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A PROOF OF SCHWARTZ'S KERNEL THEOREM

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1. Introduction.

We let O_x and O_y denote open sets in the Euclidean spaces R^n and R^m respectively and write O_{xy} for the product $O_x \times O_y$. As usual, $\mathscr{D}(O_x)$, $\mathscr{D}(O_y)$ and $\mathscr{D}(O_{xy})$ stand for the spaces of infinitely differentiable functions with compact supports (in O_x, O_y and O_{xy} respectively), equipped with the standard topology ([3]). The strong duals of these spaces will be denoted by $\mathscr{D}'(O_x)$, etc.

We identify a locally integrable function T in, e.g., $\mathscr{D}(O_x)$ with the distribution

$$f \to \langle T,f \rangle = \int T(x)\, f(x)\, dx \;,$$

and we shall use the integral as a notation for the value $\langle T,f \rangle$ of T at f also when T is an arbitrary distribution.

Consider the space $\mathscr{A}$ of all separately continuous bilinear functionals A on $\mathscr{D}(O_x) \times \mathscr{D}(O_y)$ with the topology of uniform convergence on products of bounded sets in $\mathscr{D}(O_x)$ and $\mathscr{D}(O_y)$. Any distribution T in $\mathscr{D}'(O_{xy})$ gives rise to such a functional A by specialisation to products of functions of x and y:

$$(1) \qquad (\Lambda T)(f,g) = \langle T, f(x) g(y) \rangle = A(f,g) \,.$$

The kernel theorem says that *the mapping*

$$T \to \Lambda T = A$$

is a linear homeomorphism between $\mathscr{D}'(O_{xy})$ *and* $\mathscr{A}$. In particular, there exists to any A in $\mathscr{A}$ precisely one "kernel" T in $\mathscr{D}'(O_{xy})$ such that

$$A(f,g) = \int T(x,y)\, f(x)\, g(y)\, dx dy \,.$$

The theorem was first proved by Schwartz [4], and a much simplified proof has then been given by Ehrenpreis [2]. Our proof is close to that of Ehrenpreis but has the advantage that we can easily show that Λ in (1) is one-to-one and *onto* $\mathscr{A}$, before proving that it is a topological isomorphism. The proof of the former half of the theorem can thus be

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made independent of most of the topological concepts necessary for the latter. A by-product of the proof is also a simple estimate of the local order of T in terms of the local order of A.

The theorem remains true if, e.g., $\mathscr{D}$ is replaced by the space $\mathscr{E}$ of infinitely differentiable functions, or if $O_x = R^n$, $O_y = R^m$ and $\mathscr{D}$ is replaced by the space $\mathscr{S}$ of Schwartz (see [3]). The proof given here is easily adapted to these cases.

In what follows, U and V will always denote compact sets in O_x and O_y respectively. By $\mathscr{D}(U)$ we shall denote the subspace of $\mathscr{D}(O_x)$ whose elements have their supports in U, and analogously for $\mathscr{D}(V)$ and $\mathscr{D}(U \times V)$.

2. Some estimates for Fourier series.

To any given U and V we can find a number a, such that U and V are contained in cubes with sides $2a-2$, say, in R^n and R^m. We let K_x and K_y stand for the corresponding cubes with side $2a$ and write

$$e_p(x) = \gamma_n \cdot \exp\{i\pi a^{-1}(p_1x_1 + p_2x_2 + \ldots + p_nx_n)\}$$

and

$$e_q(y) = \gamma_m \cdot \exp\{i\pi a^{-1}(q_1y_1 + q_2y_2 + \ldots + q_my_m)\} .$$

Here p and q are n- and m-tuples of integers and the constants γ_n and γ_m are chosen such that the functions are orthonormal. Any function $h(x,y)$ in $\mathscr{D}(U \times V)$ can be expanded in a Fourier series in $K_x \times K_y$:

$$h(x,y) = \sum a_{pq} e_p(x) e_q(y) , \tag{2}$$

the sum taken over all p and q. The coefficients in (2) are given by well-known integral formulas, and integrations by parts in these formulas yield the classical estimates

$$|a_{pq}| \leqq C_j |h|_j (1 + |p| + |q|)^{-j} . \tag{3}$$

Here C_j is a constant independent of h, j is an arbitrary positive integer, $|p|$ and $|q|$ are defined as $\sum |p_\nu|$ and $\sum |q_\nu|$ respectively and $|h|_j$ stands for $\sup_x \sum_{|\alpha| \leqq j} |D^\alpha h(x,y)|$. If $\varphi(x)$ and $\psi(y)$ are in $\mathscr{D}(K_x)$ and $\mathscr{D}(K_y)$ and identically one on U and V, the expansion

$$h(x,y) = \sum a_{pq} \varphi(x) e_p(x) \psi(y) e_q(y) \tag{4}$$

holds in the whole of $O_x \times O_y$. We shall write (4) in the form

$$h(x,y) = \sum \lambda_{pq} f_p(x) g_q(y) \tag{5}$$

with $f_p(x)$ and $g_q(y)$ proportional to $\varphi(x)e_p(x)$ and $\psi(y)e_q(y)$:

$$\alpha_{pq} f_p(x) = \varphi(x) e_p(x) ,$$
$$\beta_{pq} g_q(y) = \psi(y) e_q(y) .$$

With these notations the coefficients in (5) are given by

$$\lambda_{pq} = \alpha_{pq} \beta_{pq} a_{pq} .$$

The proportionality factors shall be chosen in a suitable way, expressed in the following lemmas.

