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CIRCUMSPHERES AND INNER PRODUCTS

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In the present note we show that among normed linear spaces of dimension $\geqq 3$, the inner-product spaces are characterized by certain conditions involving circumspheres or circumradii of sets. Most of the reasoning is finite-dimensional, and the results are apparently new even for three-dimensional spaces. Our principal theorem asserts that *for a normed linear space E, the following three conditions are equivalent: E is an inner-product space or is two-dimensional; if a subset Y of E lies in a cell of radius < 1, then it lies in some cell of unit radius centered at a point of* $\operatorname{conv} Y$; *if a subset Z of E lies in a cell of radius < 1, then it is intersected by every cell of unit radius centered at a point of* $\operatorname{conv} Z$. In terminology of E. A. Michael [4], the third condition asserts that the space E is 1-paraconvex.

Consider a metric space M with distance function ϱ. For $p \in M$ and $r > 0$, we denote by $S(p,r)$ the *cell* in M having center p and radius r, that is,

$$S(p,r) = \{q \in M:\ \varrho(p,q) \leqq r\}.$$

For a bounded subset X of M, the *M-radius* $r_M X$ is the greatest lower bound of all numbers r such that $X \subset S(p,r)$ for some $p \in M$. An *M-center* of X is a point $y \in M$ such that $X \subset S(y, r_M X)$; the set of all such centers will be denoted by $C_M X$. For sets in normed linear spaces, these and related notions have been discussed by Brodskiĭ and Milman [1], Routledge [5], and the author [5].

For a bounded *convex* set X in a normed linear space E, two cases are of special interest in the above setting, namely the cases $M = X$ and $M = E$. It is natural to wonder whether $r_X X = r_E X$, for such is the case in Euclidean n-space. We shall prove that in three or more dimensions, the inner-product spaces are characterized by this equality, although the equality subsists whenever E is two-dimensional. Also obtained are some similar results involving centers of convex sets.

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1.

THEOREM. *For a normed linear space E, the following four assertions are equivalent:*

(i) *E is an inner-product space or is two-dimensional;*

(ii) *whenever $\varepsilon > 0$ and X is a convex subset of the unit cell U of E, then U contains a translate of X whose distance from the origin is $< \varepsilon$;*

(iii) *for each bounded convex subset X of E, $r_X X = r_E X$;*

(iv) *for each two-dimensional plane P in E and each bounded convex subset X of P, $r_P X = r_E X$.*

PROOF. We show first that (i) implies (ii). Suppose condition (i) holds and X is a convex subset of U. Let K denote the union of all translates of X which lie in U. Then K is convex, and in order to establish (ii) we wish to show that the closure of K includes the origin θ. Suppose θ is not in the closure of K. Then by the basic separation theorem for convex sets there exists $f \in E^*$ (the conjugate space of E) such that $\|f\| = 1$ and $\inf fK = a > 0$. Let H denote the hyperplane $f^{-1}0$ and let $Q = f^{-1}[a, \infty[$. We claim there exists a continuous linear projection π of E onto H which maps the set $Q \cap U$ into $H \cap U$. In the two-dimensional case, the kernel $\pi^{-1}\theta$ of π is a line through θ which is parallel to a line supporting U at an endpoint of the segment $H \cap U$. Here it is trivial to check that $\pi(Q \cap U) \subset H \cap U$. When E is an inner-product space, the projection π is obtained as follows: Let $\bar{E}$ be the completion of E and x a point of $\bar{E} \sim \{\theta\}$ which is orthogonal to $\bar{H}$. Then under the projection of $\bar{E}$ onto $\bar{H}$ whose kernel is the line Rx, the set $\bar{Q} \cap \bar{U}$ maps into $t(\bar{H} \cap \bar{U})$ for some $t \in]0, 1[$. Thus for $z \in E$ and z sufficiently close to x, the set $\bar{Q} \cap \bar{U}$ maps into $\bar{H} \cap \bar{U}$ under the projection of $\bar{E}$ onto $\bar{H}$ whose kernel is Rz. Denoting this projection by π, we see at once that $\pi E = H$ and consequently $\pi(Q \cap U) \subset H \cap U$.

