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THE RELATION BETWEEN TWO GENERALISATIONS OF THE NOTION "SURFACE OF CURVATURE $\leqq$K"

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Surfaces of bounded total (Gaussian) curvature $\leqq K$ have interesting properties. A. D. Alexandrow has investigated such properties in [2] and especially in [3] and extended them to spaces of curvature $\leqq K$ in a more general sense. A region R in a locally compact space of arbitrary dimension with intrinsic metric is called an R_K [3, pp. 36, 41], if for every triangle T in R, the sum of the "upper angles" is not greater than the sum of the angles of a triangle T^K with sides of the same lengths on a surface of constant curvature K. (If $K>0$ it is postulated that the perimeter of any triangle T in the R_K is not greater than $2\pi K^{-\frac{1}{2}}$, so that T^K exists. For the definition of "upper angle" see [3, p. 35] or [2, p. 492].) Then a metric space, in which every point has a neighbourhood which is an R_K, is said to be "of curvature $\leqq K$" [3, p. 36]. Let us here for the sake of brevity (and to avoid confusion in the sequel) call such a space an R_K-space.

In [6] I gave for a surface of total curvature $\leqq K$ an estimate for the maximal deviation of a curve AB of given length from the geodesic AB. Then I generalised the notion "curvature $\leqq K$" in a way different from Alexandrow's—but analogous to Beurling's generalisation for $K=0$ in [4]—by means of the class $C(K)$ of "functions of curvature $\leqq K$". A real-valued function $u(z)$ of a complex variable z is said to belong to $C(K)$ in a region D if it is continuous in D and satisfies

$$(1) \qquad L(u,z_0,r)-u(z_0) \geqq -\tfrac{1}{2}K\int_0^r \varrho A(e^{2u},z_0,\varrho)\,d\varrho$$

for every $z_0 \in D$ and all sufficiently small r [6, p. 318]. Here $L(u,z_0,r)$ and $A(u,z_0,r)$ denote, as usual in the theory of subharmonic functions, the mean values of $u(z)$ on the circle $|z-z_0|=r$ and the circular disk $|z-z_0|<r$, respectively (cf. [6, p. 317] or Radó [9, p. 3]). Then I called the metric

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(2) $$ds = e^{u(z)}|dz| ,$$

defined on D, "of curvature $\leqq K$" if $u \in C(K)$ in D [6, p. 322].

In this paper we shall see how this notion "surface of curvature $\leqq K$" is related to Alexandrow's notion, here called "R_K-space". It will be found that a surface of curvature $\leqq K$ in the sense of [6] is also an R_K-space. But a 2-dimensional R_K-space (or even an R_K) need not be a "surface of curvature $\leqq K$", because the continuity of $u(z)$ assumed in the definition of the class $C(K)$ is not necessary. However, upper semicontinuity is necessary and sufficient. Furthermore, a general R_K-space does not correspond to a region D in the complex z-plane but to an arbitrary open Riemann surface R. Therefore it is convenient to introduce here a wider class $C'(K)$ of functions $u(z)$ defined on an open Riemann surface R:

Definition. $u(z) \in C'(K)$ if
a) $u(z)$ is upper semi-continuous,
b) $u(z)$ satisfies (1) for every point z_0 of R and all sufficiently small r.

Then we have the following two mutually converse theorems:

Theorem 1. *A surface given by a metric* (2), *defined on an open Riemann surface R by a real-valued function $u(z) \in C'(K)$, is an R_K-space.*

Theorem 2. *Every orientable 2-dimensional R_K-space is isometric to an open Riemann surface R with the metric* (2), *where $u(z) \in C'(K)$.*

Proof of theorem 1. We shall first show that any $u(z)$ in $C'(K)$ can locally be represented as a logarithmic potential. Let D be a region of R, which may be considered as situated in a z-plane. Let us introduce in D the auxiliary function

$$v(z) = -(2\pi)^{-1} \iint_D \ln|z-\zeta| \, K e^{2u(\zeta)} \, d\xi \, d\eta ,$$

where we have put $\zeta = \xi + i\eta$. Then the function $u-v$ is upper semicontinuous, because u is so and v is continuous. And for a sufficiently small circle in D we find

$$L(u-v,z_0,r) - u(z_0) + v(z_0) = L(u,z_0,r) - u(z_0) - [L(v,z_0,r) - v(z_0)] \geqq 0 ,$$

because u satisfies (1) and a calculation of $L(v,z_0,r) - v(z_0)$ yields exactly the right member of (1). Indeed, we have

$$\int_0^{2\pi} \ln|z_0 - \zeta + re^{i\theta}| \, d\theta = \begin{cases} 2\pi \ln|z_0-\zeta| & \text{for } |z_0-\zeta| > r , \\ 2\pi \ln r & \text{for } |z_0-\zeta| \leqq r , \end{cases}$$

which may be obtained by applying Jensen's formula (cf. e.g. [1, p. 185]) to the analytic function $f(z)=z-(\zeta-z_0)$, and thus

$$\begin{aligned}
&L(v,z_0,r)-v(z_0)\\
&= -(2\pi)^{-2}K\int_0^{2\pi}\left[\iint_D (\ln|z_0+re^{i\theta}-\zeta| - \ln|z_0-\zeta|)\, e^{2u(\zeta)}\, d\xi\, d\eta\right] d\theta\\
&= -(2\pi)^{-1}K\int_0^r\int_0^{2\pi} (\ln r-\ln\sigma)\exp 2u(z_0+\sigma e^{i\varphi})\,\sigma\, d\sigma\, d\varphi\\
&= -(2\pi)^{-1}K\int_0^r\int_0^{2\pi}\int_\sigma^r \varrho^{-1}d\varrho\,\exp 2u(z_0+\sigma e^{i\varphi})\,\sigma\, d\sigma\, d\varphi\\
&= -(2\pi)^{-1}K\int_0^r\left[\varrho^{-1}\int_0^\varrho\int_0^{2\pi}\exp 2u(z_0+\sigma e^{i\varphi})\,\sigma\, d\sigma\, d\varphi\right] d\varrho\\
&= -\tfrac{1}{2}K\int_0^r \varrho A(e^{2u},z_0,\varrho)\, d\varrho\,.
\end{aligned}$$