LEMMA 1. *Let h be a function in $\mathscr{D}(U \times V)$ and k and l given positive integers. Then f_p and g_q in* (5) *can be chosen such that*

$$|f_p|_k \leqq 1$$

and

$$|g_q|_l \leqq 1$$

for all p and q and

$$\sum |\lambda_{pq}| \leqq C|h|_{k+l+n+m+2}$$

with a constant C independent of h.

PROOF. We write (4) as

$$h(x,y) = \sum a_{pq}(1+|p|)^k(1+|q|)^l\,[(1+|p|)^{-k}\varphi(x)e_p(x)]\,[(1+|q|)^{-l}\psi(y)e_q(y)] ,$$

and choose the functions in square brackets for f_p and g_q. The estimates (3) and some straight-forward calculations then give the lemma.

LEMMA 2 (Ehrenpreis). *Let $\{a_\nu\}_1^\infty$ be a sequence of positive real numbers and h any function in $\mathscr{D}(U \times V)$ satisfying*

$$|h|_\nu \leqq a_\nu, \qquad \nu = 1, 2, \ldots .$$

Then there exists another sequence $\{b_\nu\}_1^\infty$, depending only on the original one and not on h, such that with a suitable choice of f_p and g_q in (5) *we have*

$$|f_p|_\nu \leqq b_\nu, \quad |g_q|_\nu \leqq b_\nu, \qquad \nu = 1, 2, \ldots ,$$

for all p and q, and also

(6) $$\sum |\lambda_{pq}| \leqq 1 .$$

PROOF. The lemma expresses the fact that if h is in some bounded set in $\mathscr{D}(U \times V)$, the expansion (5) can be made such that (6) holds with f_p and g_q in fixed bounded sets in $\mathscr{D}(U)$ and $\mathscr{D}(V)$ respectively. For the proof, put

$$f_p(x) = |a_{pq}|^{\frac{1}{4}}\varphi(x)e_p(x) ,$$
$$g_q(y) = |a_{pq}|^{\frac{1}{4}}\psi(y)e_q(y)$$

and hence

$$\lambda_{pq} = a_{pq}|a_{pq}|^{-\frac{1}{2}}$$

in (4). Then the rapid decrease of the coefficients a_{pq}, expressed by (3), is easily seen to give the lemma.

3. The kernel theorem.

As stated above, $\mathscr{A}$ denotes the set of all separately continuous bilinear functionals on $\mathscr{D}(O_x) \times \mathscr{D}(O_y)$, and in what follows we shall simply write $\mathscr{T}$ for $\mathscr{D}'(O_{xy})$.

It is a classical fact that the restriction to $\mathscr{D}(U) \times \mathscr{D}(V)$ of any $A \in \mathscr{A}$ is continuous (see, e.g., [1, p. 83]) and so there exist a constant C and integers k and l (depending, of course, on A) for which

$$|A(f,g)| \leqq C|f|_k|g|_l, \qquad f \in \mathscr{D}(U), \quad g \in \mathscr{D}(V) . \tag{7}$$

As stated in the introduction, there is a mapping Λ of $\mathscr{T}$ into $\mathscr{A}$, defined by (1). We shall now first prove that the range of this mapping is the whole of $\mathscr{A}$ and that it is one-to-one.

THEOREM 1. *For any separately continuous functional A on $\mathscr{D}(O_x) \times \mathscr{D}(O_y)$ there exists precisely one distribution T in $\mathscr{T}$ such that*

$$(\Lambda T)(f,g) = \langle T, f(x)g(y)\rangle = A(f,g) \tag{8}$$

for all (f,g) in $\mathscr{D}(O_x) \times \mathscr{D}(O_y)$.

PROOF. We begin by restricting A to $\mathscr{D}(U) \times \mathscr{D}(V)$, U and V compact, and write an arbitrary h in $\mathscr{D}(U \times V)$ in the form given by lemma 1. If k and l are integers such that (7) holds for our given A we find

$$\sum |\lambda_{pq}|\, |A(f_p,g_q)| \leqq C|h|_{k+l+n+m+2} \tag{9}$$

with C independent of h. We define T by

$$\langle T,h\rangle = \sum \lambda_{pq} A(f_p,g_q) \tag{10}$$

and conclude from (9) that T is a distribution on $U \times V$ of order $\leqq k+l+n+m+2$. It is clear that (8) holds for this T and also that T is uniquely determined by A, for if A vanishes we infer from (10) that $T(h)=0$ on all finite sums $h=\sum\lambda_{pq}f_p g_q$, and as the set of such sums is total in $\mathscr{D}(U \times V)$ the distribution T must vanish.