Now the kernel of π has the form Rw for some $w \in f^{-1}1$. And by the definitions of a and of K, there exist a translate Y of X and a point q of Y such that $Y \subset K$ and $fq < 3a/2$. Each point $y \in Y$ admits a unique expression in the form $y = y_1 + y_2 w$ with $y_1 = \pi y \in H$ and $a \leqq y_2 \in R$. Then for arbitrary $y \in Y$ we have

$$y - aw = y_1 + (y_2 - a)w \in [y, \pi y] \subset U,$$

whence $Y - aw \subset K$. But $f(q - aw) < a/2$, contradicting the fact that $a = \inf fK$. It follows that (i) implies (ii).

Now to prove that (ii) implies (iii) it suffices to show that $r_X X = 1$ for each convex subset X of U. Consider an arbitrary $\varepsilon > 0$. Condition (ii) implies the existence of points $p_\varepsilon \in E$ and $x_\varepsilon \in X$ such that $X + p_\varepsilon \subset U$

and $\|p_\varepsilon + x_\varepsilon\| < \varepsilon$. But then for each $u \in U$ we have $\|u-(p + x_\varepsilon)\| < 1+\varepsilon$, and since $X \subset U - p_\varepsilon$ it follows that $X \subset S(x_\varepsilon, 1+\varepsilon)$ and consequently $r_X X \leqq 1+\varepsilon$. Thus (ii) implies (iii).

It is evident that (iii) implies (iv) and remains only to prove that (iv) implies (i).

Now let us suppose that (iv) holds and E is at least 3-dimensional. To show that E is an inner-product space it suffices, in view of known characterizations (Jordan–von Neumann [2], Kakutani [3]), to show that if J is a 3-dimensional subspace of E and P is a 2-dimensional subspace of J, then J admits a linear projection of norm 1 onto P. Let T be the class of all translates of P in $J \sim P$, and for each $t \in T$ let $X_t = t \cap U$. Then $r_E X_t \leqq r_J X_t \leqq 1$,and from condition (iv) it follows that $r_t X_t \leqq 1$, whence (by bounded compactness of t) there exists $p_t \in t$ with $X_t \subset S(p_t, 1)$. (Though p_t may not be unique, that fact causes no trouble. Any definite choice of p_t is satisfactory for our purpose, so long as $p_t \in t$ and $X_t \subset S(p_t, 1)$.) Let π_t be the projection of J onto P whose kernel is the line Rp_t, so that $\pi_t X_t \subset P \cap U$. Let Y_t be the intersection with U of the halfspace in J which is bounded by t and misses P. We claim that $\pi_t Y_t \subset U$, and justify this as follows: Consider an arbitrary point $z \in Y_t \sim t$, and suppose $\pi_t z \notin P \cap U$. Let v be the point of the segment $[\theta, \pi_t z] \cap U$ which is nearest to $\pi_t z$ (that is, the endpoint other than θ) and let $w \in [v,z] \cap t$. Then $w \in X_t$ and hence $\pi_t w \in U$. But it is clear also that $\pi_t w \in \,]v, \pi_t z[$, contradicting the choice of v. Thus $\pi_t Y_t \subset U$.

Now let q be a fixed point of $U \sim P$. For all t ($\in T$) sufficiently close to P it is true that $q \in Y_t \cup - Y_t$ and hence $\pi_t q \in U$. Thus there is in T a sequence t_α converging to P for which the sequence $\pi_{t_\alpha} q$ converges to a point $q_0 \in P \cap U$. Let π be the linear projection of J onto P which takes q onto q_0. Then π_{t_α} converges to π_0. Since always $\pi_{t_i} Y_{t_i} \subset U$, and since

$$\bigcup_i Y_{t_i} \cup \bigcup_i (-Y_{t_i}) = (J \sim P) \cap U,$$

it follows that $\pi(J \cap U) = P \cap U$. Consequently π is of norm 1 and the proof of Theorem 1 is complete.

It may happen that $r_X X = r_E X$ even though both of the sets $C_X X$ and $C_E X$ are empty. (For example, let E be the Euclidean plane and X an open half-disc in E; or let E be an incomplete inner-product space and X the intersection with E of a cell centered in $\bar{E} \sim E$.) But clearly $X \cap C_E X \subset C_X X$, and thus when $C_E X$ intersects X it follows that $r_X X = r_E X$; for compact X, the reverse implication holds. The following two assertions are equivalent to (1i)–(1iv) above: (ii′) *each compact convex set* $X \subset U$ *admits a translate* X' *for which* $\theta \in X' \subset U$; (iii′) *for all* *compact convex* $X \subset E$, $C_E X$ *intersects* X. Completeness of the inner-product space is needed to produce condition (iii′) for all bounded closed convex $X \subset E$. In fact, the following two results are easy consequences of Theorem 1.

