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PARTIALLY HYPOELLIPTIC DIFFERENTIAL EQUATIONS OF FINITE TYPE

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Introduction.

Consider the partial differential equation

$$P(D)u = 0, \tag{1}$$

where $P(D)$ is a linear partial differential operator with constant coefficients, operating on functions $u=u(x)$ with $x \in R^n$. The characteristic polynomial belonging to $P(D)$ can be defined by

$$P(\xi) = e^{-i\langle x, \xi\rangle} P(D)\, e^{i\langle x, \xi\rangle},$$

where ξ is a vector in the dual space to R^n. It is convenient to introduce the set

$$V(P) = \{\zeta \mid P(\zeta)=0\}$$

of all complex zeroes of the characteristic polynomial. As is well known the shape of the set $V(P)$ can be used to characterize important classes of equations (1) with regard to properties shared by all their solutions. Thus, let A be a class of functions, closed for differentiation. Then it may be possible to prove that A contains all solutions of (1) if and only if the set $V(P)$ satisfies a certain algebraic condition. Those operators $P(D)$ that only have solutions in the class A will in their turn constitute a new class, which we denote by $\tilde{A}$.

When A is the class $\mathscr{E}$ of all regular (i.e. indefinitely differentiable) functions, then the operators in $\tilde{A}$ are called hypoelliptic operators. An important result by Hörmander [3] is that $P(D)$ is hypoelliptic if and only if

$$\operatorname{Im}\zeta \to \infty \quad \text{when} \quad \zeta \to \infty \text{ and } \zeta \in V(P). \tag{2}$$

When A is the class of all analytic functions of the real variable $x \in R^n$, then the operators in $\tilde{A}$ are called elliptic operators.

Petrowsky [8] has proved that $P(D)$ is elliptic if and only if the principal part of $P(\xi)$ has no real zeroes outside the origin. It is easy to see (Hörmander [5]) that this condition is equivalent to

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(3) $$|\mathrm{Re}\,\zeta| \leqq C(1+|\mathrm{Im}\,\zeta|) \qquad \text{for all } \zeta \in V(P)\,.$$

Evidently (3) implies (2) so that every elliptic operator is also hypoelliptic. (Of course all solutions of (1) are regular if they are already known to be analytic.)

The reverse statement is not true, but it may be replaced by a weaker result. This is based on the observation that the condition (2) for hypoellipticity can be used to prove the following inequality, of the same type as (3):

(4) $$|\mathrm{Re}\,\zeta| \leqq C(1+|\mathrm{Im}\,\zeta|)^a \qquad \text{for all } \zeta \in V(P)\,,$$

where a is a constant $\geqq 1$, depending on P. Using (4) to make explicit estimates of a fundamental solution of (1), it is then possible to show (Hörmander [3]) that every solution of (1) belongs to the "hypoanalytic" class A_a, defined as follows. A_1 is the class of all analytic functions of the real variable $x \in R^n$. A_a consists of all functions $u(x)$ such that for every compact set $K \subset R^n$

(5) $$|D^k u(x)| \leqq C^{k+1}(k!)^a, \qquad k = 0,1,\ldots,\ x \in K\,,$$

where $|D^k u|^2$ stands for the sum of the squares of all derivatives of u of order k and where C is a constant depending on K and u. Since for a given $P(D)$ satisfying (4) the inequality (5) is best possible, it follows that $\tilde{A}_a$ consists of all operators $P(D)$ satisfying (4). The heat conduction operator is a classical example of an operator in $\tilde{A}_2$.

Hörmander has also obtained a more precise result. Let y be a given vector in R^n. If $P(D)$, in addition to (4), satisfies the relation

(6) $$|\langle y, \mathrm{Re}\,\zeta\rangle| \leqq C(1+|\mathrm{Im}\,\zeta|)^{b(y)} \qquad \text{for all } \zeta \in V(P)\,,$$

for some constant $b(y)$ with $1 \leqq b(y) \leqq a$, then (5) can be complemented by the better estimate

(7) $$|\langle y, D\rangle^k u(x)| \leqq C^{k+1}(k!)^{b(y)}, \qquad k = 0,1,\ldots,\ x \in K\,.$$

The estimate is again in a sense the best possible.

Using a method of Schwartz one can extend the methods of Hörmander to cover the case of an inhomogeneous equation $P(D)u = f$, if f is at least as regular as one wants u to be (Schwartz [9]).

A generalization of the ideas above has recently been given by Gårding and Malgrange [1] [2]. Suppose that there is given a decomposition $R^n = R^{n-m} \times R^m$ of R^n, so that the vectors in R^n can be written as $x = (x', x'')$ with $x' \in R^{n-m}$, $x'' \in R^m$. A distribution v in $\mathscr{D}_{x',x''}$ is then called regular in x', if it is a regular function in x' with values in $\mathscr{D}'_{x''}$, i.e. if $\int v(x',x'')\varphi(x'')dx''$ is a function in $\mathscr{E}_{x'}$ for every φ in $\mathscr{D}_{x''}$. If v is regular in x' and if furthermore $\int v(x',x'')\varphi(x'')dx''$ is an analytic function of x' for every $\varphi \in \mathscr{D}_{x''}$, then v is called analytic in x'. A necessary and sufficient condition for $P(D)$ to have all solutions of $P(D)u=0$ analytic in x' is that (with obvious notations)

$$(8) \qquad |\mathrm{Re}\,\zeta'| \leqq C(1+|\mathrm{Im}\,\zeta|), \qquad \zeta = (\zeta',\zeta'') \in V(P)\,.$$

$P(D)$ is then called partially elliptic in x'.

Similarly, with A equal to the class of functions regular in x', $\tilde{A}$ is the class of all operators that are partially hypoelliptic in x'. These are characterized by the algebraic condition

$$(9) \qquad |\mathrm{Re}\,\zeta'| \leqq C(1+|\mathrm{Im}\,\zeta|+|\mathrm{Re}\,\zeta''|)^a, \qquad \text{for some } a \text{ and all } \zeta \in V(P)\,.$$

The results of Gårding and Malgrange are obtained without the aid of fundamental solutions. Instead are used simple estimates of suitably chosen norms. Analogous norms have been used for similar purposes by e.g. Morrey and Nirenberg [7] and Hörmander [6].

We are now going to show how the methods of Gårding and Malgrange can be used to get results concerning estimates of the high order derivatives of solutions of partially hypoelliptic equations. An advantage of the method is that without modifications it yields results also in the case of nonhomogeneous equations. The idea to make this investigation was given to me by L. Gårding, who also put the manuscript of the paper [2] at my disposal.

1. Polynomials of finite type.

The operators that we are going to study are those with characteristic polynomials "of finite type in ξ'", i.e. satisfying

$$(10) \qquad |\mathrm{Re}\,\zeta'| \leqq C(1+|\mathrm{Im}\,\zeta|)^{b'}, \qquad \text{for some } b' \geqq 1 \text{ and all } \zeta \in V(P)\,.$$

We note that (10) implies that $P(D)$ is partially hypoelliptic in x'. The reverse is not true. Indeed, the wave operator is an example of an operator, partially hypoelliptic in x^1, but with a characteristic polynomial that is not of finite type in the corresponding direction. In section 5 of this paper we will show, through examples, that there are polynomials that are not hypoelliptic but of finite type in some direction.

