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ASYMPTOTES AND PROJECTIONS OF CONVEX SETS

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We denote by $\mathfrak{C}$ the class of all closed convex subsets of a finite-dimensional Euclidean space E. The paper [2] studies those sets $C \in \mathfrak{C}$ which admit neither boundary ray nor asymptote, where an *asymptote* of C is a (closed) ray $\varrho \subset E \sim C$ such that $\delta(\varrho, C) = 0$. (We define $\delta(X, Y) = \inf_{x \in X, y \in Y} \|x-y\|$. In general, we follow the notation and terminology of [2] although explicit dependence on [2] is slight.) In the present note, we consider a different (but closely related) sort of asymptote. Although our discussion is quite elementary, it may throw some interesting light on the structure of unbounded convex sets.

An *f-asymptote* of $C \in \mathfrak{C}$ is a flat $F \subset E \sim C$ such that $\delta(F, C) = 0$; an f-asymptote of dimension j will be called a j-asymptote. The 1-asymptotes are merely the lines determined by asymptotes. The relationships among f-asymptotes of various dimensions are seen to be especially simple when C admits no boundary ray; in the general case, an unsolved problem is raised. The sets admitting no f-asymptotes are exactly those which have exclusively closed projections. The class $\mathfrak{G}$ of all such sets contains the class $\mathfrak{K}$ of all compact $K \in \mathfrak{C}$ and the class $\mathfrak{P}$ of all (not necessarily bounded) polyhedra $P \in \mathfrak{C}$; it turns out that $\mathfrak{G}$ resembles both $\mathfrak{K}$ and $\mathfrak{P}$ in certain important respects. Probably our most interesting result is that an f-asymptote of $C_1 \cap C_2$ must contain an f-asymptote of C_1 or an f-asymptote of C_2.

Let us commence with a simple but useful remark from [2]:

1.

LEMMA. *Suppose $C \in \mathfrak{C}$, $p \in C$, $q \in E \sim \{\theta\}$, and there are sequences x_α in C and t_α in $]0, \infty[$ such that $t_\alpha \to 0$ and $t_\alpha x_\alpha \to q$. Then C contains the ray $p + [0, \infty[q$.*

PROOF. Consider an arbitrary $r > 0$. Then for all $t_i \leqq 1/r$ we have

$$(1 - rt_i)p + rt_i x_i \in C ,$$

whence $p + rq \in C$.

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From Lemma 1 it follows easily that the line determined by an asymptote of C must be a 1-asymptote.

2.

Proposition. *Suppose $C \in \mathfrak{C}$ and F is an* f*-asymptote of C.* Then C *admits a boundary ray or asymptote parallel to F. And for each flat G which contains F and intersects C, the set $G \cap C$ contains a ray parallel to F.*

Proof. (The second part of Proposition 2 is quite similar to a separation theorem in [3].) We may assume that $\theta \in G \cap C$. Since F is an f-asymptote there exists sequences x_α in C and y_α in F such that $x_\alpha - y_\alpha \to \theta$, $\|x_\alpha\| \to \infty$, and $x_\alpha/\|x_\alpha\| \to x \in E \sim \{\theta\}$. Then $[0,\infty[x \subset C$ by Lemma 1. But $y_\alpha/\|x_\alpha\| \to x$, whence it follows that the ray $[0,\infty[x$ is parallel to F and thus is contained in G (for it is parallel to $F \subset G$ and includes the point $\theta \in G$). This proves the second assertion of Proposition 2 and implies the existence of a point $v \in C$ and a ray ϱ, emanating from θ and parallel to F, such that $v + \varrho \subset C$. Let $u \in F$ and let m denote the greatest lower bound of all numbers $t > 0$ such that C meets the ray $u + t(v-u) + \varrho$. Then the ray $(1-m)u + mv + \varrho$ must be a boundary ray or asymptote of C and is of course parallel to F. The proof is complete.

