1-\theta)\gamma$. Then*

$$U_\alpha^\mu(x) \leqq [U_\beta^\mu(x)]^\theta [U_\gamma^\mu(x)]^{1-\theta}$$

for every $x \in R^n$ and every positive measure μ.

We interpret $(+\infty)^\theta$ as $+\infty$ or 1 according as $\theta > 0$ or $\theta = 0$. The proof of the lemma consists in a straightforward application of Hölder's inequality

$$\int f^\theta g^{1-\theta} d\mu \leqq \left(\int f d\mu\right)^\theta \left(\int g d\mu\right)^{1-\theta}$$

to the functions $f(y) = |x-y|^{\beta-n}$ and $g(y) = |x-y|^{\gamma-n}$.

4. The Lebesgue classes.

For any number $p \geqq 1$ we denote by $\mathscr{L}^p$ the class of all Lebesgue measurable functions f on R^n such that $\int |f(x)|^p dx < \infty$. Moreover, we write $\mathscr{L}_{\mathrm{loc}}^p$ for the wider class of all measurable functions f on R^n such that $\int_K |f(x)|^p dx < \infty$ for every compact set $K \subset R^n$. We use the notations

$$\|f\|_{L^p} = \left[\int |f(x)|^p dx\right]^{1/p}, \qquad \|f\|_{L^p(E)} = \left[\int_E |f(x)|^p dx\right]^{1/p}.$$

We denote the conjugate exponent of p by $p' = p/(p-1)$.

Lemma 4.1. *Let $p > 1$, $f \in \mathscr{L}^p$, and $\alpha p < n$. Then $f \in \mathscr{M}_\alpha$, and*

$$U_\alpha^f = |x|^{\alpha-n} * f \in \mathscr{L}^{np/(n-\alpha p)}.$$

This result is known as *Sobolev's lemma.* As to the proof, see Sobolev [16, théorème III]. In the sequel we shall merely use the fact that any function of class $\mathscr{L}^p$, $p \geqq 1$, is finite of order α provided $\alpha p < n$, and this is easily verified directly by application of Hölder's inequality.

There is a local version of Sobolev's lemma asserting that

$$U_\alpha^f \in \mathscr{L}_{\mathrm{loc}}^{np/(n-\alpha p)} \quad \text{provided} \quad f \in \mathscr{L}_{\mathrm{loc}}^p \cap \mathscr{M}_\alpha, \quad p > 1, \ \alpha p < n.$$

The following more elementary lemma which can be extracted from Frostman [9, no. 2], gives a less precise extension of this local version to the cases $p = 1$ or $\alpha p \geqq n$.

LEMMA 4.2.

(a) *If* $\mu \in \mathcal{M}_\alpha$, *then* $U_\alpha^\mu \in \mathcal{L}^q_{\mathrm{loc}}$ *for any* $q < n/(n-\alpha)$.

(b) *If* $f \in \mathcal{L}^p \cap \mathcal{M}_\alpha$ *and* $\alpha p = n$, *then* $U_\alpha^f \in \mathcal{L}^q_{\mathrm{loc}}$ *for any* $q < +\infty$.

(c) *If* $f \in \mathcal{L}^p \cap \mathcal{M}_\alpha$ *and* $\alpha p > n$, *then* U_α^f *is finite and continuous everywhere.*

PROOF. As usual, it suffices to consider the case where μ or f has compact support, in which case the result follows from the fact that the kernel function $|x|^{\alpha-n}$ is of class $\mathcal{L}^s_{\mathrm{loc}}$ for any $s < n/(n-\alpha)$. Writing $\psi = \min(1, |x|^{\alpha-n})$ and $\varphi = |x|^{\alpha-n} - \psi$, we have $|x|^{\alpha-n} = \varphi + \psi$, where $\varphi \in \mathcal{L}^s$ and ψ is bounded and continuous. (In case (b), one applies L. Schwartz [15, ch. VI, § 1, formula 2] to φ and f. In case (c) one applies Hölder's inequality to φ and f. The continuity of $\psi * f$ follows easily by application of Lebesgue's theorem on dominated convergence.)