A particular case of (10) is when the x'-space is one-dimensional so that we get the inequality

$$(6) \qquad |\langle y, \mathrm{Re}\,\zeta\rangle| \leqq C(1+|\mathrm{Im}\,\zeta)^{b(y)}, \qquad \text{for all } \zeta \in V(P)\,.$$

The best possible choice of $b(y)$ is always a rational number. This can be shown, arguing as in the proof of lemma 3.9 in Hörmander [3], (See also the proof of lemma 1 in [4].) From the simple fact that

$$b(t_1y^1+t_2y^2) \leqq \max\bigl(b(y^1),b(y^2)\bigr)$$

for all real numbers t_1 and t_2, it is easy to deduce that there is an increasing sequence of subspaces of R^n corresponding to an increasing sequence of exponents in (10). This makes it possible to reduce the general case (10) to the case (6) of a polynomial of finite type $b(y)$ in the direction y.

Let now $P(\xi)=P(\xi_1,\ldots,\xi_n)$ be a given polynomial. The derivatives of $P(\xi)$ will then be denoted by

$$P_\alpha(\xi) = (\partial/\partial\xi)^\alpha P(\xi) = (\partial/\partial\xi_1)^{\alpha_1}\ldots(\partial/\partial\xi_n)^{\alpha_n}P(\xi)\,.$$

If we put $|\alpha|=\sum\alpha_j$ and if μ is the degree of $P(\xi)$ then the degree of $P_\alpha(\xi)$ is obviously $\mu-|\alpha|$, if $|\alpha|\leqq\mu$.

THEOREM 1. *$P(\xi)$ is a polynomial* (6) *of finite type $b(y)\geqq 1$ in the direction y, if and only if*

$$(11)\quad \sum_{|\alpha|>0}|P_\alpha(\xi)|^2\,|\langle y,\xi\rangle|^{2|\alpha|q} \leqq C\sum_{|\alpha|\geqq 0}|P_\alpha(\xi)|^2, \qquad \text{for all } \xi\in R_n;\ q=b(y)^{-1}\,.$$

PROOF. Suppose first that $P(\xi)$ is of finite type $b(y)$ in the direction y. We then start with an estimate of $P(\xi)^{-1}\langle\eta,\partial/\partial\xi\rangle^kP(\xi)$, where ξ and η are fixed real vectors. If we introduce a new polynomial in one variable

$$Q(t) = P(\xi+t\eta)$$

then the expression to estimate is

$$(12)\qquad P(\xi)^{-1}\,\langle\eta,\partial/\partial\xi\rangle^k\,P(\xi) = Q(0)^{-1}\,Q^{(k)}(0)\,.$$

Put

$$Q(t) = \sum_{j=0}^{\mu}a_jt^j = a_\mu\prod_{i=1}^{\mu}(t+t_i)\,.$$

Then

$$Q(0) = a_\mu\prod_{i=1}^{\mu}t^i$$

and

$$Q^{(k)}(0) = k!\,a_k = k!\,a_\mu\,e_{\mu-k}(t_1,\ \ldots,t_\mu)\,,$$

where $e_{\mu-k}(t_1,\ldots,t_\mu)$ is the sum of all products of $\mu-k$ of the quantities t_i. Thus the number of terms in $e_{\mu-k}$ is

$$\binom{\mu}{k}\,.$$

The expression in (12) can then be written as

$$Q(0)^{-1}\, Q^{(k)}(0) \;=\; k!\, e_k(t_1^{-1}, \ldots, t_\mu^{-1})$$

which gives the obvious estimate

$$(13)\qquad |Q(0)^{-1}\, Q^{(k)}(0)| \;\leqq\; k!\binom{\mu}{k}(\min|t_i|)^{-k}\,.$$

But $P(\xi - t_i\eta) = Q(-t_i) = 0$ so that $\zeta_{(i)} = \xi - t_i\eta$ is a vector in $V(P)$. So, if $|\eta| = 1$, then

$$(14)\qquad \min|t_i| \;=\; \min|\zeta_{(i)} - \xi| \;\geqq\; \inf|\zeta - \xi|\,,$$

the infimum taken over all $\zeta \in V(P)$.

We now recall the assumption that $P(\xi)$ is of type $b(y)$ in the direction y, and that $q = b(y)^{-1}$ is a number with $0 < q \leqq 1$. It is no restriction to assume that $|y| = 1$. Then we have, since ξ as well as y are supposed to be real,

$$(15)\quad |\zeta - \xi|^2 \;=\; |\mathrm{Im}\,\zeta|^2 + |\mathrm{Re}\,\zeta - \xi|^2 \;\geqq\; |\mathrm{Im}\,\zeta|^2 + |\langle y, \mathrm{Re}\,\zeta\rangle - \langle y, \xi\rangle|^2\,.$$

If now

$$|\langle y, \mathrm{Re}\,\zeta\rangle - \langle y, \xi\rangle| \;\geqq\; \tfrac{1}{2}|\langle y, \xi\rangle| \quad\text{and}\quad |\langle y, \xi\rangle| \;\geqq\; 1$$

then

$$4|\zeta - \xi|^2 \;\geqq\; |\langle y, \xi\rangle| \;\geqq\; |\langle y, \xi\rangle|^{2q}$$

and

$$|\zeta - \xi| \;\geqq\; \tfrac{1}{2}|\langle y, \xi\rangle|^{q}\,.$$

But if instead

$$|\langle y, \mathrm{Re}\,\zeta\rangle - \langle y, \xi\rangle| \;<\; \tfrac{1}{2}|\langle y, \xi\rangle|$$

then

$$(16)\qquad |\langle y, \mathrm{Re}\,\zeta\rangle| \;>\; \tfrac{1}{2}|\langle y, \xi\rangle|\,.$$

On the other hand we have assumed that

$$|\langle y, \mathrm{Re}\,\zeta\rangle| \;\leqq\; C(1 + |\mathrm{Im}\,\zeta|)^{b(y)} \qquad \text{for } \zeta \in V(P)$$

which gives the estimate of $|\mathrm{Im}\,\zeta|$

$$|\mathrm{Im}\,\zeta| \;\geqq\; C_1|\langle y, \mathrm{Re}\,\zeta\rangle|^{q} - 1\,.$$

This inequality together with (15) and (16) now gives

$$|\zeta - \xi| \;\geqq\; |\mathrm{Im}\,\zeta| \;\geqq\; C_1|\langle y, \mathrm{Re}\,\zeta\rangle|^{q} - 1 \;>\; C|\langle y, \xi\rangle|^{q}$$

provided that $|\langle y, \xi\rangle| > A$ for a suitable constant A, that can be assumed to be > 1. (We let C denote a constant that is not always the same during the course of the proof, but that does not depend in ξ.) We have now proved that

$$(17)\quad |\zeta - \xi| \;>\; C|\langle y, \xi\rangle|^{q} \qquad\text{when}\qquad \zeta \in V(P) \text{ and } |\langle y, \xi\rangle| > A\,.$$

Comparing (12), (13), (14) and (17) we can finally conclude that

$$(18)\qquad |P(\xi)^{-1}\langle \eta, \partial/\partial\xi\rangle^k P(\xi)| \leqq C|\langle y,\xi\rangle|^{-kq}, \qquad \text{when } |\langle y,\xi\rangle| > A .$$