When E is n-dimensional ($E = E^n$), we denote the corresponding classes of sets by $\mathfrak{C}_n$, $\mathfrak{G}_n$, etc.

3.

Theorem. *Suppose $C \in \mathfrak{C}_n$, C admits no boundary ray, and $1 \leqq j \leqq k \leqq n-1$. Then every j-asymptote of C lies in a k-asymptote of C and is parallel to a 1-asymptote of C.*

Proof. For the first assertion, use the second part of Proposition 2 in conjunction with the fact that every f-asymptote of C lies in the bounding hyperplane of a closed halfspace which contains C. For the second assertion, use the first part of Proposition 2.

It is evident that a set $C \in \mathfrak{C}_n$ admits a j-asymptote if one of its $(n-j)$-projection fails to be closed, where a *k-projection* of C is the image of C under an affine projection of E onto a k-dimensional subflat. From Theorem 3 we see that if $C \in \mathfrak{C}_n$, C admits no boundary ray, and $1 \leqq k \leqq n-1$, then all of C's projections are closed if all its k-projections are closed. A similar situation occurs for a closed convex cone Y in E^n. Of course Y admits no $(n-1)$-asymptote, so all its 1-projections must be closed; but it is known for $2 \leqq k \leqq n-1$ that all of Y's projections are closed (and, in fact, Y is polyhedral) if all its k-projections are closed. (See [4]. Also [1], [5] for the case $k=2$.)

For an n-dimensional $C \in \mathfrak{C}_n$, let us denote by αC the set of all integers j between 1 and $n-1$ such that C admits a j-asymptote. When C admits no boundary ray, then $\alpha C = \emptyset$ or $\alpha C = \{1, \ldots, n-1\}$. When C is a cone, then $\alpha C = \emptyset$ or $\alpha C = \{1, \ldots, n-2\}$. It would be of interest to determine all possibilities for αC, but we have been unable to do this. To the obvious examples in E^3 we add the set

$$J = \{(x,y,z) \in E^3 \colon\ x \geqq 0,\ xy \geqq 1,\ z \geqq (x+y)^2\} .$$

It can be verified that $\alpha J = \{2\}$. Thus for $n \leqq 3$ it is true that every subset of $\{1, \ldots, n-1\}$ can be realized as the set αC for appropriately constructed n-dimensional $C \in \mathfrak{C}_n$. We do not know whether this is true for larger values of n. In particular, can the sets $\{1\}$ and $\{1,3\}$ be realized as αC for $C \in \mathfrak{C}_4$? (It is easy to verify that certain sets of consecutive integers can always be realized in E^n, namely the sets $\{j, j+1, \ldots, n-2\}$ for $1 \leqq j \leqq n-2$ and $\{j, j+1, \ldots, n-1\}$ for $1 \leqq j \leqq n-1$. When $n=4$, these cover all possibilities except $\{1\}$ and $\{1,3\}$.)

We mention also the set

$$Q = \{(x,y,z) \in E^3 \colon\ x \geqq 0,\ xyz = 1,\ z \geqq (x+y)^2\} .$$

Although Q admits no boundary ray and $\alpha Q = \{1,2\}$, there are 2-asymptotes of Q which contain no 1-asymptote. This is of interest relative to the second assertion of Theorem 3.

We next discuss f-asymptotes of the intersection of two sets. Consider the closed convex cone

$$C = \{(x,y,z) \in E^3 \colon\ x \geqq 0,\ y \geqq 0,\ xy \geqq z^2\};$$

let D denote the closed halfspace $\{(x,y,z) \colon z \leqq 1\}$ and F the plane $\{(x,y,z) \colon x = 0\}$. Then F is a 2-asymptote of the set $C \cap D$, although neither C nor D admits a 2-asymptote. However, we are able to prove

4.