Theorems 4.1 and 4.2 below may be regarded as refinements of Theorems 3.1 and 3.2, respectively, in the particular case where μ has a density f of class $\mathcal{L}^p$, $p > 1$. References concerning these two theorems were given in the introduction (Part 1° and Part 2°, respectively). Throughout the present section we suppose $\alpha p < n$. (The case $\alpha p = n$ leads to the logarithmic capacity and will be treated in § 7. The case $\alpha p > n$ is of no interest in view of Lemma 4.2 (c).)—A glance at Theorems 4.1 and 4.2 reveals that quite precise results are obtained only for $p \leqq 2$ (in Theorem 4.1) and for $p \geqq 2$ (in Theorem 4.2). As we shall see, there is, in a certain sense, a duality between the two cases $p \leqq 2$ of Theorem 4.1 and $p \geqq 2$ of Theorem 4.2. If we define

$$(1) \qquad p' = p/(p-1), \qquad \alpha' = \alpha(p-1),$$

the transformation $(p, \alpha) \to (p', \alpha')$ is involutory and preserves the crucial product αp, whereas the inequality $p \leqq 2$ is transformed into $p' \geqq 2$, and conversely.

THEOREM 4.1. *Let* $f \geqq 0$, $f \in \mathcal{L}^p$, $0 < \alpha p < n$, *and write*

$$E = \{x \in R^n : U_\alpha^f(x) = +\infty\}.$$

Then

(i) $\mathrm{cap}^*_{\alpha p} E = 0$ *provided* $1 \leqq p \leqq 2$,

(ii) $\mathrm{cap}^*_{\alpha p - \varepsilon} E = 0$ *for every* $\varepsilon > 0$, $\varepsilon < \alpha p$, *provided* $p > 2$.

We consider the two cases (i) and (ii) separately and apply the method described in du Plessis [13]. The case $p = 1$ is covered by Theorem 3.1.

(i) *The case* $p \leqq 2$. We shall assume $p > 1$, and we begin with a lemma.

LEMMA 4.3. *Let* $1 < p \leqq 2$, $0 < \alpha p < n$, $p' = p/(p-1)$. *For any positive measure* μ *on* R^n,

$$\|U_\alpha^\mu\|_{L^{p'}} \leqq w_{\alpha p}{}^{1/p'} \left[\sup_{x \in R^n} U_{\alpha p}^\mu(x)\right]^{1/p} \mu(R^n)^{1/p'}.$$

PROOF. Since $0<\alpha p/2 \leqq \alpha < \alpha p < n$, we may apply Lemma 3.2 with

$$\beta = \alpha p/2, \quad \gamma = \alpha p, \quad \theta = 2/p' = 2-2/p .$$

Raising the resulting equations to the power p' and integrating over R^n, we obtain

$$\begin{aligned}\int (U_\alpha^\mu)^{p'} dx &\leqq \int (U_{\alpha p/2}^\mu)^2 (U_{\alpha p}^\mu)^{p'-2}\, dx \\ &\leqq \left[\sup_{x\in R^n} U_{\alpha p}^\mu(x)\right]^{p'-2} \int (U_{\alpha p/2}^\mu)^2\, dx .\end{aligned}$$

Using (5) and (6), § 2, we obtain, moreover,

$$\int (U_{\alpha p/2}^\mu)^2\, dx = w_{\alpha p} \int U_{\alpha p}^\mu\, d\mu \leqq w_{\alpha p} \left[\sup_{x\in R^n} U_{\alpha p}^\mu(x)\right] \mu(R^n) ,$$

and the stated inequality follows.

In order to prove Theorem 4.1 in case (i), we introduce the open sets

$$G_t = \{x \in R^n : U_\alpha^f(x) > t\}, \qquad t > 0 .$$

Since $G_t \supset E$, it suffices to prove that $\operatorname{cap}_{\alpha p} G_t \to 0$ as $t \to +\infty$. Denoting by λ the capacitary distribution of order αp on an arbitrary compact set $K \subset G_t$, we obtain from (5), § 2,

$$t\int d\lambda \leqq \int U_\alpha^f d\lambda = \int U_\alpha^\lambda f\, dx \leqq \|U_\alpha^\lambda\|_{L^{p'}} \|f\|_{L^p} . \tag{2}$$

Applying the above lemma to $\mu = \lambda$, and inserting $\lambda(R^n) = \operatorname{cap}_{\alpha p} K$ and $\sup_x U_{\alpha p}^\lambda(x) \leqq 2^{n-\alpha p} < 2^n$ (cf. (11) and (15), § 2), we obtain

$$t \operatorname{cap}_{\alpha p} K \leqq w_{\alpha p}{}^{1/p'}\, 2^{n/p}\, (\operatorname{cap}_{\alpha p} K)^{1/p'} \|f\|_{L^p} ,$$

from which it follows that

$$\operatorname{cap}_{\alpha p} G_t \leqq 2^n w_{\alpha p}{}^{p-1} \int f^p dx \cdot t^{-p} \qquad (\to 0 \text{ as } t \to +\infty) .$$