Since trivially

$$|\langle \eta, \partial/\partial\xi\rangle^k P(\xi)|\ |\langle y,\xi\rangle|^{kq} \leqq C|\langle \eta, \partial/\partial\xi\rangle^k P(\xi)|, \qquad \text{when } |\langle y,\xi\rangle| \leqq A ,$$

we see that (18) is equivalent to

$$(19)\qquad |\langle \eta, \partial/\partial\xi\rangle^k P(\xi)|^2\ |\langle y,\xi\rangle|^{2kq} \leqq C\sum_{|\alpha|\geqq 0} |P_\alpha(\xi)|^2 .$$

The last step from (19) to (11) is very simple. We observe that η is an arbitrarily given real vector. Suppose that we choose N different such vectors $\eta_{(i)}$. Then we have

$$(20)\qquad \langle \eta_{(i)}, x\rangle^k = \sum_{|\alpha|=k} c_{k,\alpha}\eta_{(i)}{}^{\alpha}\, x^{\alpha}, \qquad i = 1,\ldots,N .$$

If N is big enough and if the $\eta_{(i)}$ are suitably chosen it is possible to solve the system (20) for x^α. Thus we get every x^α with $|\alpha|=k$ expressed as a finite sum of terms of the type $\langle \eta_{(i)}, x\rangle^k$. Replacing x by $\partial/\partial\xi$ then gives $(\partial/\partial\xi)^\alpha$ with $|\alpha|=k$ as a linear combination of $\langle \eta_{(i)}, \partial/\partial\xi\rangle^k$ and consequently

$$|P_\alpha(\xi)| \leqq C\sum_{i=1}^{N} |\langle \eta_{(i)}, \partial/\partial\xi\rangle^k P(\xi)| .$$

A simple application of Cauchy's inequality together with (19) now finishes the first half of the proof of theorem 1.

Let us now show that inversely every polynomial for which (11) is valid is of type at most $b(y)=1/q$ in the direction y. The proof is an immediate extension of the corresponding proof by Gårding and Malgrange in the partially elliptic case $q=1$. The inequality (11) can be written in the form

$$\sum_{|\alpha|>0} |P_\alpha(\xi)|^2(|\langle y,\xi\rangle|^{2|\alpha|q} - C) \leqq C|P(\xi)|^2, \qquad \xi\in R_n ,$$

which shows that (11) is equivalent to

$$(21)\qquad |P_\alpha(\xi)|\ |\langle y,\xi\rangle|^{|\alpha|q} \leqq C|P(\xi)|, \qquad \xi\in R_n,\ |\langle y,\xi\rangle| > A,\ \text{all } |\alpha|>0 .$$

Since some $P_\alpha(\xi)$, $|\alpha|=\mu$, must be a nonvanishing constant, we get from (21) the special case

$$(22)\qquad |P(\xi)| \geqq B|\langle y,\xi\rangle|^{\mu q}, \qquad \text{for } \xi\in R_n,\ |\langle y,\xi\rangle| > A .$$

Consider now the set W_t of complex vectors $\zeta=\xi+i\eta$ satisfying

$$|\eta| \leqq t|\langle y,\xi\rangle|^q ,$$

where t is a positive number. Then

$$\begin{aligned}|P(\zeta)| = |P(\xi+i\eta)| &= |P(\xi)+\sum_{|\alpha|>0} c_\alpha P_\alpha(\xi)(i\eta)^\alpha| \\ &\geqq |P(\xi)| - C\sum_{|\alpha|>0} |P_\alpha(\xi)|\,|\eta|^\alpha \\ &\geqq |P(\xi)| - C\sum_{|\alpha|>0} t^{|\alpha|}|P_\alpha(\xi)|\;|\langle y,\xi\rangle|^{|\alpha|q} \\ &\geqq |P(\xi)|\left(1-C\sum_{|\alpha|>0} t^{|\alpha|}\right),\end{aligned}$$

if $\zeta \in W_t$ and if (21) is valid. But we know from (22) that $|P(\xi)|$ is positive if $|\langle y,\xi\rangle| > A$. With t sufficiently small we are therefore able to conclude that

$$P(\zeta) \neq 0 \quad \text{if } \zeta \in W_t \text{ and } |\langle y,\xi\rangle| > A\,,$$

i.e. if

$$|\langle y,\xi\rangle| > C(1+|\eta|)^{1/q}\,.$$

But this is just another way of saying that $P(\xi)$ is of type $\leqq 1/q$ in the direction y. That ends the proof of theorem 1.

Lemma 1. *Suppose that $P(\xi)$ is a polynomial of type $b(y)=r/s \geqq 1$ in the direction y, r and s being positive integers without common factors and $r \geqq s$. Then*

$$(23) \qquad \sum_{|\alpha|>0} \|P_\alpha(D)^r \langle y,D\rangle^{|\alpha|s} v(x)\|^2 \leqq C\sum_{|\alpha|\geqq 0} \|P_\alpha(D)^r v(x)\|^2$$

for all $v(x) \in \mathscr{D}$, where the norm is the usual L^2 norm.

Proof. When $b=1$ we prove the lemma immediately, multiplying each side of the inequality (11) by the square of the Fourier transform $\hat{v}(\xi)$ of $v(x)$ and using Parseval's formula. When b is not an integer this method does not work without modification. But from

$$|P_\alpha(\xi)|^2 |\langle y,\xi\rangle|^{2|\alpha|s/r} \leqq C\sum_{|\alpha|\geqq 0} |P_\alpha(\xi)|^2$$

follows

$$|P_\alpha(\xi)|^{2r} |\langle y,\xi\rangle|^{2|\alpha|s} \leqq C^r\left\{\sum |P_\alpha(\xi)|^2\right\}^r$$

and estimating the second member by Hölder's inequality we get with a new constant C

$$|P_\alpha(\xi)|^{2r} |\langle y,\xi\rangle|^{2|\alpha|s} \leqq C\sum |P_\alpha(\xi)|^{2r}\,.$$

We can then multiply by $|\hat{v}(\xi)|^2$ and prove (23) as in the simple case $b=1$ by Parseval's formula.

2. The norm $|u, V|_h$.

Let V be any given set in R^n. We then define the norm $\|u, V\|^2$ by

$$\|u,V\|^2 = \int_V |u(x)|^2 dx .$$

When V coincides with R^n we simply write $\|u\|^2$. Fundamental for the following is the introduction of a new norm $|u, V|_h$, defined by

$$|u, V|_h^2 = \sum \|P_{\alpha_1}(D) \ldots P_{\alpha_r}(D)\langle y,D\rangle^k u,\ V\|^2\ h^{2bk-2\Sigma|\alpha_i|}, \qquad 0 < h \leqq 1 .$$

The sum is to be taken over all index sets $\alpha_i = (\alpha_i^1, \ldots \alpha_i^n)$ with

$$0 < |\alpha_1| \leqq \ldots \leqq |\alpha_r| \leqq \mu$$

and over all integers k with

$$0 \leqq k < s \min|\alpha_i| = s|\alpha_1| .$$

$\big(b = b(y) = r/s \geqq 1$ and μ is the degree of $P(\xi)$.$\big)$ We shall also use the simpler norm

$$|u, V| = |u, V|_1 .$$

Observe that the exponent of h^2 in $|u, V|_h$ is $\leqq 0$ because

$$bk - \sum|\alpha_i| \leqq bs|\alpha_1| - \sum|\alpha_i| = r|\alpha_1| - \sum|\alpha_i| = \sum(|\alpha_1| - |\alpha_i|) .$$