2.

COROLLARY. *For a normed linear space* E, *the following three assertions are equivalent*:

(i) E *is a complete inner-product space or is two-dimensional*;

(ii) *each closed convex set* $X \subset U$ (*the unit cell*) *admits a translate* X' *for which* $\theta \in X' \subset U$;

(iii) *for each bounded closed convex* $X \subset E$, $C_E X$ *intersects* X.

3.

COROLLARY. *For a normed linear space* E, *the following two assertions are equivalent*:

(i) E *is a complete inner-product space or is two-dimensional and strictly convex*;

(ii) *for each bounded closed convex* $X \subset E$, $C_E X$ *is a nonempty subset of* X.

And of course we may replace "bounded closed" by "compact" in (3ii) provided "complete" is omitted in (3i).

Our principal result (stated in the first paragraph) rests on Theorem 1 and on the following:

4.

PROPOSITION. *Suppose* J *is a three-dimensional normed linear space with unit cell* U, P *is a plane through the origin* θ *in* J, *and* K *is the intersection of* U *with one of the closed halfspaces bounded by* P. *Let* π *be a projection of smallest possible norm of* J *onto* P. *Then either* $\pi K \subset K$ (*that is,* $\|\pi\| = 1$) *or* θ *is interior to the convex hull of* $\pi K \sim K$ (*relative to* P).

PROOF. Let $C = U \cap P$. Then $\pi K \subset \|\pi\| C$, and the set $\tau K \sim mC$ is nonempty for each $m \in [0, \|\pi\|[$ and each projection τ of J onto P. Suppose $\theta \notin \operatorname{int} \operatorname{conv}(\pi K \sim K)$, whence there is a line L through θ such that $\pi K \sim K$ lies in one of the closed halfplanes in P bounded by L. Let w and $-w$ be the endpoints of the segment $L \cap C$. Let $z \in P \sim \{\theta\}$ be such that $\pi K \sim K \subset L + [0, \infty[z$ and such that C is supported at w by the line $w + Rz$; if there is more than one supporting line of C at w, choose z further so that $(w + Rz) \cap C = \{w\}$. We claim that

$$\pi K \sim K \subset [-rw, rw] + [0, \infty[z \qquad \text{for some } r < \|\pi\|,$$

that is, that πK lies in the *open* strip bounded by the lines $\|\pi\|w+Rz$ and $-\|\pi\|w+Rz$. To establish this, we consider two cases:

Case 1: *C is smooth at w.* Then $w+Rz$ is the only line which supports C at w. If y is any point of the halfplane $]1,\infty[\,w+Rz$, there is a point $y' \in C$ for which the segment $[y,y']$ intersects the set $(L+]-\infty,0[z) \sim C$. But of course πK is a convex set containing C, and

$$\pi K \sim K = \pi K \sim C \subset L+[0,\infty[z ,$$

so it follows that the point y as described cannot lie in πK. The same reasoning involving $-w$ leads, then, to the conclusion that

$$\pi K \sim K \subset [-w,w]+[0,\infty[z.$$

Case 2: *C is not smooth at w.* Suppose the desired conclusion fails. Then, since $\pi K \subset \|\pi\|C$ and since the set $\|\pi\|C$ is intersected only at $\|\pi\|w$ by the line $\|\pi\|w+Rz$, it follows from compactness of πK that $\|\pi\|w \in \pi K$. But πK is convex and contains C, so this contradicts the fact that $\pi K \sim K \subset L+[0,\infty[z$.