(ii) *The case* $p>2$. Since α is no longer located between $\alpha p/2$ and αp, Lemma 4.3 is inapplicable and must be replaced by the following less satisfactory result:

LEMMA 4.4. *Let* $p>q>2$, $0<\alpha q<n$, $p'=p/(p-1)$. *For any number* $R>0$ *there is a constant* $C(R)$, *depending further on* n, p, q, *and* α, *such that the inequality*

$$\|U_\alpha^\mu\|_{L^{p'}(B)} \leqq C(R) \left[\sup_{x\in B} U_{\alpha q}^\mu(x)\right]^{1/p} \mu(B)^{1/p'}$$

holds for any ball B *of radius* R *and any positive measure* μ *concentrated on* B.

PROOF. Applying Lemma 3.2 with $\beta=\alpha(p-q)/(p-1)$, $\gamma=\alpha q$, $\theta=1/p'=1-1/p$, we obtain after raising to the power $p'=1/\theta$ and integrating over B

$$\int_B [U_\alpha^\mu(x)]^{p'}\,dx \leqq \int_B U_\beta^\mu(x)\,[U_{\alpha q}^\mu(x)]^{p'/p}\,dx$$

$$\leqq \left[\sup_{x\in B} U_{\alpha q}^\mu(x)\right]^{p'/p} \int_B U_\beta^\mu(x)\,dx\,.$$

Denote by $C_\beta(R)$ the supremum of the bounded potential U_β^φ of the characteristic function $\varphi=\varphi_B$ associated with the ball B. It follows from (5), § 2, that

$$\int_B U_\beta^\mu\,dx = \int U_\beta^\mu \varphi\,dx = \int U_\beta^\varphi\,d\mu \leqq C_\beta(R)\int d\mu\,.$$

This implies the assertion of the lemma with $C(R)=C_\beta(R)^{1/p'}$.

In the proof of Theorem 4.1 in the remaining case $p>2$ we may suppose that $\varepsilon<\alpha(p-2)$. We define $q=p-\varepsilon/\alpha$ and observe that $2<q<p$ and $\alpha q=\alpha p-\varepsilon$. It suffices to prove that $\operatorname{cap}_{\alpha q}^*(E\cap A)=0$ for every closed ball A. Let B denote a ball of radius R containing A in its interior, and put $\varphi=\varphi_B$. Since the potential U_α^f differs from $U_\alpha^{\varphi f}$ by a function $U_\alpha^{f-\varphi f}$ which is finite in A, we have $U_\alpha^{\varphi f}=+\infty$ in $E\cap A$. Thus we might as well suppose from the beginning that f vanishes outside B. Writing

$$G_t = \{x\in A : U_\alpha^f(x)>t\}\,,$$

we proceed as in the case $p\leqq 2$ and conclude, after application of Lemma 4.4, that

$$\operatorname{cap}_{\alpha q}^*(E\cap A) \leqq \operatorname{cap}_{\alpha q} G_t \leqq 2^n C(R)^p \int f^p\,dx\cdot t^{-p}\,,$$

from which it follows that $\operatorname{cap}_{\alpha q}^*(E\cap A)=0$. This completes the proof of Theorem 4.1.

REMARK. An example in N. du Plessis [13] shows that one cannot conclude $\operatorname{cap}_{\alpha p}^* E=0$ in case (ii).

THEOREM 4.2. *Let $p>1$, $0<\alpha p<n$, $E\subset R^n$, and suppose $\operatorname{cap}_{\alpha p}^* E=0$. Then there exists a function $f\geqq 0$ such that $f\in\mathscr{L}^q$ for all $q<p$, and*

$$U_\alpha^f(x) = +\infty \qquad \textit{for every } x\in E\,.$$

If $p\geqq 2$, f may be so chosen that, in addition, $f\in\mathscr{L}^p$.

(i) *The case $p\geqq 2$.* The following lemma is simply the dual of Lemma 4.3 under the involution (1).