On the other hand, the smallest value of the exponent is $-r\mu$. This value is assumed when $k = 0$ and $|\alpha_1| = \ldots = |a_r| = \mu$. Hence, in view of the assumption that $0 < h \leqq 1$ we get

$$(24) \qquad |u, V| \leqq |u, V|_h \leqq |u, V| h^{-r\mu} .$$

We will now try to estimate $|u, V|$ by norms that do not depend on $P(\xi)$. We first notice that there is some α with $|\alpha| = \mu$ and $P_\alpha(\xi) = c \neq 0$. This implies that $|u, V|$ contains terms of the type $\|c^r\langle y,D\rangle^k u, V\|^2$ for all k with $0 \leqq k < s\mu$. Therefore

$$C_1 \sum_{k<s\mu} \|\langle y,D\rangle^k u, V\|^2 \leqq |u, V|^2 .$$

On the other hand the total degree of the polynomial

$$P_{\alpha_1}(\xi) \ldots P_{\alpha_r}(\xi)\langle y,\xi\rangle^k, \qquad 0 \leqq |\alpha_1| \leqq \ldots \leqq |\alpha_r| \leqq \mu,\ k < s|\alpha_1|$$

is smaller than

$$\max_{|\alpha_i|>0} \{r\mu - \sum|\alpha_i| + s \min|\alpha_i|\} = r\mu - (r-s) .$$

Thus

$$(25) \qquad C_1 \sum_{k<s\mu} \|\langle y,D\rangle^k u, V\|^2 \leqq |u, V|^2 \leqq C_2 \sum_{|\alpha|<r\mu-(r-s)} \|D^\alpha u, V\|^2 .$$

REMARK. $r\mu - (r-s) > s\mu$ except in the trivial cases $\mu = 1$ or $r = s = q = 1$.

3. The fundamental inequality.

LEMMA 2. *Let $Q_i(D)$, $i=1,\ldots,p$ be arbitrarily given linear differential operators with constant coefficients. Then there is a constant C so that*

$$\|Q_1(D)\ldots Q_p(D)v\|^2 \leqq C\sum_{i=1}^{p}\|Q_i(D)^p v\|^2, \qquad \text{for all } v\in\mathscr{D}\,.$$

PROOF. Parseval's lemma shows that the inequality is equivalent to

$$|Q_1(\xi)\ldots Q_p(\xi)|^2 \leqq C\sum_{i=1}^{p}|Q_i(\xi)|^{2p}\,.$$

But this is exactly the inequality for the geometric and arithmetic means of $|Q_1(\xi)|^{2p},\ldots,|Q_p(\xi)|^{2p}$.

We are now able to establish the fundamental inequality.

THEOREM 2. *Suppose that $P(\xi)$ is a polynomial of type $b(y)=1/q=r/s$ in the direction y. Let K, L be two relatively compact domains in R^n with $K\subset L$ and suppose that the minimum distance between the boundaries of K and L is h, $0<h\leqq 1$. Then there is a constant C such that*

$$(26)\quad h^b|\langle y,D\rangle u,K|_h \leqq C|u,L|_h + Ch^{\mu-r\mu}\sum_{|\alpha|\leqq r\mu-\mu}\|D^\alpha P(D)u,L\|, \qquad \text{all } u\in\mathscr{E}\,.$$

PROOF. Let φ be a regular function with support in L $\big(\varphi\in\mathscr{D}(L)\big)$ and with $\varphi(x)=1$ on K, such that

$$\sup|D^\alpha\varphi(x)| \leqq c_\alpha h^{-|\alpha|} \qquad \text{for all } \alpha\,.$$

A function of this type can easily be constructed by means of regularization. If now u is a function in $\mathscr{E}$ and if $v=\varphi u$ then

$$v = \varphi u\in\mathscr{D}(L) \qquad \text{and} \qquad v = u \text{ on } K\,.$$

Further, according to the formula of Leibniz we have for every linear differential operator $Q(D)$

$$(27)\qquad \begin{aligned} |Q(D)v(x)|^2 &= \big|\textstyle\sum a_\alpha D^\alpha\varphi(x)\,Q_\alpha(D)u(x)\big|^2\\ &\leqq \big\{\textstyle\sum |a_\alpha|\,c_\alpha h^{-|\alpha|}\,|Q_\alpha(D)\,u(x)|\big\}^2\\ &\leqq C\textstyle\sum |Q_\alpha(D)\,u(x)|^2 h^{-2|\alpha|}\,. \end{aligned}$$

The quantity that we are going to estimate is

$$h^{2b}|\langle y,D\rangle u,K|_h{}^2 = \sum \|P_{\alpha_1}(D)\ldots P_{\alpha_r}(D)\langle y,D\rangle^{k+1}u,K\|^2\; h^{2b(k+1)-2\Sigma|\alpha_i|}$$

with

$$0 \leqq k < s\min|\alpha_i| = s|\alpha_1|, \qquad \text{all } |\alpha_i|>0\,.$$

We can split the sum above in two parts so that in the first part always

$k+1<s|\alpha_1|$ while in the second part $k+1=s|\alpha_1|$. In the first sum there is a finite number of terms, each of which is also contained in $|u,K|_h{}^2$. This part is therefore majorated by $C|u,K|_h{}^2$ and a fortiori by $C|u,L|_h{}^2$. It remains to estimate the terms where $k+1=s|\alpha_1|$. Each such term is treated as follows.

$$\begin{aligned}
&\|P_{\alpha_1}(D)\ldots P_{\alpha_r}(D)\langle y,D\rangle^{s|\alpha_1|}u,K\|^2\ h^{2bs|\alpha_1|-2\Sigma|\alpha_i|}\\
&\qquad = \|P_{\alpha_1}(D)\ldots P_{\alpha_r}(D)\langle y,D\rangle^{s|\alpha_1|}u,K\|^2\ h^{-2\Sigma(|\alpha_i|-|\alpha_1|)}\\
&\qquad \leqq \|P_{\alpha_1}(D)\ldots P_{\alpha_r}(D)\langle y,D\rangle^{s|\alpha_1|}v\|^2\ h^{-2\Sigma(|\alpha_i|-|\alpha_1|)}\\
&\qquad \leqq C\sum\|P_{\alpha_i}(D)^r\langle y,D\rangle^{s|\alpha_1|}v\|^2\ h^{-2r(|\alpha_i|-|\alpha_1|)}.
\end{aligned}$$

The first inequality is based on the fact that $v=u$ on K, while the second estimate is an application of lemma 2 with

$$Q_i(D) = P_\alpha(D)h^{-(|\alpha_i|-|\alpha_1|)}.$$

The last sum is again composed of terms of two different kinds. One type is

$$\|P_\alpha(D)^r\langle y,D\rangle^{s|\alpha|}v\|^2,$$

the other is

$$\|P_\alpha(D)^r\langle y,D\rangle^{sk}v\|^2\ h^{-2r(|\alpha|-k)}, \qquad \text{with } |\alpha|>k.$$

Suppose we have a term of the first type. Then we can use lemma 1 and the formula (27) and so

$$\begin{aligned}
\|P_\alpha(D)^r\langle y,D\rangle^{s|\alpha|}v\|^2 &\leqq C\sum_{|\alpha|\geqq 0}\|P_\alpha(D)^r v\|^2\\
&\leqq C\sum_{|\beta_i|\geqq 0}\ \sum_{|\alpha|\geqq 0}\|P_{\alpha+\beta_1}(D)\ldots P_{\alpha+\beta_r}(D)u,L\|^2\ h^{-2\Sigma|\beta_i|}.
\end{aligned}$$