Now, O_x and O_y are unions of compact sets in each of which the existence of a unique T has been proved, and the theorem follows.

THEOREM 2. *The mapping Λ defined by* (8) *is a linear homeomorphism.*

PROOF. The topologies on $\mathscr{A}$ and $\mathscr{T}$ as introduced above are defined by the seminorms

$$\varrho_{B_xB_y}(A) = \sup|A(f,g)|, \qquad f \in B_x, \quad g \in B_y\,,$$
$$\sigma_{B_{xy}}(T) = \sup|\langle T,h\rangle|, \qquad h \in B_{xy}\,,$$

where B_x, B_y and B_{xy} are bounded sets in $\mathscr{D}(O_x)$, $\mathscr{D}(O_y)$ and $\mathscr{D}(O_{xy})$ respectively.

It is clear that Λ is linear. In order that Λ be a homeomorphism it is necessary and sufficient that it is bicontinuous, i.e. that Λ and Λ^{-1} are both continuous.

Let $\varrho_{B_xB_y}$ be an arbitrary seminorm on $\mathscr{A}$. Then

$$\varrho_{B_xB_y}(\Lambda T) = \varrho_{B_xB_y}(A) = \sup_{f\in B_x,\, g\in B_y}|A(f,g)| = \sup_{f\in B_x,\, g\in B_y}|\langle T,fg\rangle|\,.$$

It is easy to see that for any bounded sets $B_x \subset \mathscr{D}(O_x)$ and $B_y \subset \mathscr{D}(O_y)$ there exists a bounded set $B_{xy} \subset \mathscr{D}(O_{xy})$ such that all products fg are in B_{xy} whenever f is in B_x and g in B_y. Hence

$$\sup_{f\in B_x,\, g\in B_y}|\langle T,fg\rangle| \leqq \sup_{h\in B_{xy}}|\langle T,h\rangle| = \sigma_{B_{xy}}(T)\,,$$

and so Λ is continuous. Conversely, if $\sigma_{B_{xy}}$ is a seminorm on $\mathscr{T}$ we find

$$\sigma_{B_{xy}}(\Lambda^{-1}A) = \sigma_{B_{xy}}(T) = \sup_{h\in B_{xy}}|\langle T,h\rangle| = \sup_{h\in B_{xy}}|\textstyle\sum \lambda_{pq}A(f_p,g_q)|\,,$$

where h has been expanded as in lemma 2, and thus

$$\sup_{h\in B_{xy}}|A(f_p,g_q)| \leqq \sup_{f\in B_x,\, g\in B_y}|A(f,g)| = \varrho_{B_xB_y}(A)\,,$$

if B_x and B_y are those bounded sets in $\mathscr{D}(O_x)$ and $\mathscr{D}(O_y)$ which contain all f_p and g_q according to lemma 2. From (6) we now conclude

$$\sigma_{B_{xy}}(\Lambda^{-1}A) \leqq \sup_{h\in B_{xy}}|A(f_p,g_q)| \textstyle\sum |\lambda_{pq}| \leqq \varrho_{B_xB_y}(A)\,,$$

and thus Λ^{-1} is also continuous and the theorem is proved.

REMARK. For a given A in $\mathscr{A}$ we can define a functional L on $\mathscr{D}(O_y)$ by

$$\langle L,g\rangle = A(f,g), \qquad f \in \mathscr{D}(O_x), \qquad g \in \mathscr{D}(O_y)\,,$$

and L is immediately seen to be a distribution in $\mathscr{D}'(O_y)$ for every f. Hence every A in $\mathscr{A}$ gives rise to a mapping Γ from $\mathscr{D}(O_x)$ to $\mathscr{D}'(O_y)$:

$$(\Gamma f)(g) = A(f,g)\,, \tag{11}$$

and it is easily checked that this mapping is linear and continuous. Conversely, via formula (11) every such linear continuous mapping is seen to define a bilinear functional in $\mathscr{A}$.

The usual strong topology on the space $\mathscr{L}$ of all continuous linear mappings from $\mathscr{D}(O_x)$ to $\mathscr{D}'(O_y)$ is defined by the seminorms

$$\theta(\Gamma) = \sup_{f \in B_x} \tau(\Gamma f) ,$$

where τ is a seminorm on $\mathscr{D}'(O_y)$ and B_x a bounded set in $\mathscr{D}(O_x)$. Therefore

$$\theta(\Gamma) = \sup_{f \in B_x} \tau(\Gamma f) = \sup_{f \in B_x,\, g \in B_y} |\langle \Gamma f, g\rangle| = \sup_{f \in B_x,\, g \in B_y} |A(f,g)| = \varrho_{B_x B_y}(A) \, .$$

Thus the kernel theorem states that the spaces $\mathscr{T}$ and $\mathscr{L}$ are homeomorphic, and this formulation of the theorem was the one used in the previous proofs.

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