Thus $u-v$ is subharmonic in D. In a region D', which together with its boundary is contained in D, $u-v$ is then the potential of a non-positive mass-distribution $-\mu$ plus a harmonic function [9, p. 42]:

$$u(z)-v(z) = \iint_{D'} \ln|z-\zeta|\,\mu(dE_\zeta) + h(z)\,. \tag{3}$$

On the other hand, if we divide the integral over D defining v into two parts, one over D' and the other over $D-D'$, the latter part gives in D' a harmonic function, and we get in D'

$$\begin{aligned}
v(z) &= -(2\pi)^{-1}\iint_{D'} \ln|z-\zeta|\, K e^{2u(\zeta)}\, d\xi\, d\eta + h_1(z)\\
&= -(2\pi)^{-1}K\iint_{D'} \ln|z-\zeta|\, j(dE_\zeta) + h_1(z)\,,
\end{aligned} \tag{4}$$

where $j(E)$ denotes the area (Lebesgue measure) of the set E in the metric (2). Adding (3) and (4) we find

$$u(z) = -(2\pi)^{-1}\iint_{D'} \ln|z-\zeta|\,\omega(dE_\zeta) + h_2(z)$$

with $h_2(z)$ harmonic in D' and

$$\omega(E) = Kj(E)-2\pi\mu(E)\,. \tag{5}$$

Now according to a theorem of Reschetnjak [10] (cf. [2, pp. 503 ff.]), the metric (2), with $u(z)$ a difference between two subharmonic functions, is "of bounded curvature" in the sense of [2, p. 493], and the curvature corresponding to a set E is $\omega(E)$. As $\mu(E) \geqq 0$, we get from (5)

$$\omega(E) \leqq Kj(E) \tag{6}$$

for every Borel set E in D'. However, (6) then holds also for any Borel set E in D. Indeed such an E can be approximated from within by a closed E' for which the differences $|\omega(E)-\omega(E')|$ and $|j(E)-j(E')|$ are arbitrarily small. If $\omega(E)-Kj(E)=2\varepsilon>0$, we could thus find an E' in a D' so that

$$\omega(E') > Kj(E)+\varepsilon > Kj(E') ,$$

which is impossible since (6) has been proved for E'. Thus (6) must hold for any Borel set in D. An arbitrary region E in R can be divided into parts E_i for which this yields $\omega(E_i) \leqq Kj(E_i)$. Adding, we get (6) for E or

$$\omega(E)/j(E) \leqq K ,$$

that is: The metric is of specific curvature $\leqq K$ in the sense of [2, p. 431]. According to Alexandrow [2, p. 513], the surface is then an R_K-space. (In [2] this statement is proved only for convex surfaces (cf. theorem 4, p. 433, and its proof on pp. 442f. in [2]). For that part of theorem 4 with which we are concerned the method of this proof is however general.)

Proof of theorem 2. According to Alexandrow [3, p. 43], an R_K-space is also "of bounded curvature". Huber has recently proved [7, p. 100] (completing a result of Reschetnjak [10]) that an orientable 2-dimensional space "of bounded curvature" is isometric to a Riemann surface R with the metric

$$ds = e^{u(z)}|dz| , \tag{2}$$

where $u(z)$ is the difference between two subharmonic functions. R is open, because Alexandrows definition of R_K-space (mentioned above) assumes that every point in the space has a neighbourhood. We have to prove that $u(z)$ a) satisfies (1) and b) is upper semi-continuous.

a) (1) is trivial, if $u(z_0) = -\infty$. Further $u(z_0) = +\infty$ is impossible in an R_K, as will follow from b). Here we may thus assume $u(z_0)$ finite. Then we prove (1) for $r < \frac{1}{2}$ so small that the closed disk $|z-z_0| \leqq r$ is contained in R. Let C_r denote the interior of such a disk. Then $u(z)$ may be written (cf. formula (3) in [2, p. 504])

$$u(z) = -(2\pi)^{-1} \iint_{C_r} \ln|z-\zeta| \; \omega(dE_\zeta) + h(z) , \tag{7}$$

where $h(z)$ is a function harmonic in C_r, and $\omega(E)$ denotes the curvature of the set in our R_K-space which corresponds to E in R (The curvature of a Borel set being defined in [2, p. 496]).

As $L(h, z_0, r) - h(z_0) = 0$, it is sufficient to study the left member of (1) for

$$u_1(z) = u(z) - h(z) = -(2\pi)^{-1} \iint_{C_r} \ln |z-\zeta| \, \omega(dE_\zeta) .$$

Inserting this expression in $L(u_1, z_0, r)$, we obtain

$$L(u_1, z_0, r) = (2\pi)^{-1} \int_0^{2\pi} u_1(z_0 + re^{iv}) \, dv = -(2\pi)^{-2} \int_0^{2\pi} \left[\iint_{C_r} \ln |z_0 + re^{iv} - \zeta| \, \omega(dE_\zeta) \right] dv .$$

Here we interchange the order of integration. This is legitimate according to well-known theorems of Tonelli and Fubini (cf. e.g. [8, p. 151]), because the integrand is measurable and <0 in C_r (in virtue of the assumption $r < \frac{1}{2}$) and the integral with respect to the measure $|\omega|$ is finite. This last point is verified at the end of our calculation. Thus we get

$$L(u_1, z_0, r) = -(2\pi)^{-2} \iint_{C_r} \left[\int_0^{2\pi} \ln |z_0 + re^{iv} - \zeta| \, dv \right] \omega(dE_\zeta) .$$

The integral in brackets has (as mentioned above p. 340) the value $2\pi \ln r$, since $|\zeta - z_0| < r$ in C_r, and we get

$$L(u_1, z_0, r) = -(2\pi)^{-1} \ln r \iint_{C_r} \omega(dE_\zeta) = -(2\pi)^{-1} \ln r \, \omega(C_r)$$

(cf. the analogous result 4.29 in [9, p. 30]). Now we see that the corresponding integral with respect to $|\omega|$ has the value $-(2\pi)^{-1} \ln r |\omega|(C_r)$. That $|\omega|(C_r)$ is finite is contained in the statement that the space is "of bounded curvature" [2, p. 493].