LEMMA 4.5. *Let $p \geqq 2$, $\alpha p < n$, $p' = p/(p-1)$. For any positive measure μ on R^n,*

$$\|U^{\mu}_{\alpha(p-1)}\|_{L^p} \leqq w_{\alpha p}{}^{1/p}\left[\sup_{x \in R^n} U^{\mu}_{\alpha p}(x)\right]^{1/p'} \mu(R^n)^{1/p} .$$

In order to prove Theorem 4.2 in the case $p \geqq 2$, we proceed as in Cartan's proof of Theorem 3.2. We choose open sets $G_k \supset E$ so that $\operatorname{cap}_{\alpha p} G_k < 2^{-k}$, and denote by λ_k the interior capacitary distribution of order αp associated with G_k (Theorem 2.1). Then $\lambda_k(R^n) = \operatorname{cap}_{\alpha p} G_k$, and $U^{\lambda_k}_{\alpha p} \leqq 2^{n-\alpha p} < 2^n$. Defining

$$f_k = U^{\lambda_k}_{\alpha(p-1)}, \qquad f = \sum_{k=1}^{\infty} f_k ,$$

we obtain from (4) and (14), § 2,

$$U^{f_k}_{\alpha} = c_{\alpha,\,\alpha p-\alpha}\, U^{\lambda_k}_{\alpha p} \geqq c_{\alpha,\,\alpha p-\alpha} \qquad \text{everywhere in } G_k \supset E ,$$

and hence $U^{f}_{\alpha} = +\infty$ everywhere in E. Next, we infer from Lemma 4.5 (with μ replaced by λ_k) that $\sum_k \|f_k\|_{L^p} < \infty$, and hence $f \in \mathscr{L}^p$. In fact,

$$\|f_k\|_{L^p} \leqq w_{\alpha p}{}^{1/p}\, 2^{n/p'}\, 2^{-k/p} .$$

The function f has the required properties except possibly for the property $f \in \mathscr{L}^q$ for all $q < p$. If we replace $f(x)$ by $f(x)/(1+|x|)^N$ for a suitably large N, we obtain a new function possessing all the desired properties.

(ii) *The case* $1 < p < 2$. Applying Lemma 4.4 with p, q, and α replaced by $q' = q/(q-1)$, $p' = p/(p-1)$, and $\alpha' = \alpha(p-1)$, respectively, we obtain the following lemma:

LEMMA 4.6. *Let $1 < q < p < 2$, $0 < \alpha p < n$, $q' = q/(q-1)$. For any number $R > 0$ there is a constant $C'(R)$, depending further on n, p, q, and α, such that the inequality*

$$\|U^{\mu}_{\alpha(p-1)}\|_{L^q(B)} \leqq C'(R)\left[\sup_{x \in B} U^{\mu}_{\alpha p}(x)\right]^{1/q'} \mu(B)^{1/q}$$

holds for any ball B of radius R and any positive measure μ concentrated on B.

In the proof of Theorem 4.2 in the case $1 < p < 2$ we may suppose without loss of generality that E is contained in some closed ball A.[5]

[5] In the general case we cover R^n by a sequence of congruent closed balls A_j, $j = 1, 2, \ldots$, and denote by B_j a larger ball concentric with A_j. Choose functions $f_j \geqq 0$ so that $f_j \in \mathscr{L}^q$ for all $q < p$ and $U^{f_j}_{\alpha} = +\infty$ everywhere in $E \cap A_j$. As above, we may suppose that f_j vanishes outside B_j. Replacing, if necessary, f_j by a suitably small multiple thereof, we may assume, moreover, that $\|f_j\|_{L^{p-1/j}} < 2^{-j}$. We proceed to verify that $f = \sum_j f_j$ has the properties stated in Theorem 4.2 for the case $p < 2$. It is clear that

Let $B \supset A$ denote a larger ball concentric with A. Choose open subsets $G_k \supset E$ of B so that $\mathrm{cap}_\alpha G_k < 2^{-k}$, and denote by λ_k the interior capacitary distribution of order αp associated with G_k. As in the case $p \geqq 2$, write $f_k = U^{\lambda_k}_{\alpha(p-1)}$, and define $f = \sum_k f_k$. Then $U^f_\alpha = +\infty$ everywhere in E, and it follows from Lemma 4.6 that $\sum_k \|f_k\|_{L^q(B)} < \infty$, and hence $f \in \mathscr{L}^q(B)$. Replacing f by $\varphi_B \cdot f$, we obtain a function possessing all the stated properties.