Notice in particular that the norms are taken with respect to L, since L is the support of $v=\varphi u$. Recalling that $0<h\leqq 1$ we now see that all terms in the sum with $|\alpha+\beta_i|>0$, $i=1,\ldots,r$, can be majorated by $C|u,L|_h{}^2$. The remaining terms, with $|\alpha+\beta_i|=0$ for some i, are certainly smaller than

$$Ch^{-2\mu(r-1)}\sum_{|\alpha|\leqq\mu(r-1)}\|D^\alpha P(D)u,L\|^2.$$

Finally, consider terms of the form

$$\|P_\alpha(D)^r\langle y,D\rangle^{sk}v\|^2\ h^{-2r(|\alpha|-k)}, \qquad \text{with } |\alpha|>s\geqq 0.$$

Using (27) again we get the majorating sum

$$C\sum\|P_{\alpha+\beta_1}(D)\ldots P_{\alpha+\beta_r}(D)\langle y,D\rangle^j u,L\|^2\ h^{-2\Sigma|\beta_i|-2(sk-j)-2r(|\alpha|-k)},$$

where $0\leqq j\leqq sk$. The exponent of h^2 can be split into

$$\left(bj-\sum|\alpha+\beta_i|\right)+(b-1)(sk-j),$$

and since $b \geqq 1$ and $sk \geqq j$ it cannot be smaller than $bj-\sum|\alpha+\beta_i|$. Moreover, $j \leqq sk$ and $s<|\alpha|$ implies

$$j \;<\; s|\alpha| \;\leqq\; s\min|\alpha+\beta_i|\;.$$

This proves that the majorating sum above is itself majorated by $C|u,K|_h$.

Summing up all the different estimates we finally get

$$h^{2b}|\langle y,D\rangle u,K|_h{}^2 \;\leqq\; Ch^{-2\mu(r-1)} \sum_{|\alpha|\leqq\mu(r-1)} \|D^\alpha P(D)u,L\|^2\;.$$

We have now only to use the inequality $x^2+y^2 \leqq (|x|+|y|)^2$ a number of times to get the desired estimate of theorem 2.

4. Differential equations corresponding to polynomials of finite type.

We will now apply theorem 2 to the solutions of the equation $P(D)u=0$, when $P(\xi)$ is assumed to be of finite type.

LEMMA 3. *Let $P(\xi)$ be a polynomial of type $b=r/s$ in the direction y, and let μ be the degree of $P(\xi)$. Suppose that Ω is an open domain in R^n. Then there is a constant C such that the inequality*

$$\sum_{k<s\mu}\|\langle y,D\rangle^{j+k}u,K\| \;\leqq\; C^j(j!)^b d^{-jb-r\mu} \sum_{|\alpha|<r\mu-(r-s)} \|D^\alpha u,L\|, \qquad j\;=\;0,1,\ldots,$$

is valid for every solution $u \in \mathscr{E}(\Omega)$ of $P(D)u=0$ and every pair of relatively compact domains K and L with $K\subset L\subset \bar{L}\subset\Omega$. The minimum distance d between the boundaries of K and L is supposed to satisfy $0<d\leqq 1$.

The constant C depends on $P(\xi)$ but not on K or L.

PROOF. Now suppose that K and L are given with $d(K,L)=d$, $0<d\leqq 1$. Then there exists an increasing sequence of relatively compact domains $K_0,K_1,\ldots,K_j$ with

$$K \;=\; K_0 \subset K_1 \subset \ldots \subset K_j \;=\; L \qquad\text{and}\qquad d(K_i,K_{i+1}) \;=\; d(K,L)/j \;=\; h\;.$$

Thus every pair K_i, K_{i+1} satisfies the conditions imposed on K and L in theorem 2. If now $u\in\mathscr{E}(\Omega)$ is a solution of the equation $P(D)u=0$ then also $\langle y,D\rangle^i\, u\in\mathscr{E}(\Omega)$ and

$$P(D)\langle y,D\rangle^i u \;=\; \langle y,D\rangle^i P(D)u \;=\; 0, \qquad i \;=\; 0,1,\ldots\;.$$

Successive applications of theorem 2 to K_i, K_{i+1} and $\langle y,D\rangle^{j-i}u$, $i=0,1,\ldots,$ show that

$$h^{jb}|\langle y,D\rangle^j u,K_0|_h \leqq h^{(j-1)b}C|\langle y,D\rangle^{j-1}u,K_1|_h \leqq \ldots$$

and finally

$$h^{jb}|\langle y,D\rangle^j u,K_0|_h \leqq C^j|u,K_j|_h .$$

We can now make use of (24) to eliminate h from the norms;

$$|\langle y,D\rangle^j u,K| \leqq C^j h^{-jb}h^{-r\mu}|u,L| ,$$

where h still depends on j, $h=d/j$. Thus we have proved the following inequality, which now is valid for $j=1,2,\ldots$

$$|\langle y,D\rangle^j u,K| \leqq C^j j^{jb+r\mu}d^{-jb-r\mu}|u,L| .$$

But the Taylor expansion $e^j=\sum j^k/k!>j^j/j!$ gives the estimate $j^j<e^j j!$, $j^{jb}<(e^b)^j(j!)^b$, and there is a constant B so that $j^{r\mu}<B^j$ for $j=1,2,\ldots$ Consequently we have with a new constant C

$$|\langle y,D\rangle^j u,K| \leqq C^j(j!)^b d^{-jb-r\mu}|u,L|, \qquad j = 0,1,\ldots$$

(The inequality is trivially true for $j=0$.)

The rest of the proof of the lemma is a direct application of (25).

Corollary. u belongs to the hypoanalytic class $A_{b(y)}$, defined by (7), in Ω. Indeed, we can apply lemma 3 to $D^\beta u$, $|\beta|\leqq[\frac{1}{2}n]+1$, and after that use Sobolev's lemma to ge a majoration of $|\langle y,D\rangle^j u(x)|$ on any compact, interior to K.

We now turn to the corresponding problem for the inhomogeneous equation.

Lemma 4. *If in lemma* 3 *the equation* $P(D)u=0$ *is replaced by* $P(D)u=f$ *in* Ω, *where* f *and its derivatives of order* $\leqq N$ $(N=r\mu-\mu+[\frac{1}{2}n]+1)$ *belong to* $A_{a(y)}$, *then*

$$u\in A_{c(y)} \text{ in } \Omega; \qquad c(y) = \max\big(a(y),b(y)\big) .$$

Proof. The proof proceeds in strict analogy to the proof of lemma 3. But since $P(D)u$ does not vanish any longer we get some extra terms when we apply theorem 2 to K_i, K_{i+1} and $\langle y,D\rangle^{j-i}u$. Thus

$$h^{(k-j+1)b}C^{j-k-1}|\langle y,D\rangle^{k+1}u,K_{j-k-1}|_h$$
$$\leqq h^{(k-j)b}C^{j-k}|\langle y,D\rangle^k u,K_{j-k}|_h + h^{(k-j)b}C^{j-k}h^{\mu-r\mu}\sum_{|\alpha|\leqq r\mu-\mu}\|\langle y,D\rangle^k D^\alpha f,K_{j-k}\| ,$$

for $k=0,1,\ldots,j-1$. Summation of these inequalities gives

$$(28)\quad |\langle y,D\rangle^j u,K|_h$$
$$\leqq C^j h^{-jb}|u,L|_h + h^{\mu-r\mu}\sum_{k=0}^{j-1}h^{(k-j)b}C^{j-k}\sum_{|\alpha|\leqq r\mu-\mu}|\langle y,D\rangle^k D^\alpha f,K_{j-k}| ,$$

with $h=d/j\leqq 1$. But we have assumed that

$$D^\alpha f\in A_{a(y)} \text{ in } \Omega, \quad |\alpha|\leqq r\mu-\mu\,.$$

That means e.g.