Now, when $u_1(z_0)$ is finite, the left member of (1) is,

$$(8) \quad L(u_1, z_0, r) - u_1(z_0) = -(2\pi)^{-1} \ln r \iint_{C_r} \omega(dE_\zeta) + (2\pi)^{-1} \iint_{C_r} \ln |z_0 - \zeta| \, \omega(dE_\zeta)$$

$$= -(2\pi)^{-1} \iint_{C_r} \ln \frac{r}{|z_0 - \zeta|} \, \omega(dE_\zeta) .$$

However, any R_K-space is also of specific curvature $\leqq K$, that is

$$(6) \qquad \omega(E) \leqq Kj(E)$$

for every region E. (This fact is stated by Alexandrow in [2, p. 513]. A proof is given here in an appendix.) Then (6) must hold for every Borel set E in R. Applying it to the last member of (8), we get,

$$(9)\ L(u,z_0,r)-u(z_0) = L(u_1,z_0,r)-u_1(z_0) \geqq -(2\pi)^{-1}K\iint\limits_{C_r}\ln\frac{r}{|z_0-\zeta|}j(dE_\zeta)\,,$$

because $\ln(r/|z_0-\zeta|)>0$ in C_r.

By the definition of $A(u,z_0,r)$ the right member of (1) is

$$-\tfrac{1}{2}K\int\limits_0^r\left(\varrho\frac{1}{\pi\varrho^2}\iint\limits_{C_\varrho}e^{2u(\zeta)}d\xi\,d\eta\right)d\varrho = -(2\pi)^{-1}K\int\limits_0^r\left[\varrho^{-1}\iint\limits_{C_\varrho}j(dE_\zeta)\right]d\varrho\,.$$

This is a triple integral over the cone $|\zeta-z_0|<\varrho$, $0<\varrho<r$, which may be written

$$-(2\pi)^{-1}K\iint\limits_{C_r}\left[\int\limits_{|z_0-\zeta|}^r\varrho^{-1}d\varrho\right]j(dE_\zeta) = -(2\pi)^{-1}K\iint\limits_{C_r}\ln\frac{r}{|z_0-\zeta|}j(dE_\zeta)\,.$$

This is the right member of (9), and thus, (1) is proved.

b) We start the proof that $u(z)$ is upper semi-continuous by splitting the curvature ω into its positive and negative parts: $\omega=\omega^++\omega^-$. The corresponding parts of the potential u are denoted by u^+ and u^- resp. Then u^- is a subharmonic function and thus upper semi-continuous. It remains to prove the upper semi-continuity of u^+. In fact we can prove much more, namely that $u^+(x+iy)$ has partial derivatives u_x^+ and u_y^+, which satisfy Lipschitz conditions of every order $1-\varepsilon<1$. Since $u^+\equiv 0$ if $K\leqq 0$, we may assume $K>0$ in the sequel.

In a circular disk C in R we have

$$(10)\qquad u^+(z) = -(2\pi)^{-1}\iint\limits_C\ln|z-\zeta|\,\omega^+(dE_\zeta) + h(z)\,,$$

where $h(z)$ is harmonic in C. Introducing a positive number $\varepsilon<1$, we can also write

$$u^+(z) = \frac{\omega^+(C)}{2\pi\varepsilon}\iint\limits_C\ln|z-\zeta|^{-\varepsilon}\frac{\omega^+(dE_\zeta)}{\omega^+(C)} + h(z)\,.$$

The integral here is the logarithm of a geometric mean. The inequality between the arithmetic and geometric means then gives

$$u^+(z) \leqq \frac{\omega^+(C)}{2\pi\varepsilon}\ln\iint\limits_C|z-\zeta|^{-\varepsilon}\frac{\omega^+(dE_\zeta)}{\omega^+(C)} + h(z)\,.$$

For a given $\varepsilon>0$ we can choose C with given centre z_0 so that $\omega^+(C)\leqq\pi\varepsilon$. Indeed, when the radius a of C tends to 0, $\omega^+(C)$ tends to $\omega^+(z_0)=0$.

In an R_K, $\omega^+(z_0) > 0$ is impossible. (This follows e.g. from the fact that no shortest line ("Kürzeste") could pass through such a point. However, according to theorem 6, p. 54, in [3], in an R_K a shortest line varies continuously with its endpoints. Hence, between two shortest lines, AB and AC, passing near z_0 on opposite sides, there must be an intermediate one passing through z_0.) Thus, if $a < \frac{1}{2}$, we have

$$u(z) \leqq \tfrac{1}{2} \ln \iint_C |z-\zeta|^{-\varepsilon} \frac{\omega^+(dE_\zeta)}{\omega^+(C)} + h(z) + u^-(z)\,.$$

And because $h(z)$ and $u^-(z)$ have upper bounds in C, we get in C

$$e^{2u(z)} \leqq k \iint_C |z-\zeta|^{-\varepsilon}\, \omega^+(dE_\zeta)\,,$$

where k is a constant.