Remark. The question remains open whether there is, in the case $1 \leqq p < 2$, a function $f \geqq 0$, $f \in \mathscr{L}^p$, such that $U^f_\alpha = +\infty$ everywhere in E (when it is given that $\mathrm{cap}^*_{\alpha p} E = 0$).

5. The logarithmic potential in R^n.

In order to extend the results of § 3 and § 4 to the case $\alpha = n$ and $\alpha p = n$, respectively, we shall make use of certain properties of the logarithmic kernel $-\log|x-y|$ in R^n which are established in [12].

The *logarithmic potential* U^μ of a measure μ of compact support is defined by

$$U^\mu(x) = -\int \log|x-y|\, d\mu(y) = U^{\mu^+}(x) - U^{\mu^-}(x)$$

at any point $x \in R^n$ where $U^{\mu^+}(x)$ and $U^{\mu^-}(x)$ are not both $= +\infty$, thus in particular if $\mu \geqq 0$. Instead of the restriction to measures μ of compact support it would suffice to assume that μ be *logarithmically finite*, that is,

$$\int_{|x| \geqq 1} \log|x|\, |d\mu(x)| < +\infty\,. \tag{1}$$

The *mutual logarithmic energy* $\langle \mu, \nu \rangle$ of two positive measures μ and ν of compact supports is defined by

$$\langle \mu, \nu \rangle = -\iint \log|x-y|\, d\mu(x)\, d\nu(y) = \int U^\mu d\nu = \int U^\nu d\mu\,.$$

For $\nu = \mu$ we obtain the *logarithmic energy* $\langle \mu, \mu \rangle$.

The inequalities (2) and (3) below, serving as substitutes for (4) and (6), § 2, respectively, are contained in [12, Lemmas 3.2 and 3.3]. By A and B we denote two concentric balls in R^n of radii ϱ and 3ϱ, respectively.

$U^f_\alpha = +\infty$ everywhere in each $E \cap A_j$, and hence everywhere in E. To see that $f \in \mathscr{L}^q$ for any given $q < p$, we observe that $\sum_j \|f_j\|_{L^q} < +\infty$. Restricting the attention to indices $j > (p-q)^{-1}$, that is $q < p - 1/j$, we obtain, in fact, by application of Hölder's inequality, denoting the measure of each B_j by V (which we may assume to be > 1):

$$\|f_j\|_{L^q} = \left[\int_{B_j} (f_j)^q dx\right]^{1/q} \leqq \|f_j\|_{L^{p-1/j}} V^{1/q - 1/(p-1/j)} \leqq V^{1/q}\, 2^{-j}\,.$$

$M(\varrho)$ is a certain constant (depending only on ϱ, α, and n, $0<\alpha<n$), and ω_n is the surface of the unit sphere in R^n. We write φ_E for the characteristic function associated with a set E.

For any measure $\mu \geqq 0$ supported by A we have, writing $f=\varphi_B \cdot U^{\mu}_{n-\alpha}$,

$$(2) \qquad \frac{1}{\omega_n} U^f_\alpha(x) - M(\varrho)\mu(R^n) \leqq U^\mu(x) \leqq \frac{1}{\omega_n} U^f_\alpha(x) + M(\varrho)\mu(R^n).$$

If, in addition, $\langle \mu,\mu\rangle < +\infty$, then

$$(3) \qquad \left| \langle \mu, \mu\rangle - \frac{1}{\omega_n}\int_B [U^\mu_{n/2}]^2\, dx \right| \leqq M(\varrho)\mu(R^n)^2 .$$

From now on we shall restrict the attention mostly to measures supported by some fixed closed ball $A \subset R^n$ of sufficiently small radius ϱ. The *restriction of the logarithmic kernel* to such a ball A is a *positive* kernel if $\varrho < \frac{1}{2}$; and it follows from the main result of [12] that this restricted kernel is *perfect* in the sense of [11] (in particular *strictly positive definite*) provided $\varrho < a_n$. Here a_n denotes a certain constant. The positivity ($\varrho < \frac{1}{2}$) implies that the usual concepts of *interior* and *exterior* (Wiener) *capacity* are well defined (with reference to this restricted logarithmic kernel), and the exterior capacity, cap*, is countably subadditive. Also, the concepts "*nearly everywhere*" and "*quasi-everywhere*" are well defined for this kernel. We refer to [11, § 2.3] for details. The perfectness ($\varrho < a_n$) implies the existence of a unique *interior capacitary distribution* to be associated with any given set $E \subset A$, cf. [11, Theorem 4.1]. This distribution is a positive measure λ supported by A (but not necessarily by E unless E is closed), for which

$$(4) \qquad \lambda(R^n) = \langle \lambda,\lambda\rangle = \mathrm{cap}_* E ,$$

and

$$(5) \qquad U^\lambda \geqq 1 \quad \text{nearly everywhere in } E ,$$

$$(6) \qquad U^\lambda \leqq 1 \quad \text{everywhere in the support of } \lambda .$$