$$\|\langle y,D\rangle^k D^\alpha f, K_{j-k}\|\leqq C_1 \sup_{\substack{x\in L\\ |\alpha|\leqq r\mu-\mu}} |\langle y,D\rangle^k D^\alpha f(x)|\leqq C_1 C_2^{k+1}(k!)^{a(y)}\,.$$

Then the sum in (28) is certainly smaller than

$$C_1 C_2 h^{\mu-r\mu}\sum_{k=0}^{j=1} h^{(k-j)b} C^{j-k} C_2^k (k!)^a\,.$$

For $k\neq 0$ is $k!\leqq k^k<j^k$, since only terms with $k<j$ appear in the sum. For $k=0$ the inequality $k!\leqq j^k$ is trivial. Replacing h by d/j and introducing a new constant C again we then get the majorant

$$C^{j+1}\sum_{k=0}^{j-1} j^{(j-k)b} j^{ka}\leqq C^{j+1}\sum_{k=0}^{j-1} j^{(j-k)c} j^{kc} = C^{j+1} j j^{jc}\leqq C_3^{j+1}(j!)^c\,,$$

with $c=\max(a,b)$. As the first term in the second member of (28) is already known to be smaller than $C_4^j(j!)^b$ we have now proved that

$$|\langle y,D\rangle^j u,K|\leqq |\langle y,D\rangle^j u,K|_h\leqq C_4^j(j!)^b+C_3^{j+1}(j!)^c\,,$$

that is,

$$|\langle y,D\rangle^j u,K|\leqq C^{j+1}(j!)^c\,.$$

The rest of the proof proceeds as the verifications of lemma 3 and its corollary.

We can also extend the results of lemma 3 in another direction (cf. section 1), namely to polynomials of finite type in ξ', when $\xi=(\xi',\xi'')$, $x=(x',x'')$ correspond to a decomposition of R^n in $R^n=R^{n-m}\times R^m$. Then $D^{\alpha'}$ will denote derivatives in x' only.

THEOREM 3. *Suppose that $P(\xi)$ is a polynomial of degree μ and of type $b'=r/s$ in ξ'. Let Ω be an open domain in R^n. Then*

$$(29)\quad \sum_{|\alpha'|<s\mu}\sum_{|\beta'|=j}\|D^{\alpha'+\beta'}u,K\|\leqq C^j(j!)^{b'}d^{-jb'-r\mu}\sum_{|\alpha|<r\mu-(r-s)}\|D^\alpha u,L\|,\qquad j=0,1,\ldots$$

for every solution $u\in\mathscr{E}(\Omega)$ of $P(D)u=0$ in Ω and every pair of relatively compact domains K,L with $K\subset L\subset\bar{L}\subset\Omega$ and $d=d(K,L)\leqq 1$.

PROOF. We recall that $P(\xi)$ is of type b' in ξ' if

$$(10)\qquad |\mathrm{Re}\,\zeta'|\leqq C(1+|\mathrm{Im}\,\zeta|)^{b'},\qquad \text{for all } \zeta\in V(P)\,.$$

But then, a fortiori,

$$|\langle y,\mathrm{Re}\,\zeta\rangle|\leqq C(1+|\mathrm{Im}\,\zeta|)^{b'},\qquad \text{for all } \zeta\in V(P)\,,$$

and for any real vector $y=(y',y'')$ with $y''=0$. Accordingly, $P(\xi)$ is of type b' in the direction y for every such vector y. By the same technique as in the end of the proof of theorem 1, we can now show that there is a set of vectors $y_{(i)}$, $i=1,\ldots,N$, with $y''=0$ and with the following property; every differentiation operator $D^{\beta'}$ with $|\beta'|=p\leqq s\mu$ can be expressed as a linear combination

$$\sum_{i=1}^{N} c_{i,\beta'}\langle y_{(i)},D\rangle^p .$$

Since there is only a finite number of indices $|\beta'|$ with $|\beta'|\leqq s\mu$, we realize that

$$\sum_{i=1}^{N}\sum_{|\beta'|\leqq s\mu}|c_{i,\beta'}|^2$$

is bounded. Consequently, from

$$(23)\qquad \sum_{|\alpha|>0}\|P_\alpha(D)^r\langle y,D\rangle^{|\alpha|s}v\|^2 \leqq C\sum_{|\alpha|\geqq 0}\|P_\alpha(D)^r v\|^2, \qquad \text{for } v\in\mathscr{D},$$

we can deduce that

$$\sum_{|\alpha|>0}\sum_{|\beta'|=s|\alpha|}\|P_\alpha(D)^r D^{\beta'}v\|^2 \leqq C\sum_{|\alpha|\geqq 0}\|P_\alpha(D)^r v\|, \qquad \text{for } v\in\mathscr{D}.$$

According to this new form of (23) we now also change the definition of the norm $|u,V|_h$:

$$|u,V|_h{}^2 = \sum\|P_{\alpha_1}(D)\ldots P_{\alpha_r}(D)D^{\beta'}u,V\|^2\, h^{2b|\beta'|-2\Sigma|\alpha_i|}$$

where the sum is to be taken over all α_i and β' with

$$0<|\alpha_1|\leqq\ldots\leqq|\alpha_r|\leqq\mu,\qquad |\beta'|<s|\alpha_1|.$$

Then we can proceed as before to get the proof of theorem 3. For instance, the fundamental inequality (26) now takes the form

$$h^b\sum_{|\beta'|=1}|D^{\beta'}u,K|_h \leqq C|u,L|_h+\ldots.$$

EXAMPLE. An elliptic operator has a characteristic polynomial of type 1 in every direction. In this case we get a much simpler form of the inequality of therem 3, because $s\mu=r\mu-(r-s)=\mu$. The inequality (29) can then be written as

$$\sum_{|\beta|=j}|||D^\beta u,K||| \leqq C^j(j!)d^{-j-\mu}|||u,L|||,$$

where

$$|||u,V||| = \sum_{|\alpha|<\mu}\|D^\alpha u,V\|.$$

REMARK 1. Of course it is also possible to prove corresponding generalizations of lemma 4 and of the corollary of lemma 3.

Remark 2. If $P(\xi)$ is of different type in different directions, then it is possible to get estimates of the form

$$C^{j_1+\ldots+j_p}\,(j_1!)^{b_1}\ldots(j_p!)^{b_p} \qquad \text{or} \qquad AB^{j_1+\ldots+j_p}\,\Gamma(j_1 b_1+\ldots+j_p b_p)$$

for the mixed derivatives of the solutions of $P(D)u=0$. This is made simply by repeated application of lemma 3.