We can now estimate $\omega^+(\gamma_r)$ for an arbitrary circular disk γ_r in C with radius r. Using (6), the last inequality for e^{2u} and that $K > 0$ has been assumed, we get

$$\begin{aligned}
\omega^+(\gamma_r) \leqq Kj(\gamma_r) &= K \iint_{\gamma_r} e^{2u(z)}\, dx\, dy \\
&\leqq K \iint_{\gamma_r} \left[k \iint_C |z-\zeta|^{-\varepsilon}\, \omega^+(dE_\zeta) \right] dx\, dy \\
&= Kk \iint_C \left(\iint_{\gamma_r} |z-\zeta|^{-\varepsilon}\, dx\, dy \right) \omega^+(dE_\zeta) \\
&\leqq Kk \iint_C \left(\int_0^r 2\pi \varrho^{1-\varepsilon} d\varrho \right) \omega^+(dE_\zeta) \\
&= \frac{2\pi}{2-\varepsilon} K k r^{2-\varepsilon} \iint_C \omega^+(dE_\zeta) = k' r^{2-\varepsilon}\,.
\end{aligned}$$

We have thus proved that, for every point z_0 in R and every $\varepsilon > 0$, there exists a circle C with centre z_0 such that

$$w^+(\gamma_r) \leqq k' r^{2-\varepsilon} \tag{11}$$

for every circle γ_r in C.

Studying the regularity of $u^+(z)$ at a point z_0, we need only the values of $u^+(z)$ in a corresponding circle C. In the expression (10) for $u^+(z)$ we may also disregard the function $h(z)$, which has partial derivatives of the second order, and the constant $-(2\pi)^{-1}$. Instead of $u^+(z)$ we thus study

$$t(z) = \iint_C \ln|z-\zeta|\, \omega^+(dE_\zeta) = \tfrac{1}{2} \iint_C \ln[(x-\xi)^2+(y-\eta)^2]\, \omega^+(dE_{\xi+i\eta}) \,.$$

For the derivatives of $t(x,y)$ we get

$$t_x = \iint_C \frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}\, \omega^+(dE_{\xi+i\eta})$$

and an analogous expression for t_y, or in one formula

$$t_1 = t_x - it_y = \iint_C \frac{\bar{z}-\bar{\zeta}}{|z-\zeta|^2}\, \omega^+(dE_\zeta) = \iint_C \frac{\omega^+(dE_\zeta)}{z-\zeta}.$$

The differentiation under the integral sign may be justified by inverting the order of integration in

$$\iint_C \left(\int_{x_1}^{x_2} \frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}\, dx \right) \omega^+(dE_\zeta) \,.$$

We put $|z-z_0|=r$, suppose $4r<a$ (radius of C), and denote by C' the circular disk with centre z_0 and radius $2r$. Now we can estimate $t_1(z)-t_1(z_0)$ in the following way:

$$\begin{aligned} t_1(z)-t_1(z_0) &= \iint_C \left(\frac{1}{z-\zeta} - \frac{1}{z_0-\zeta} \right) \omega^+(dE_\xi) \\ &= \iint_{C'} \frac{1}{z-\zeta}\, \omega^+(dE_\zeta) - \iint_{C'} \frac{1}{z_0-\zeta}\, \omega^+(dE_\zeta) + \iint_{C-C'} \frac{z_0-z}{(z-\zeta)(z_0-\zeta)}\, \omega^+(dE_\zeta) \\ &= I_1+I_2+I_3 \,. \end{aligned}$$

For I_2 we get the estimate

$$|I_2| \leqq \iint_{C'} \frac{\omega^+(dE_\zeta)}{|z_0-\zeta|}.$$

The most unfavourable mass-distribution compatible with (11) is

(12) $$\omega^+(C_\varrho) = k'\varrho^{2-\varepsilon}$$

for every circle C_ϱ with centre z_0 and radius ϱ. This gives

$$|I_2| \leqq \int_0^{2r} k'(2-\varepsilon)\varrho^{1-\varepsilon}\varrho^{-1}\, d\varrho = \frac{k'(2-\varepsilon)}{1-\varepsilon}(2r)^{1-\varepsilon} = c_2 r^{1-\varepsilon} \,.$$

For I_1 we have an analogous estimate $|I_1| \leqq c_1 r^{1-\varepsilon}$. For I_3 we find

$$|I_3| = r\left|\iint\limits_{C-C'} \frac{\omega^+(dE_\zeta)}{(z-\zeta)(z_0-\zeta)}\right| \leqq 2r \iint\limits_{C-C'} \frac{\omega^+(dE_\zeta)}{|z_0-\zeta|^2}.$$

In this case the most unfavourable mass-distribution is given by (12) for $\varrho > 2r$ and has the mass $k'(2r)^{2-\varepsilon}$ concentrated on the circle $|\zeta - z_0| = 2r$. This yields

$$|I_3| \leqq 2r\left[k'(2r)^{2-\varepsilon}(2r)^{-2} + \int_{2r}^{a} k'(2-\varepsilon)\varrho^{1-\varepsilon}\varrho^{-2}\,d\varrho\right]$$

$$= k'(2r)^{1-\varepsilon} + 2r\frac{k'(2-\varepsilon)}{-\varepsilon}\left[a^{-\varepsilon}-(2r)^{-\varepsilon}\right] \leqq c_3 r^{1-\varepsilon}.$$

With these estimates of I_1, I_2 and I_3 we get

$$|t_1(z)-t_1(z_0)| \leqq |I_1|+|I_2|+|I_3| \leqq (c_1+c_2+c_3)r^{1-\varepsilon},$$

which is the desired Lipschitz condition ($\varepsilon > 0$ can be made arbitrarily small). (The estimation of $t_1(z)-t_1(z_0)$ can be carried out in a more elegant way by the method used by Carleson in [5, pp. 17–18 (II)].)

Remark 1. The result of b) also shows, that the superharmonic part $u^+(z)$ of any function $u(z) \in C'(K)$ has partial derivatives u_x^+ and u_y^+, which satisfy Lipschitz conditions of every order $1-\varepsilon < 1$.