The following two consequences of (5) and (6), respectively, can be obtained in the same way as in the case of potentials of order α (§ 2):

$$(7) \qquad U^\lambda \geqq 1 \qquad \text{everywhere in the interior of } E ,$$

$$(8) \qquad U^\lambda \leqq 1 + (\log 2)\, \mathrm{cap}_* E \qquad \text{everywhere in } R^n .$$

Remark. Although the exterior Wiener capacity cap*E has been defined above only for sets contained in a ball of radius $\varrho < \frac{1}{2}$, we shall need the concept of a set E of zero exterior capacity (with respect to the logarithmic kernel in R^n) without any such restriction. Following a convention introduced by Deny [4, p. 165], we shall write

$$\operatorname{cap}^* E = 0$$

if and only if $\operatorname{cap}^*(E \cap A) = 0$ for every ball $A \subset R^n$ of radius $\varrho < \frac{1}{2}$. (It suffices to verify this condition for a family of such closed balls covering R^n. This follows from the subadditivity of the exterior capacity on subsets of A.)—There is of course a corresponding extension of the concept "quasi-everywhere". It is easily shown that, for a bounded set $E \subset R^n$, the relation $\operatorname{cap}^* E = 0$ (in this extended sense) is equivalent to the condition $\gamma^*(E) = 0$. Here γ^* denotes the exterior logarithmic capacity, cf. [12, § 2].

In the last two sections the results of § 3 and § 4 are carried over to the case $\alpha = n$ and $\alpha p = n$, respectively. Briefly speaking, the extension consists in interpreting the potential U_β^μ and capacity cap_β of order $\beta = n$ as the logarithmic potential U^μ and the corresponding Wiener capacity cap. (Only in one instance, viz. Lemma 3.2, we use the trivial interpretation $\int d\mu$ of U_n^μ.)

6. Extension of the results of § 3 to the case $\alpha = n$.

The extension of Theorems 3.1 and 3.2 to the logarithmic case $\alpha = n$ will be undertaken, for simplicity, only when the support of μ, resp. the set E, is bounded.

THEOREM 6.1. *For any positive measure μ of compact support, the logarithmic potential of μ is finite quasi-everywhere:*

$$\operatorname{cap}^* \{x \in R^n \colon U^\mu(x) = +\infty\} = 0 .$$

PROOF. It suffices to consider the case $S(\mu) \subset A$, where A denotes a closed ball of radius $\varrho < a_n$ (cf. § 5). The set $\{x \in R^n \colon U^\mu(x) = +\infty\}$ is then contained in A, and the method used in the proof of Theorem 3.1 is now applicable.

It follows from Theorem 6.1 that the logarithmic potential U^μ of *any* measure μ of compact support is defined and finite quasi-everywhere in R^n. The same is true in case of a logarithmically finite measure μ (cf. § 5).

THEOREM 6.2. *For any bounded set $E \subset R^n$ with $\operatorname{cap}^* E = 0$, there is a measure $\mu \geqq 0$ of compact support and of finite logarithmic energy, such that the logarithmic potential U^μ equals $+\infty$ everywhere in E.*

PROOF. Since E may be covered by a finite system of closed balls, each of radius $\varrho < a_n$, $\varrho < \frac{1}{2}$, it suffices to consider the case where E is contained in one such ball A. The method used in the proof of Theorem 3.2 is now applicable.

LEMMA 6.1. $\operatorname{cap}^* E = 0$ *implies* $\operatorname{cap}_\beta^* E = 0$ *for every order* $\beta < n$.

Again, it suffices to consider the case $E \subset A$, and the previous method applies.

Finally, we observe that Lemma 3.2 holds with unchanged proof even for $\beta = n$, $\gamma < n$, provided U_n^μ is interpreted simply as $\int |x-y|^0 d\mu(y) = \int d\mu$, *not* as the logarithmic potential.