Now let r be as described above, and let

$$(1) \qquad W = C \cup (\|\pi\|C) \cap ([-rw,rw]+[0,\infty[z) \supset \pi K .$$

It is easy to verify the existence of $b>0$ and $m_1 \in]0,\|\pi\|[$ such that $W-bz \subset mC$. For $\varepsilon \in]0,b[$ we have

$$W-\varepsilon z = \frac{\varepsilon}{b}(W-bz)+\left(1-\frac{\varepsilon}{b}\right)W \subset \frac{\varepsilon}{b}m_1C+\left(1-\frac{\varepsilon}{b}\right)\|\pi\|C = m_\varepsilon C ,$$

where

$$m_\varepsilon = \|\pi\|-\frac{\varepsilon}{b}(\|\pi\|-m_1) \in]0,\|\pi\|[.$$

It follows that

$$(2) \qquad W-[\varepsilon,b]z \subset m_\varepsilon C \qquad \text{for all} \quad \varepsilon \in]0,b[.$$

Let f be the linear functional on J for which $f^{-1}0=P$ and $\sup fK=1$. Let $q \in f^{-1}1$ with $\pi q=\theta$, so that the line Rq is the kernel of π. For each $t>0$, let π_t be the projection of J onto P whose kernel is the line $R(q+tz)$. It can be verified that

$$(3) \qquad \pi_t x = \pi x-t(fx)z \qquad \text{for all} \quad x \in J .$$

We will show that $\|\pi_t\|<\|\pi\|$ for a sufficiently small $t>0$, a contradiction completing the proof of Proposition 4.

Since the section $S_a=K\cap f^{-1}a$ varies continuously with $a \in [0,1]$, and since π is continuous and $\pi S_0=S_0=C$, it is clear that there exists $\delta \in]0,1[$ and $m' \in]1,\|\pi\|[$ such that

$$\pi S_a + [0,\delta](-z) \subset m'C \qquad \text{for all} \quad a \in [0,\delta] .$$

Now choose $t \in]0,1[$ such that $t < b$. For $x \in K \cap f^{-1}[0,\delta]$ we have $x \in S_a$ for some $a \in [0,\delta]$, and $t(fx) \in [0,\delta]$, whence (using (3))

$$\pi_t x = x - t(fx)z \subset \pi S_a + [0,\delta](-z) \subset m'C .$$

And for $x \in K \cap f^{-1}[\delta,1]$, we see by (1)–(3) that

$$\pi_t x = \pi x - t(fx)z \subset \pi K - [t\delta,t]z \subset W - [t\delta,b]z \subset m_{t\delta}C .$$

Thus $\pi_t K \subset mC$ for $m = \min(m', m_{t\delta}) < \|\pi\|$, whence $\|\pi_t\| < \|\pi\|$ and the proof is complete.

With J, P, and K as in Proposition 4, it follows easily that *the following two assertions are equivalent*: *J admits no projection of norm* 1 *onto P*; *J admits a projection π onto P for which $\theta \in \operatorname{int}\operatorname{conv}(\pi K \sim K)$.*

5.

THEOREM. *For a normed linear space E, the following three assertions are equivalent*:

(i) *E is an inner-product space or is two-dimensional*;

(ii) *if a subset Y of E lies in a cell of radius* <1, *then Y lies in some cell of unit radius centered at a point of* conv Y;

(iii) *if a subset Z of E lies in a cell of radius* <1, *then Z is intersected by every cell of unit radius centered at a point of* conv Z.

PROOF. Equivalence of conditions (i) and (ii) is immediate from Theorem 1 as applied to the set $X = \operatorname{conv} Y$.

Now suppose (i) holds and let U denote the unit cell of E, $y \in E \sim \{\theta\}$. To establish (iii) it suffices to show that $\theta \notin \operatorname{conv}\big((y+U) \sim U\big)$. When E is two-dimensional, let L be a line through θ such that U admits supporting lines parallel to Ry at the endpoints of the segment $L \cap U$. It can be verified that $(y+U) \sim U$ lies in the open halfplane $L +]0,\infty[\,y$ and hence $\theta \notin \operatorname{conv}\big((y+U) \sim U\big)$. When E is an inner-product space, let H be the orthogonal supplement of the line Ry. From the two-dimensional result in conjunction with the symmetry of U about the line Ry it follows that $(y+U) \sim U \subset H +]0,\infty[\,y$. Consequently (i) implies (iii).

Finally, suppose condition (iii) holds an E is at least 3-dimensional. To prove that E is an inner-product space it suffices [2, 3] to show that if J is a 3-dimensional subspace of E, P a 2-dimensional subspace of J, and π a projection of smallest possible norm of J onto P, then $\|\pi\| = 1$. Suppose $\|\pi\| > 1$ and let U and K be as in Proposition 4, so that

$\theta \in \operatorname{int} \operatorname{conv}(\pi K \sim K)$. Let q be the nonzero endpoint of the segment $K \cap \pi^{-1}\theta$, so that Rq is the kernel of π.