Remark 2. The estimate—mentioned in the introduction—for the deviation of a curve AB from the geodesic AB has been proved for an arbitrary R_K by Alexandrow [3, p. 82]. For curves γ of given length l in an R_K, connecting endpoints with given goedesic distance r, the deviation is greatest when the R_K is of constant curvature K and γ consists of two geodesics of equal length $\frac{1}{2}l$. In my previous paper [6], I was not able to generalise this estimate to a metric (2) with $u \in C(K)$ for $K > 0$ (cf. theorem 1, p. 317, and theorem 3, p. 326, in [6]). Theorem 1 above shows that this case, too, is contained in Alexandrow's result.

I wish to thank Professors Carleson and Ganelius for valuable help and advice in the preparation of the manuscript. In particular, I am indebted to Carleson for the idea of part b) of the proof of theorem 2.

Appendix: On the notion of area in a 2-dimensional R_K.[1]

On p. 343 we used the fact, stated by A. D. Alexandrow in [2, p. 513], that a 2-dimensional R_K-space is of specific curvature $\leqq K$. Since Alexandrow's proof has not yet been published, a proof is given here with his

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consent. This proof is intimately connected with the notion of area in an R_K. Indeed, if ω and j as above denote curvature and area, respectively, we have to prove

$$\omega(E) \leqq Kj(E) \tag{6}$$

for any region E. According to the definition of an R_K, we know that for any triangle T the "excess relative K" $\delta_K(T)$ is $\leqq 0$. In the simplest case this is equivalent to $\omega(T) \leqq Kj(T^K)$, where T^K, as above (p. 339) and everywhere in the following, denotes a triangle in a "K-plane" (surface of constant curvature K) [3, p. 34] with sides of the same lengths as the sides of T. According to theorem 1, p. 71 in [3], $j(T^K) \geqq j(T)$. Then for $K \leqq 0$, the inequality (6) follows for certain triangles. But for $K > 0$ we need an estimate of the difference $j(T^K) - j(T)$ corresponding to the theorem on p. 399 in [2]. Therefore we must carry out a discussion corresponding to § 1 of chap. X in [2, pp. 391 ff.]. This will also lead to the conclusion that the notion of area defined in [3, pp. 70 f.], for a 2-dimensional R_K may be understood also in the stronger sense of [2, chap. X].

We shall need the following elementary estimate of the area of a triangle in a K-plane in terms of one angle and the greatest side.

LEMMA 1. *The area of a triangle in a K-plane with greatest side $d < |K|^{-\frac{1}{2}}$ and one angle v is less than vd^2.*

PROOF. The area of the triangle is at most $v/2\pi$ times the area of a circle with radius d. The area of this circle is for $K > 0$:

$$2\pi K^{-1}[1 - \cos(dK^{\frac{1}{2}})] < \pi d^2 ,$$

for $K < 0$:

$$2\pi K^{-1}[1 - \cosh(d|K|^{\frac{1}{2}})] = \pi d^2 \cosh(\theta d|K|^{\frac{1}{2}}) < \pi d^2 \cosh 1 ,$$

(and for $K = 0$: πd^2). In all cases we get even better estimates than stated in the lemma.

Now we begin the investigation corresponding to Alexandrow's. The curvature ω has here to be replaced by the excess relative K, that is

$$\delta_K(T) = \alpha + \beta + \gamma - \alpha^K - \beta^K - \gamma^K ,$$

where α, β, γ denote the angles of T and α^K, β^K, γ^K are the angles of the "corresponding triangle" T^K in a K-plane. Instead of the lemma of [2, p. 392] on triangles with polyedric metric, we have a lemma on triangles with "concavely K-polyedric" metric. We call an intrinsic metric of a 2-dimensional manifold S concavely K-polyedric, if every point of S has a neighbourhood, which is isometric to a cone in a space of constant curvature K, and the "full angle" [2, p. 38] of any point of S is $\geqq 2\pi$.

LEMMA 2. *If T is a triangle on a 2-dimensional manifold with concavely K-polyedric metric, $j(T)$ and $j(T^K)$ are the areas of T and the corresponding triangle T^K, respectively, $d < |K|^{-\frac{1}{2}}$ is the diameter of T and $\delta_K(T)$ is the relative excess of T, then*

$$\delta_K(T)d^2 \leqq j(T) - j(T^K) \leqq 0 . \tag{13}$$

PROOF. The right inequality is well known—even for any R_K (theorem 1, p. 71, in [3]). The problem is to prove the left inequality.

The difference between T and T^K is due to the presence of "conic points" in the interior and on the sides of T. For the following proof it is important to observe that no interior point has a full angle $\geqq 3\pi$. In fact, if T (with vertices A, B, C) should contain a point P with full angle $\geqq 3\pi$, $\sphericalangle APB$ or one of the analogous angles would have to be $\geqq \pi$. This is impossible.

Now any interior conic point P can be removed or displaced to the boundary of T by the following construction, (cf. fig. 1 and [2, pp. 394f.]).

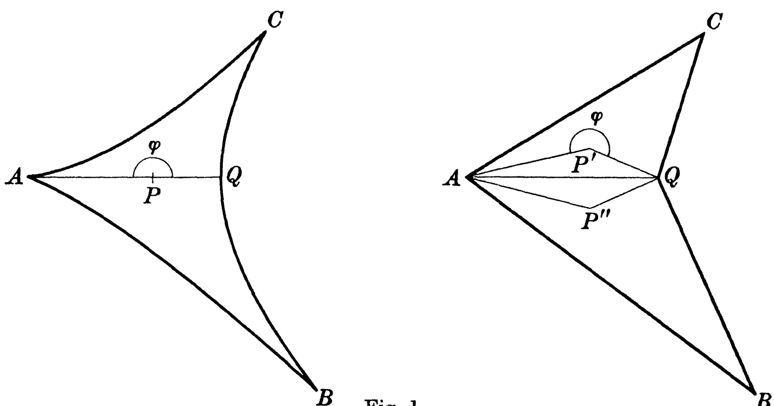

Fig. 1.