We have now proved that for every operator with a characteristic polynomial of type b' in ξ' all the regular solutions of $P(D)u=0$ are in the hypoanalytic class $A_{b'(x')}$, consisting of all regular functions v such that one has for every compact $K\subset R^n$

$$(30)\quad |D^{\beta'}v(x',x'')| \leqq C^{k+1}(k!)^{b'}, \qquad k=0,1,\ldots, \qquad x=(x',x'')\in K\,.$$

This result can immediately be generalized to all distribution solutions of $P(D)u=0$. But this requires an extension of the definition of $A_{b'(x')}$. In analogy with the definitions made by Gårding [2] we say that a distribution $v\in\mathscr{D}'_{x',x''}(\Omega)$ belongs to $A_{b'(x')}$ if and only if, for all $V\times W\subset\Omega$,

$$v_\varphi(x') = \int v(x',x'')\,\varphi(x'')\,dx''$$

belongs to the class $A_{b'(x')}$, when $x'\in V$ and $\varphi\in\mathscr{D}(W)$. We can then state our main theorem:

Theorem 4. *All distribution solutions of $P(D)u=0$ in Ω belong to the class $A_{b'(x')}$ if and only if $P(\xi)$ is partially hypoelliptic of type b' in ξ'.*

In other words, the class $\tilde{A}_{b'(x')}$ consists exactly of all polynomials of type b' in ξ'.

Proof. We begin by proving that if $P(\xi)$ is of type b' in ξ', then every distribution v with $P(D)v=0$ is in $A_{b'(x')}$. But, as $P(\xi)$ is of finite type in ξ', it is also partially hypoelliptic in x' and so the distribution v must be regular in x'. Then one can show (see [2]) that the function

$$w_\varphi(x',x'') = \int v(x',x''+y'')\,\varphi(y'')\,dy''$$

is regular for x'' contained in a small neighbourhood of the origin. The function $w_\varphi(x',x'')$ also satisfies the equation $P(D)w_\varphi=0$, so that theorem 3 gives estimates of the x'-derivatives of w_φ. But, since

$$w_\varphi(x',0) = v_\varphi(x') = \int v(x',x'')\,\varphi(x'')\,dx''$$

this shows that $v_\varphi\in A_{b'}$, that is, v is a distribution in $A_{b'(x')}$.

The other half of the proof of theorem 4 is very close to the proof of

Gårding and Malgrange for the partially elliptic case and will therefore be given without details. Suppose that all distribution solutions of $P(D)u=0$ belong to $A_{b'(x')}$. Then it follows from Baire's theorem that there is a constant c so that

$$\sum_{|\alpha'|=j} |D^{\alpha'} u_\varphi(0)| \leqq c^{j+1} j^{jb'} |u|_1 |\varphi|_2, \qquad j = 0,1,2,\ldots \tag{31}$$

for every solution u, regular in a fixed compact $C' \times C''$ and every $\varphi \in \mathscr{D}(C'')$. $|u|_1$ and $|\varphi|_2$ stand for suitable norms of u and φ. In particular, if $u = e^{i(x'\zeta'+x''\zeta'')}$ then according to (31)

$$|\hat{\varphi}(\zeta'')| \, |\zeta'|^j \leqq C^{j+1} j^{jb'} (1+|\zeta'|+|\zeta''|)^k e^{k(|\mathrm{Im}\,\zeta'|+|\mathrm{Im}\,\zeta''|)} |\varphi|_2 ,$$

where $\hat{\varphi}(\zeta'')$ is the Fourier transform of φ and k some constant. Choosing φ carefully we obtain with new constants

$$|\zeta'|^j \leqq C^{j+1} j^{jb'} (1+|\zeta'|+|\zeta''|)^k e^{k(|\mathrm{Im}\,\zeta'|+|\mathrm{Im}\,\zeta''|)}$$

or

$$|\zeta'| \leqq C j^{b'} e^{A/j}, \qquad A = k\{\log(1+|\zeta'|+|\zeta''|) + |\mathrm{Im}\,\zeta'| + |\mathrm{Im}\,\zeta''|\}, \qquad j = 0,1,2,\ldots . \tag{32}$$

The right band side is near its minimum as a function of j if we chose j as an integer approximately equal to A. Thus we can replace (32) by the inequality

$$|\zeta'| \leqq C\{\log(1+|\zeta'|+|\zeta''|) + |\mathrm{Im}\,\zeta'| + |\mathrm{Im}\,\zeta''|\}^{b'} ,$$

which is equivalent to

$$|\mathrm{Re}\,\zeta'|/(1+|\mathrm{Im}\,\zeta'|+|\mathrm{Im}\,\zeta''|)^{b'} \leqq C \log(1+|\mathrm{Re}\,\zeta''|) .$$

So, if $m(r)$ is the maximum of the left hand side of the inequality above when $|\mathrm{Re}\,\zeta''| \leqq r$ then

$$m(r) \leqq C \log(1+r) .$$

Since $m(r)$ is an algebraic function of r for large r (Seidenberg [10]) this shows that $m(r)$ is bounded. Thus $P(\xi)$ is of type b' in ξ'.

5. Some remarks concerning polynomials of finite type.

In this section we are going to collect some rather loosely connected results about polynomials of finite type.

We have already mentioned that for any given polynomial and for any given vector y the smallest possible number $b(y) \geqq 1$ in the inequality

$$|\langle y, \mathrm{Re}\,\zeta\rangle| \leqq C(1+|\mathrm{Im}\,\zeta|)^{b(y)}, \qquad \text{all } \zeta \in V(P) ,$$

is a rational number $\leqq \infty$. We will now show by an example that, inversely, for every rational number $b \geqq 1$ there is a polynomial of type b in some direction.

Proposition. *Let $b = r/s$ be a given rational number $\geqq 1$ and let a be a nonreal number. Then the polynomial*

$$P(\xi) = \xi_1{}^r - a\xi_2{}^s$$

satisfies the inequalities

$$|\mathrm{Re}\,\zeta_1| \leqq C(1+|\mathrm{Im}\,\zeta_1|+|\mathrm{Im}\,\zeta_2|^{s/r}) \leqq C\ (1+|\mathrm{Im}\,\zeta|)\,,$$
$$|\mathrm{Re}\,\zeta_2| \leqq C(1+|\mathrm{Im}\,\zeta_1|^{r/s}+|\mathrm{Im}\,\zeta_2|) \leqq C_1(1+|\mathrm{Im}\,\zeta|)^b\,,$$

for all $\zeta \in V(P)$. Consequently, $P(\xi)$ is of type 1 in the direction $(1,0)$ and of type b in all other directions.

The proof is easy and will not be supplied here.

Remark. The characteristic polynomial $\xi_1{}^2 + i\xi_2$ of the heat conduction operator is of this kind.

We now turn to the following problem: If $P(\xi)$ is a polynomial of finite type in some direction, are there some classes of polynomials Q and R for which PQ respectively $P+R$ are of the same type as P? This question will be partially answered by the following lemmas.

Lemma 5. *Suppose that $P_1(\xi)$ and $P_2(\xi)$ are of type b_1 and b_2 respectively in the direction y. Then the product $P_1(\xi)P_2(\xi)$ is of type $\max(b_1, b_2)$ in the same direction.*

Proof. The lemma can be deduced from theorem 1 but also and simpler as follows. Let u be an arbitrary solution of the equation $P_1(D)P_2(D)u = 0$. Then we put $v = P_2(D)u$ and get $P_1(D)v = 0$ which shows that $v \in A_{b_1(y)}$. Since $P_2(D)u$ is in $A_{b_1(y)}$ we then get from lemma 4 that $u \in A_{c(y)}$ with $c(y) = \max(b_1, b_2)$.