7. Extension of the results of § 4 to the case $\alpha p = n$.

We leave out the case $p = 1$, which was treated in the preceding section. Hence $\alpha = n/p < n$, so that the potential U_α^f of order α preserves its original sense (§ 2). We begin by extending Theorem 4.1 to the logarithmic case $\alpha p = n$.

THEOREM 7.1. *Let* $f \geqq 0, f \in \mathscr{L}^p$, $\alpha p = n$, *and suppose f is finite of order* α. *Writing* $E = \{x \in R^n\colon U_\alpha^f(x) = +\infty\}$, *we have*

(i) $\operatorname{cap}^* E = 0$ *provided* $1 < p \leqq 2$,

(ii) $\operatorname{cap}_{n-\varepsilon}^* E = 0$ *for every* $\varepsilon > 0$, $\varepsilon < n$, *provided* $p > 2$.

PROOF. The proof of case (ii) of Theorem 4.1 covers the present case (ii) since f is now *assumed* to be finite of order $\alpha = n/p$. In case (i) we shall need the following substitute for Lemma 4.3:

LEMMA 7.1. *Let* $1 < p \leqq 2$, $\alpha p = n$, $p' = p/(p-1)$. *Consider two concentric, closed balls A and B of radii ϱ and 3ϱ, respectively. For any positive measure μ supported by A,*

$$\frac{1}{\omega_n} \int_B (U_\alpha^\mu)^{p'} dx \leqq [\langle \mu, \mu \rangle / \mu(A)^2 + M(\varrho)]\, \mu(A)^{p'} .$$

Here $M(\varrho)$ denotes the constant mentioned in § 5. The proof of Lemma 7.1 is quite parallel to that of Lemma 4.3. One merely has to replace $U_{\alpha p}^\mu$ by $\int d\mu$ (not by the logarithmic potential of μ) and to apply (3), § 5, in place of (6), § 2.

Returning to the remaining case (i) of Theorem 7.1, we show that $\operatorname{cap}^*(E \cap A_0) = 0$ for every closed ball A_0 of some given radius $\varrho_0 < a_n$. Let $A \supset A_0$ denote a closed ball concentric with A_0 and of radius $\varrho < a_n$. Replacing, if necessary, the given function f by $\varphi_A \cdot f$, we may suppose from the beginning that f is supported by A. (In fact, $f - \varphi_A \cdot f$ has a finite potential of order α everywhere in A_0.) Now there is no difficulty in copying the proof of case (i) of Theorem 4.1, the attention being confined to the ball A: Introducing the relatively open subsets

$$G_t = \{x \in A : U^f_\alpha(x) > t\}, \qquad t > 0,$$

of A, and denoting by λ the capacitary distribution (corresponding to the *logarithmic* kernel, cf. § 5) on an arbitrary compact subset K of G_t, we get

$$t \cdot \int d\lambda \leqq \int U^f_\alpha \, d\lambda = \int_A U^\lambda_\alpha f \, dx \leqq \|U^\lambda_\alpha\|_{L^{p'}(A)} \|f\|_{L^p} .$$

Raising to the power p', and applying Lemma 7.1 with $\mu = \lambda$, we obtain, since $A \subset B$,

$$\omega_n^{-1} t^{p'} \leqq [(\operatorname{cap} K)^{-1} + M(\varrho)] \left(\int f^p \, dx\right)^{p'/p} .$$

The same inequality subsists with $\operatorname{cap} G_t$ in place of $\operatorname{cap} K$, and we conclude that $\operatorname{cap} G_t \to 0$ as $t \to +\infty$, whence $\operatorname{cap}^*(E \cap A_0) = 0$.

Theorem 7.2 *Let* $p > 1$, $\alpha p = n$, $E \subset R^n$, *and suppose* $\operatorname{cap}^* E = 0$, i.e., E is of zero exterior capacity with respect to the logarithmic kernel. *Then there exists a function* $f \geqq 0$ *such that* $f \in \mathscr{L}^q$ *for all* $q < p$, *and*

$$U^f_\alpha(x) = +\infty \qquad \textit{for every } x \in E .$$

If $p \geqq 2$, f *may be so chosen that, in addition,* $f \in \mathscr{L}^p$.

(Note that f is automatically finite of order α because $\alpha q < \alpha p = n$, cf. Lemma 4.1.) In proving Theorem 7.2, it suffices to consider the case where E is contained in some closed ball (no matter how small). This follows by the argument employed in the proof of Theorem 4.2 in the case $p < 2$.