We connect P with one vertex A of T by the geodesic AP, and suppose that no other conic point is situated on AP. (Otherwise we begin with the first conic point on AP.) Then we draw a geodesic from P so that the angles which it forms with PA are equal, say φ. We have $\pi < \varphi < \frac{3}{2}\pi$. This geodesic is extended until it in Q meets either the side BC or a conic point. We make a cut along APQ and insert between its sides two triangles $AP'Q$ and $AP''Q$ from a K-plane with the sides $AP' = AP'' = AP$ and $P'Q = P''Q = PQ$ and the angles at P' and P'' equal to $2\pi - \varphi$. If then the sides AQ of the two triangles are identified, T is transformed into a new triangle T'. The excess — and because T^K is unchanged, also the excess relative K — of the triangle has increased as much as the angle at A, say by $2v$. The area of the triangle has increased by the two congruent triangles $AP'Q$ and $AP''Q$. According to lemma 1, either of these triangles has an area less than vd^2. Thus, if (13) should not be true for T, we would have

$$j(T')-j(T^K) < j(T)-j(T^K)+2vd^2 < \delta_K(T)d^2+2vd^2 = \delta_K(T')d^2,$$

that is, (13) could not hold for T' either. (The diameter of T' is also d, because for any not too large triangle in an R_K the diameter is the greatest side. This follows from the corresponding fact for the K-plane by theorem 2, p. 53, in [3].) The number of conic points in the interior of T' is less than in T. Repeating this process we can remove all conic points from the interior of T. Thus it will suffice to prove (13) for triangles T without conic points in the interior.

Such a triangle T may differ from T^K by the presence of extra vertices P, with angles $>\pi$, on the sides. Now any such vertex P can be removed by the following transformation of T into a new triangle T_1 (cf. fig. 2, and consider e.g. the case $K=0$ of the ordinary plane). If the vertices on AB are $A, P, D, \ldots$ in that order, we first extend DP to A' so that $PA'=PA$ (and the angle $DPA'=\pi$). Then we construct in a

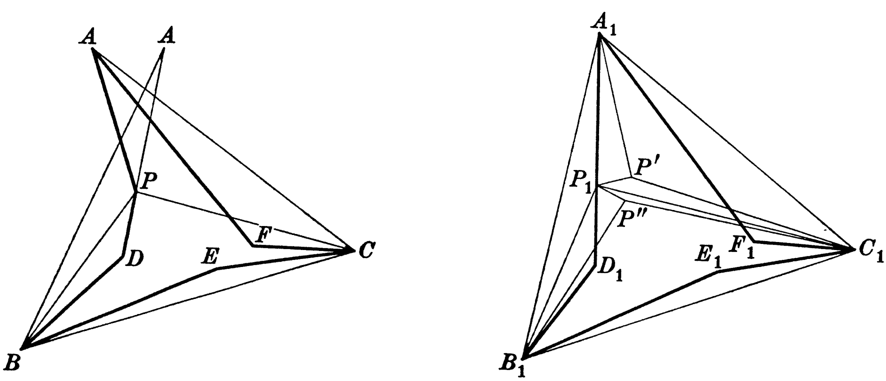

Fig. 2.

K-plane the triangle $A_1B_1C_1$ with sides equal to $A'B$, BC and CA. In this triangle the polygons $A_1P_1D_1\ldots B_1$, $B_1E_1\ldots C_1$ and $C_1F_1\ldots A_1$ are constructed congruent to $A'PD\ldots B$, $BE\ldots C$ and $CF\ldots A$, respectively. $A_1D_1\ldots B_1E_1\ldots C_1F_1\ldots A_1$ is our new triangle T_1, where the extra vertex P is removed but all other extra vertices remain with unchanged angles. Using arguments similar to the proof of lemma 2 in

[3, pp. 51f.], we find that each angle of T_1 is greater than the corresponding angle of T. In fact

$$\begin{aligned}
\sphericalangle BPA' > \sphericalangle BPA \; &\Rightarrow B_1A_1 = BA' > BA \\
&\Rightarrow \sphericalangle B_1C_1A_1 > \sphericalangle BCA \Rightarrow \sphericalangle C_1 > \sphericalangle C , \\
\sphericalangle APC > \sphericalangle A'PC \; &\Rightarrow A_1C_1 = AC > A'C \\
&\Rightarrow \sphericalangle A_1B_1C_1 > \sphericalangle A'BC \Rightarrow \sphericalangle B_1 > \sphericalangle B , \\
\sphericalangle P_1B_1C_1 > \sphericalangle PBC &\Rightarrow P_1C_1 > PC \\
&\Rightarrow \sphericalangle P_1A_1C_1 > \sphericalangle PAC \Rightarrow \sphericalangle A_1 > \sphericalangle A .
\end{aligned}$$

For comparison of the areas of T_1 and T we also construct in T_1 the triangles A_1C_1P' and B_1C_1P'' congruent to ACP and BCP, respectively. The difference Δj between the areas $j(T_1)$ and $j(T)$ is, in fact, equal to the difference between the sums $j(A_1C_1P_1)+j(B_1C_1P_1)$ and $j(ACP)+j(BCP)=j(A_1C_1P')+j(B_1C_1P'')$, because these sums differ from $j(T_1)$ and $j(T)$, respectively, by areas of polygons which are congruent in pairs. Δj is thus equal to the sum of the areas of the triangles A_1P_1P', C_1P_1P', C_1P_1P'' and B_1P_1P''. Estimating these areas by the formula $j < vd^2$ of lemma 1, we find

$$\Delta j < \psi d^2 ,$$

where $\psi = \sphericalangle P_1A_1P' + \sphericalangle P'C_1P'' + \sphericalangle P_1B_1P''$ is equal to the increase in excess $\delta(T_1)-\delta(T) = \delta_K(T_1)-\delta_K(T)$. Thus if (13) should not be true for T, we would have

$$j(T_1)-j(T^K) < j(T)-j(T^K)+\psi d^2 < \delta_K(T)\,d^2+\psi d^2 = \delta_K(T_1)d^2 ,$$

that is, (13) could not hold for T_1 either.