THEOREM. *An f-asymptote of the intersection of two closed convex sets must contain an f-asymptote of one of them.*

PROOF. In effecting the proof, the following notion will be useful: a set Z is *associated with* a sequence p_α provided Z intersects every flat W for which

$$\liminf_{n \to \infty} \inf_{i \geqq n} \delta(W, \operatorname{conv}(Z \cup \{p_i\})) = 0 .$$

Now consider two closed convex sets X and Y, and an f-asymptote F of the set $X \cap Y$. Then $X \cap Y \cap F = \emptyset$, but there is a sequence p_α in F such that

$$\delta(p_\alpha, X \cap Y) = \varepsilon_\alpha \to 0 .$$

Let $U = X \cap F$ and $V = Y \cap F$. Consider a flat $W \subset F$ for which

$$\liminf \delta(W, \operatorname{conv}(U \cup \{p_\alpha\})) = 0 .$$

Since $p_i \in S(X, \varepsilon_i)$ $(= \{z : \delta(z, X) \leqq \varepsilon_i\})$ and $U \subset X$, we have

$$\operatorname{conv}(U \cup \{p_i\}) \subset S(X, \varepsilon_i) ,$$

and thus W must be an f-asymptote of X unless W intersects X. It follows that if F contains no f-asymptote of X, then the set U is associated with the sequence p_α. A similar statement holds for the set V, and thus the proof can be completed by showing that if two members of $\mathfrak{C}_n$ are associated with the same sequence, then they must intersect. We denote this assertion by A_n and observe that A_1 is easily verified.

Suppose A_{k-1} is known, and consider members C and D of $\mathfrak{C}_n$ which are associated with the same sequence q_α. If q_α is bounded, it has a convergent subsequence whose limit must lie in the set $C \cap D$. If q_α is unbounded, we may assume without loss of generality that $\|q_\alpha\| \to \infty$ and $q_\alpha / \|q_\alpha\| \to q \in E^n \sim \{\theta\}$. For $r > 0$ and $u \in C$ we have

$$\delta(u + rq, \operatorname{conv}(C \cup \{q_i\})) \leqq \big\| u + rq - (1 - r/\|p_i\|)u - rp_i/\|p_i\| \big\|$$

for all i, and since C is associated with q_α it follows that $u + rq \in C$. A similar argument holds for D, and we thus conclude

$$(*) \qquad C = C + [0, \infty[\, q \qquad \text{and} \qquad D = D + [0, \infty[\, q .$$

By use of (*) it can be verified that the sets πC and πD are both associated with the sequence πq_α, where π is a linear projection of E^k whose kernel is Rq. But then πC must intersect πD by the inductive hypothesis, and a second application of (*) shows that C intersects D. This completes the proof of Theorem 4.

Let $\mathfrak{N}$ denote the class of all continuous convex sets in E (the sets in $\mathfrak{C}$ which admit neither boundary ray nor asymptote), and let $\mathfrak{G}$, $\mathfrak{P}$, and $\mathfrak{K}$ be as defined earlier.

5.

THEOREM. *Let $\mathfrak{X}$ be any of the subclasses $\mathfrak{K}$ (compact), $\mathfrak{N}$ (continuous), $\mathfrak{P}$ (polyhedral), or $\mathfrak{G}$ (closed projections) of $\mathfrak{C}$. Then whenever $X, Y \in \mathfrak{X}$ it is true that*

$$X \cap Y \in \mathfrak{X} ,$$
$$X + Y \in \mathfrak{X} ,$$
$$\textit{if } X \cap Y = \emptyset \textit{ then } \delta(X, Y) > 0 .$$

PROOF. For the classes $\mathfrak{K}$, $\mathfrak{N}$, and $\mathfrak{P}$ this is already known [2, 4]. There remains the class $\mathfrak{G}$, for which the intersection condition follows from Theorem 4. Now consider the assertion