There are a finite set $V = \{v_1, \ldots, v_n\} \subset \pi K \sim K$ and a neighborhood N of θ in P such that $N \subset \operatorname{conv} V$. For each i, the ray $v_i + [0, \infty[\, q$ intersects K, and since $v_i \notin K$ there exists $s_i > 0$ such that $v_i + s_i q \in K$ but $v_i + tq \in J \sim K$ for all $t \in [0, s_i[$. Let $w_0 = -q$, $w_i = v_i + s_i q$ for $i = 1, \ldots, n$, and $W = \{w_0, w_1, \ldots, w_n\} \subset U$. We claim that θ is interior to conv W relative to J. Let $\alpha = \min\{s_1, \ldots, s_n\}$, $\beta = \max\{s_1, \ldots, s_n\}$, and $\gamma = \alpha/(\alpha + \beta)$. Since $\theta \in N \subset \operatorname{conv} V$, there are numbers $a_i \geqq 0$ such that $\sum_1^n a_i = 1$ and $\sum_1^n a_i v_i = \theta$; then

$$\operatorname{conv} W \ni \sum_1^n a_i w_i = \left(\sum_1^n a_i s_i\right) q .$$

But of course $\sum_1^n a_i s_i \geqq \alpha$, and since $-q \in W$ it follows that $[-q, \alpha q] \subset \operatorname{conv} W$. We will now show that $\gamma N \subset \operatorname{conv} W$; since $\operatorname{conv}(\gamma N \cup [-q, \alpha q])$ is a neighborhood of θ in J, this will show that $\theta \in \operatorname{int} \operatorname{conv} W$ relative to J. Consider an arbitrary $x \in N \subset \operatorname{conv} V$, say $x = \sum_1^n b_i v_i$ with $b_i \geqq 0$ and $\sum_1^n b_i = 1$. Then

$$\begin{aligned} \gamma x &= \sum_1^n (\gamma b_i) w_i - \sum_1^n (\gamma b_i s_i) q \\ &= \sum_1^n (\gamma b_i) w_i + \left(1 - \sum_1^n (\gamma b_i)\right) \frac{\gamma \sum_1^n b_i s_i}{1-\gamma} w_0 . \end{aligned}$$

But $\gamma = \alpha/(\alpha + \beta)$, so

$$0 < \frac{\gamma \sum_1^n b_i s_i}{1-\gamma} \leqq \frac{\gamma\beta}{1-\gamma} = \alpha ,$$

and since $[-q, \alpha q] \subset \operatorname{conv} W$ it follows that $\gamma x \in \operatorname{conv} W$.

Now since θ is interior to conv W, there exists $\varepsilon > 0$ such that $\theta \in \operatorname{conv}\{y_0, \ldots, y_n\}$ whenever $y_i \in J$ with always $\|y_i - w_i\| \leqq \varepsilon$. Let $\delta = \min(\varepsilon/2\|q\|, \alpha)$ and let $x_i = w_i - \delta q$ for $i = 0, \ldots, n$. Then $\|x_i - w_i\| = \delta\|q\| \leqq \varepsilon/2$. Since $x_0 = -(1+\delta)q$, and $x_i = v_i + (s_i - \delta)q$ with $0 \leqq s_i - \delta < s_i$ for $i = 1, \ldots, n$, it follows that always $x_i \in J \sim U$. Thus for $\mu \in]0, 1[$ and μ sufficiently close to 1 it is true that always $\mu x_i \in J \sim U$ and $\|\mu x_i - x_i\| \leqq \varepsilon/2$, whence $\|\mu x_i - w_i\| \leqq \varepsilon$ and $\theta \in \operatorname{conv}\{\mu x_0, \ldots, \mu x_n\}$. But all the points μx_i lie in the cell $\mu U - \mu\delta q$ of radius $\mu < 1$ and it then follows from condition (iii) that some point μx_i must lie in U. This is a contradiction which completes the proof of Theorem 5.

In closing, we note that condition (5iii) above is equivalent to (5iii′): *if a subset Z of E lies in a cell of radius equal to* 1, *the Z is intersected by* *every cell of unit radius centered at a point of* conv Z. (Or, in other words, if $Z \subset S(p, 1)$ then conz $Z \subset S(Z, 1)$.) For clearly (5iii′) implies (5iii), and the discussion above actually showed that (5i) implies (5iii′).

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