Repeating this process we can successively remove all extra vertices from T. We thus arrive at the result that if (13) did not hold for T, it could not hold for T^K either. But as (13) is obviously true for T^K, it must be true also for T, and thus the proof is completed.

Corresponding to the lemma on p. 396 in [2] we have the following result:

Lemma 3. *Let T be a triangle in a 2-dimensional R_K, $d < |K|^{-\frac{1}{2}}$ its diameter, $\delta_K(T)$ its excess relative K and $j(T^K)$ the area of the corresponding triangle in the K-plane. Then for any partition of T into triangles T_i, the sum of the areas $j(T_i{}^K)$ of the corresponding triangles in the K-plane satisfies the inequality*

$$\delta_K(T)d^2 \leqq \sum_i j(T_i{}^K)-j(T^K) \leqq 0 .$$

Proof. Let T be divided into triangles T_i. If each T_i is replaced by the corresponding triangle $T_i{}^K$ in a K-plane, the $T_i{}^K$ (connected in the same way as the T_i) constitute a polygon Q with K-polyedric metric. Any angle α_i in a T_i is not greater than the corresponding angle α_i^K in the T_i^K (theorem 4, p. 54, in [3]). This has three consequences:

1° For any interior vertex of Q the full angle is $\geqq 2\pi$. In fact, it is not smaller than the full angle of the corresponding point in T, and even this is $\geqq 2\pi$.

2° At any exterior vertex of Q which does not correspond to one of the three vertices of T the angle is $\geqq \pi$. This angle is not smaller than the corresponding angle in T, which also is $\geqq \pi$. (The "Schwenkung" [2, pp. 351 ff.] of a geodesic is non-positive [2, p. 498].) Thus, considering its interior metric Q is a triangle.

3° The relative excess of Q is not smaller than the relative excess of T: $\delta_K(Q) \geqq \delta_K(T)$.

Because of 1° and 2°, Q satisfies the assumptions of lemma 2, which yields

$$0 \geqq j(Q)-j(T^K) \geqq \delta_K(Q)d^2 \geqq \delta_K(T)d^2 , \tag{14}$$

because of 3° and since the diameter of Q is equal to the greatest side of $Q=$ the greatest side of $T=d$. However, (14) is the statement of our lemma, because $j(Q)=\sum_i j(T_i^K)$.

THEOREM. *Every triangle T in a 2-dimensional R_K has an area $j(T)$ in the following sense: Let T be divided into triangles T_i with diameters $\leqq d<|K|^{-\frac{1}{2}}$. Then if d tends to zero, the sum $\sum_i j(T_i^K)$ of the areas of the triangles T_i^K (in a K-plane and with sides of the same lengths as the sides of T_i) converges to the limit $j(T)$. More precisely the inequality*

$$0 \leqq \sum_i j(T_i^K)-j(T) \leqq -\delta_\varkappa(T)d^2$$

holds, where $\delta_\varkappa(T)$ is the excess relative $\varkappa$ and $\varkappa=\max(K,0)$.

PROOF. We first consider two arbitrary partitions of T into triangles T_i with diameters $\leqq d$ and into triangles T'_h with diameters $\leqq d_1<|K|^{-\frac{1}{2}}$, respectively, and prove that

$$\delta_\varkappa(T)d_1^2 \leqq \sum_i j(T_i^K)-\sum_h j(T_h'^K) \leqq -\delta_\varkappa(T)d^2 . \tag{15}$$

This is done by means of a common subdivision of the two partitions, that is, a partition of T into triangles T''_ν so that any T''_ν is at the same time contained in one T_i and one T'_h. According to lemma 3 we have for any T_i

$$\delta_K(T_i)d^2 \leqq \sum_\nu{}' j(T_\nu''^K)-j(T_i^K) \leqq 0 ,$$

where the sum $\sum'_\nu$ is extended over those ν for which T''_ν is contained in T_i. Adding for all T_i we find

(16) $$d^2 \sum_i \delta_K(T_i) \leqq \sum_\nu j(T_\nu''^K) - \sum_i j(T_i^K) \leqq 0 .$$

Between the excesses of T and the T_i we have the relation (obtained by considering the sum of all the angles of the T_i)

(17) $$\sum_i \delta(T_i) = \delta(T) - \sum_P \tau_P - \sum_Q \omega_Q \geqq \delta(T) ,$$

because both the "Schwenkung" τ_P of one side of T at a point P (vertex of some T_i) and the curvature ω_Q of an interior vertex Q are $\leqq 0$. Thus we have

$$\sum_i \delta_K(T_i) = \sum_i [\delta(T_i) - Kj(T_i^K)] \geqq \delta(T) - K \sum j(T_i^K) .$$

For $K \geqq 0$ we now use the fact that $\sum_i j(T_i^K) \leqq j(T^K)$. This is the right inequality of lemma 3 but does not depend on the assumption made there about the diameter of T. In fact, it follows directly from theorem 1 in [3, p. 71] (cf. the proof of lemma 3). We thus find

(18) $$\sum_i \delta_K(T_i) \geqq \delta(T) - Kj(T^K) = \delta_K(T) = \delta_\varkappa(T) .$$

If $K < 0$ we can replace δ_K by δ. We find by (17)

(19) $$\sum_i \delta_K(T_i) = \sum_i [\delta(T_i) - Kj(T_i^K)] \geqq \sum \delta(T_i) \geqq \delta(T) = \delta_\varkappa(T) .$$

Inserting (18) and (19) in the left member of (16) we get

$$d^2 \delta_\varkappa(T) \leqq \sum_\nu j(T_\nu''^K) - \sum_i j(T_i^K) .$$

Using

$$\sum_\nu j(T_\nu''^K) \leqq \sum_h j(T_h'^K) ,$$

which is the right inequality (16) for the partition into the triangles T_h', we get the right inequality (15). The left inequality is proved in the same way.