A_n: whenever X and Y are disjoint members of $\mathfrak{G}_n$, then $\delta(X, Y) > 0$. Clearly A_1 is true. Now suppose A_{k-1} is known and consider disjoint members X and Y of $\mathfrak{G}_k$. If $\delta(X, Y) = 0$, there are sequences x_α in X and y_α in Y such that $x_\alpha - y_\alpha \to \theta$. With $X \cap Y = \emptyset$, we may assume without loss of generality that $\|x_\alpha\| \to \infty$ and $x_\alpha / \|x_\alpha\| \to q \in E^k \sim \{\theta\}$, whence also $y_\alpha / \|x_\alpha\| \to q$. From Lemma 1 we see that $X = X + [0, \infty[\, q$ and $Y = Y + [0, \infty[\, q$; since $X \cap Y = \emptyset$ it follows that no translate of the line Rq can intersect both X and Y. Let π be the orthogonal projection of E^k whose kernel is Rq. Then πX and πY are disjoint members of $\mathfrak{G}_{k-1}$, hence at positive distance by the inductive hypothesis. But of course $\delta(\pi X, \pi Y) \leqq \delta(X, Y)$, and the contradiction completes the proof that A_{k-1} implies A_k. Thus A_n is valid for all n.

It remains only to show that if X, $Y \in \mathfrak{G}_n$, then $X + Y \in \mathfrak{G}_n$. We show first that $X + Y$ is closed. For consider an arbitrary point $p \in \mathrm{cl}(X + Y)$. Then $\theta \in \mathrm{cl}\big(X - (p - Y)\big)$ and hence $\delta(X, p - Y) = 0$. Since X, $p - Y \in \mathfrak{G}_n$, it follows from the above result that $X \cap (p - Y) \neq \emptyset$; that is, $x = p - y$ for some $x \in X$ and $y \in Y$. But then $p = x + y \in X + Y$ and $X + Y$ must be closed. Now for each linear projection π of $X + Y$ onto a j-dimensional subset of E^n, we have $\pi(X + Y) = \pi X + \pi Y$, where the sets πX and πY are both members of $\mathfrak{G}_j$. Thus the proof of Theorem 5 can be completed by an inductive argument.

The classes $\mathfrak{K}$ and $\mathfrak{R}$ have the additional property that if X and Y are in one of these classes, then so is the set $\mathrm{conv}(X \cup Y)$. This property is shared by the class of all bounded convex polyhedra in E, but not by the class $\mathfrak{P}$ of all polyhedra; however, if X, $Y \in \mathfrak{P}$ then $\mathrm{cl\,conv}(X \cup Y) \in \mathfrak{P}$. Now let

$$C = \{(x, y, z) \in E^3 :\ y \geqq x^2,\ z = 1\} .$$

Then for each $a \neq 0$ the line $\{(x, y, z):\ x = a,\ z = 0\}$ is a 1-asymptote of the set $\mathrm{cl\,conv}(\{\theta\} \cup C)$, although of course $\{\theta\}$, $C \in \mathfrak{G}$. Here again a special role is played by the sets which have no boundary ray, for we note

6.

Proposition. *Suppose X_1 and X_2 are closed convex sets in E^n and neither admits a boundary ray. Then every f-asymptote of the set* $\mathrm{cl\,conv}(X_1 \cup X_2)$ *is parallel to some $(n-1)$-asymptote of X_1 or X_2.*

Proof. Let F be an f-asymptote of the set $\mathrm{cl\,conv}(X_1 \cup X_2)$. By the basic separation theorem for convex sets there exists a linear functional φ on E^n and a number a_1 such that

$$\varphi F = a_1 = \sup \varphi\big(\mathrm{cl\ conv}(X_1 \cup X_2)\big) .$$

We assume without loss of generality that $\sup \varphi X_2 = a_2 \leqq \sup \varphi X_1$, whenc $\sup \varphi X_1 = a_1$. Now if $\varphi x_j < a_j$ for all $x \in X_j$, then the hyperplane $\varphi^{-1} a_j$ is an $(n-1)$-asymptote of X_j. Thus to prove Proposition 6 it suffices to derive a contradiction from the simultaneous existence of $p_1 \in X_1$ with $\varphi p_1 = a_1$ and $p_2 \in X_2$ with $\varphi p_2 = a_2$.