(i) *The case* $p \geqq 2$. The following lemma is the dual of Lemma 7.1 above under the involution (1), § 4:

Lemma 7.2. *Let* $p \geqq 2$, $\alpha p = n$, $p' = p/(p-1)$. *Consider two concentric, closed balls* A *and* B *of radii* ϱ *and* 3ϱ, *respectively. For any positive measure* μ *supported by* A,

$$\frac{1}{\omega_n} \int_B [U^\mu_{\alpha(p-1)}]^p \, dx \leqq [\langle \mu, \mu \rangle / \mu(A)^2 + M(\varrho)] \; \mu(A)^p .$$

In order to prove Theorem 7.2 in the case $p \geqq 2$, $E \subset A_0$ (where A_0 denotes a closed ball of radius $\varrho_0 < a_n$), we apply Cartan's method just as in Theorem 4.2. First we choose a closed ball $A \supset A_0$ concentric with A_0 and of radius $\varrho < a_n$. Then there are open subsets $G_k \supset E$ of A such that $\operatorname{cap} G_k < 2^{-k}$. Denote by λ_k the interior capacitary distribution associated with G_k, the kernel being the logarithmic kernel on A (cf. § 5), and apply (2), § 5 with μ replaced by λ_k. Denoting by B the closed ball concentric with A and of radius $R=3\varrho$, we define

$$f_k = \varphi_B U^{\lambda_k}_{n-\alpha} ,$$

and obtain

$$\frac{1}{\omega_n} U^{f_k}_\alpha \geqq U^{\lambda_k} - M(\varrho)\int d\lambda_k .$$

According to (7), § 5, this implies

$$\frac{1}{\omega_n} U^{f_k}_\alpha \geqq 1 - M(\varrho)\operatorname{cap} G_k \qquad \text{everywhere in } G_k ,$$

in particular everywhere in E. Writing $f=\sum_k f_k$, we conclude that

$$U^f_\alpha = \sum_k U^{f_k}_\alpha = +\infty \qquad \text{everywhere in } E .$$

The preceding part of the proof was independent of the assumption $p \geqq 2$. We proceed to apply Lemma 7.2 with μ replaced by λ_k. Thus

$$\|f_k\|_{L^p} \leqq \omega_n^{1/p}(1+M(\varrho)\operatorname{cap} G_k)^{1/p} (\operatorname{cap} G_k)^{1/p'} ,$$

which shows that $\sum_k \|f_k\|_{L^p} < \infty$, and consequently $f \in \mathscr{L}^p$. Since f vanishes outside B, we conclude that $f \in \mathscr{L}^q$ for all $q<p$.

(ii) *The case* $1<p<2$. As mentioned above, we may suppose that E is contained in some closed ball A_0 of radius $\varrho_0 < a_n$. Using the notations introduced in the discussion of the case $p \geqq 2$, we have again $U^f_\alpha = +\infty$ everywhere in E. In order to prove that $f \in \mathscr{L}^q$ for every $q<p$, we merely have to use the following lemma analogous to Lemma 4.6 (with λ_k in place of μ):

Lemma 7.3. *Let* $1<q<p<2$, $\alpha p = n$. *For any number* $R>0$ *there is a constant* $K(R)$, *depending further on* n, α, *and* q, *such that the inequality*

$$\|U^\mu_{n-\alpha}\|_{L^q} \leqq K(R)\,\mu(B)$$

holds for any ball B *of radius* R *and any positive measure* μ *concentrated on* B.

Proof. Applying Lemma 3.2 with α, β, γ, θ replaced by $n-\alpha$, $n-\alpha q$, n, $1/q$, respectively, we obtain after raising to the power q and integrating over B:

$$\int_B [U^\mu_{n-\alpha}]^q\,dx \leqq \left(\int d\mu\right)^{q-1} \int_B U^\mu_{n-\alpha q}\,dx .$$

(Recall that U^μ_n should be interpreted as $\int d\mu$ in Lemma 3.2, cf. the concluding paragraph of § 5). Denoting by $C_\beta(R)$ the supremum of the bounded potential $U_\beta^{\varphi_B}$, we infer from the formula of reciprocity (5), § 2, in the manner explained in the proof of Lemma 4.4, that

$$\int_B U_\beta^\mu dx \leqq C_\beta(R) \int d\mu .$$

Hence Lemma 7.3 follows, with $K(R)=C_{n-\alpha q}(R)^{1/q}$, and the proof of Theorem 7.2 is complete.

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