For a sequence of partitions P_n of T into triangles $T_\nu^{(n)}$ with diameters $\leqq d_n$, (15) yields, if we put $\sum_\nu j(T_\nu^{(n)K}) = \sum_n$

$$\delta_\varkappa(T) d_n^2 \leqq \sum_m - \sum_n \leqq -\delta_\varkappa(T) d_m^2 .$$

If $\lim_{n\to\infty} d_n = 0$, it follows that $\lim_{n\to\infty} \sum_n = a$ exists. Then if we use $T_\nu^{(n)}$ as the T_h' in (15) and let $n \to \infty$, we get

(20) $$0 \leqq \sum_i j(T_i^K) - a \leqq -\delta_\varkappa(T) d^2$$

for the arbitrary partition into triangles T_i. The definition of area in [3, pp. 70 ff.] may—as remarked there—for T be written

$$j(T) = \inf \underline{\lim}_{q\to\infty} \sum_i j(T_i^{(q)K})$$

where the $\underline{\lim}$ is taken for an arbitrary sequence of partitions P'_q into triangles $T_i^{(q)}$ with diameters $\leqq d_q$, with $\lim_{q\to\infty} d_q = 0$, and the inf is then taken over all such sequences of partitions. Now it follows from (20) both that $j(T) = a$ and that the inequality of the theorem holds.

COROLLARY. *Since, as just observed, $j(T) = \lim_{n\to\infty} \sum_\nu j(T_\nu^{(n)K})$, application of lemma 3 to the partitions P_n yields, if the diameter of T is $d < |K|^{-\frac{1}{2}}$, the following estimate for $j(T) - j(T^K)$:*

$$\delta_K(T) d^2 \leqq j(T) - j(T^K) \leqq 0 .$$

REMARK. The inequalities given here in the theorem, the corollary and the lemmas are not the best possible. E.g. we have not made use of the factor $\frac{1}{2}$ obtained in the proof of lemma 1 for $K \geqq 0$. Also the restriction $d < |K|^{-\frac{1}{2}}$ might be weakened.

APPLICATION. We can now prove that a 2-dimensional R_K (and thus also an R_K-space) is of specific curvature $\leqq K$ in the sense of [2, p. 431], that is,

$$\omega(E) \leqq K j(E) \tag{6}$$

holds for any region E in the R_K. We begin with the case $K \geqq 0$. According to the definition of curvature [2, p. 496], we have

$$\omega(E) \leqq \omega^+(E) = \sup \sum_i \delta(T_i) , \tag{21}$$

where the supremum is taken over all sets of non-overlapping triangles T_i which are contained in E, and $\delta(T_i)$ as above is the excess of T_i. For any T_i we have

$$\begin{aligned} \delta(T_i) - K j(T_i) &= \delta(T_i) - K j(T_i^K) + K[j(T_i^K) - j(T_i)] \\ &= \delta_K(T_i) + K[j(T_i^K) - j(T_i)] \\ &\leqq \delta_K(T_i) - K \delta_K(T_i) d_i^2 \\ &= \delta_K(T_i)(1 - K d_i^2) , \end{aligned}$$

because of our corollary above. If the diameter of E is $< K^{-\frac{1}{2}}$, the same holds for d_i (the diameter of T_i). Then $1 - K d_i^2 > 0$, and as $\delta_K(T_i) \leqq 0$ by the definition of an R_K [3, p. 36], we get $\delta(T_i) \leqq K j(T_i)$ and

$$\sum_i \delta(T_i) \leqq K \sum_i j(T_i) \leqq K j(E) ,$$

which proves (6) for all sufficiently small E. A larger region E can e.g. by a few geodesics g_m be divided into sufficiently small parts E_n. Since the set function $\omega_K = \omega - Kj$ is completely additive and $\omega_K(E_n) \leqq 0$, we have

$$\omega_K(E) = \sum_n \omega_K(E_n) + \sum_m \omega_K(g_m) \leqq \sum_m \omega_K(g_m) = \sum_m \omega(g_m) .$$

This is $\leqq 0$, because $\omega(g) \leqq 0$ for any geodesic g (cf. [2, p. 498]).

If $K \leqq 0$, we have for any triangle T

$$\delta(T) = \delta_K(T) + Kj(T^K) \leqq \delta_K(T) \leqq 0 ,$$

according to the definition of an R_K. The positive part of the curvature ω^+ is thus identically 0 (cf. (21)), and the definition of curvature [2, p. 496] for a region E is simplified to

$$\omega(E) = -\omega^-(E) = \inf \sum_i \delta(T_i) ,$$

where the infimum is taken over all sets of non-overlapping triangles T_i which are contained in E. From the premise $\delta_K(T_i) \leqq 0$ we get

$$\delta(T_i) \leqq Kj(T_i{}^K) \leqq Kj(T_i) ,$$

because $j(T_i) \leqq j(T_i{}^K)$ (theorem 1, p. 71, in [3]). This yields

$$\inf \sum_i \delta(T_i) \leqq \inf \sum_i Kj(T_i) = K \cdot \sup \sum_i j(T_i) .$$

But, by the usual definition of $j(E)$, we have $\sup \sum_i j(T_i) = j(E)$, and hence

$$\omega(E) = \inf \sum_i \delta(T_i) \leqq K \cdot \sup \sum_i j(T_i) = Kj(E) ,$$

which proves our statement.

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