Since the hyperplane $\varphi^{-1} a_1$ contains the f-asymptote F of the set $\operatorname{cl} \operatorname{conv}(X_1 \cup X_2)$, there must exist sequences $x_\alpha{}^j$ in X_j and t_α in $[0, 1]$ such that

$$\varphi(t_\alpha x_\alpha{}^1 + (1-t_\alpha) x_\alpha{}^2) \to a_1 \qquad \text{and} \qquad \|t_\alpha x_\alpha{}^1 + (1-t_\alpha) x_\alpha{}^2\| \to \infty .$$

We may assume without loss of generality that

$$t_\alpha \to t \in [0,1] \qquad \text{and} \qquad x_\alpha{}^j / \|x_\alpha{}^j\| \to x^j \in E^n \sim \{\theta\} .$$

We consider two cases:

Case I: $t = 1$. Then $\varphi x_\alpha{}^1 \to a_1$ and $\varphi\big((1-t_\alpha) x_\alpha{}^2\big) \to 0$. If the sequence $x_\alpha{}^1$ is unbounded it follows that $\varphi x^1 = 0$ and then from Lemma 1 that the intersection $X_1 \cap \varphi^{-1} a_1$ contains the ray $p_1 + [0, \infty[x^1$, which must therefore be a boundary ray of X_1. If $x_\alpha{}^1$ is bounded, then $\|(1-t_\alpha) x_\alpha{}^2\| \to \infty$; with $m_i = (1-t_i)/\|x_i\|$, we have $m_\alpha \to 0$ and $m_\alpha x_\alpha{}^2 \to x^2 \in E^n \sim \{\theta\}$. Of course $\varphi x^2 = 0$ and by use of Lemma 1 we see that the ray $p_2 + [0, \infty[x^2$ is a boundary ray of X_2.

Case II: $t < 1$. When $t < 1$ we must have $a_1 = a_2$ and thus, in view of the case just treated, may assume that $t > 0$. But then $\varphi x_\alpha{}^1 \to a_1$, $\varphi x_\alpha{}^2 \to a_1$, and at least one of the sequences $t_\alpha x_\alpha{}^1$ and $(1-t_\alpha) x_\alpha{}^2$ is unbounded, whence a contradiction is reached as in Case I. The proof of Proposition 6 is complete.

For further properties of sets which admit no boundary ray, see [2, 3].

Reasoning as for certain "separation" theorems of [2], we can derive from Theorem 5 the following:

7.

Corollary. *Suppose $X_1, \ldots, X_m$ are members of $\mathfrak{G}$ which have empty intersection. Then there exist $\varepsilon > 0$ and closed halfspaces $Q_i \supset S(X_i, \varepsilon)$ such that the Q_i's have empty intersection.*

For the families $\mathfrak{K}$ and $\mathfrak{R}$, there are similar theorems involving infinite families of sets [2]. These cannot be extended to $\mathfrak{P}$ or $\mathfrak{G}$, as is seen from the following example in E^3: Let $P_1, P_2, \ldots,$ be a sequence of closed halfspaces whose intersection is the cone

$$C = \{(x,y,z) \colon x \geqq 0, \ y \geqq 0, \ xy \geqq z^2\} ,$$

and consider the line

$$P_0 = \{(x,y,z):\ x=0,\ z=1\}\,.$$

Then $\bigcap_0^\infty P_i = \emptyset$. But every closed halfspace which contains P_0 must intersect C, and consequently there are no closed halfspaces $Q_i \supset P_i$ such that $\bigcap_0^\infty Q_i = \emptyset$.

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