The idea used above gives us also the opportunity to construct examples of polynomials that are of finite type in some directions but not in all, i.e. that are not hypoelliptic. Nearly trivial is the case when

$$P(\xi', \xi'') = Q(\xi')\,H(\xi', \xi'')\,,$$

where $Q(\xi')$ is of finite type in ξ' and $H(\xi)$ is hypoelliptic. Then we can show as above that P is of finite type in ξ'. That $P(\xi)$ is not hypoelliptic follows from the observation that every function of x only, satisfies the equation $P(D)u = 0$ without even having derivatives of all orders. Less trivial examples can be constructed by adding suitable terms of lower order to polynomials of the form $P=QH$. A polynomial of this kind is $P(\xi)=\xi_1(\xi_1^2+\xi_2^2)+1$. It is easy to verify directly that this polynomial is of type 1 in ξ_1 but not of finite type in ξ_2.

Let P and Q be two polynomials. As usual we call Q weaker than P if and only if

$$\sum |Q_\alpha(\xi)| \leqq C \sum |P_\alpha(\xi)|, \tag{33}$$

where the sums are extended over all derivatives of P and Q. If P also is weaker than Q then we say that that P and Q are equally strong.

Lemma 6. *If P and Q are equally strong and if P is of type b in the direction y, then Q is too.*

Corollary. *If P is of finite type $b(y)$, then one can add to P any weaker polynomial and get a new polynomial of the same type $b(y)$.*

Proof of lemma 6. For every integer m there are numbers a_i and real vectors η_i, $i=1,2,\ldots,N(m)$, so that the identity

$$t^{|\alpha|} R_\alpha(\xi) = \sum_1^N a_i R(\xi+t\eta_i) \tag{34}$$

is valid for every polynomial of degree $\leqq m$. This is a consequence of Taylor's theorem for polynomials. If now P and Q have degree $\leqq m$, we obtain from (34), (33) and Taylor's theorem again

$$|t^{|\alpha|} Q_\alpha(\xi)| \leqq C \sum_1^N |Q(\xi+t\eta_i)| \leqq C \sum_1^N \sum_\beta |P_\beta(\xi+t\eta_i)| \leqq C \sum_{\beta,\gamma} t^{|\gamma|} |P_{\beta+\gamma}(\xi)| .$$

Let $t=|\langle y,\xi\rangle|^q$ with $q=b(y)^{-1}$. Then, using the assumptions about P, we see that

$$\begin{aligned} |\langle y,\xi\rangle|^{|\alpha| q} |Q_\alpha(\xi)| &\leqq C \sum_\alpha |\langle y,\xi\rangle|^{|\alpha| q} |P_\alpha(\xi)| \\ &\leqq C|P(\xi)| \leqq C \sum_\alpha |Q_\alpha(\xi)|, \qquad \text{for } |\langle y,\xi\rangle| \geqq 1, \end{aligned}$$

which shows that Q is of type b in the direction y.

We will end this section with a remark about the general form of a polynomial of finite type. Suppose that $P(\xi)=P(\xi',\xi'')$ is of type $b'=q^{-1}$ in ξ'. Then we first note that P is hypoelliptic in ξ' and so, by the results of Gårding and Malgrange [2], of the form

$$P(\xi',\xi'') = P_0(\xi') + \sum_1^M P_j(\xi')(\xi'')^{\beta_j},$$

where P_0 is hypoelliptic and P_j are strictly weaker than P_0, that is, to every P_j there is a positive number k_j so that

$$(35) \qquad \overline{\lim}\, |P_j(\xi')|\, |\xi'|^{k_j} / \big(|P_0(\xi')| + 1\big) = a_j \neq 0$$

(Since P_0 is hypoelliptic this implies that P_j is weaker than P_0.) We have supposed that P is of type $b' = q^{-1}$ in ξ' which means that for every derivative P_α

$$(36) \qquad |P_\alpha(\xi)|\, |\xi'|^{q|\alpha|} \leqq C |P(\xi)|, \qquad \text{if } |\xi'| > A\,.$$

But if $\alpha = (\alpha', \alpha'') = (0, \beta_j)$, then

$$P_\alpha(\xi', 0) = c_j P_j(\xi'), \qquad c_j \neq 0\,,$$

which together with (36) gives

$$|P_j(\xi')|\, |\xi'|^{q|\beta_j|} \leqq C |P(\xi', 0)|, \qquad |\xi'| > A\,.$$

Then also

$$\sum_{j=1}^{M} |P_j(\xi')|\, |\xi'|^{q|\beta_j|} \leqq C |P(\xi', 0)| \leqq C |P_0(\xi')| + C \sum |P_j(\xi')|, \qquad |\xi'| > A\,,$$

and finally for some constant B

$$\sum |P_j(\xi')|\, |\xi'|^{q|\beta_1|} \leqq C |P_0(\xi')|, \qquad |\xi'| > B\,.$$

On the other hand we observe that since $P(\xi', \xi'')$ is of type b' in ξ' then of course also

$$P(\xi', 0) = P_0(\xi') + \sum b_j P_j(\xi')$$

is of type b'. But P_j are assumed to be (strictly) weaker than P_0 and so, by lemma (6), $P_0(\xi')$ is of the same type as $P(\xi', 0)$. Hence $P_0(\xi')$ is hypoelliptic of type b'.

We have then proved

LEMMA 7. *A necessary condition for $P(\xi', \xi'')$ to be partially hypoelliptic of type b' in ξ' is that*

$$P(\xi', \xi'') = P_0(\xi') + \sum_{j=1}^{N} P_j(\xi') (\xi'')^{\beta_j}\,,$$

where $P_0(\xi')$ is hypoelliptic of type b', and where $P_j(\xi')$ are strictly weaker than $P_0(\xi')$. Further, the numbers $|\beta_j|$ are restricted by the condition

$$|\beta_j| \leqq b' k_j\,,$$

where k_j is given by

$$\overline{\lim}\, |P_j(\xi')|\, |\xi'|^{k_j} / \big(|P_0(\xi')| + 1\big) = a_j \neq 0\,.$$

Of course all $k_j \leqq \mu =$ degree of P_0 so that we get the general restriction $|\beta_j| \leqq b' \mu$.

Note that when $b'=1$, i.e. when P is partially elliptic, we get the necessary condition that

$$(37) \qquad P(\xi',\xi'') = P_0(\xi') + \sum P_j(\xi')Q_j(\xi'')$$

with $P_0(\xi')$ elliptic, $\deg P_j < \deg P_0$ and $\deg P_j + \deg Q_j \leqq \deg P_0$. This shows that the condition in lemma 7 is not sufficient in order that P be of finite type. The polynomials satisfying (37) constitute indeed a broader class, the so called conditionally elliptic polynomials. These were introduced by Gårding and Malgrange in [2].

In the special case when $\xi' = \xi_1$ the lemma tells us that

$$P(\xi) = A\xi_1{}^{\mu} + \sum_{j=1}^{\mu} \xi_1{}^{\mu-j} Q_j(\xi''),$$

where Q_j are polynomials of degree $b_1 j$ at most. We can use this result to get an estimate of b_1 when P is given. If we denote the degree of Q_j by d_j we know that

$$d_j \leqq b_1 j, \qquad j = 1,2,\ldots,\mu,$$

and thus

$$b_1 \geqq \max_{j \neq 0} d_j/j = d.$$

That equality cannot be valid in general is clear from examples, e.g.

$$P(\xi) = (\xi_1+\xi_2)^2 + i(\xi_1-\xi_2),$$

where $b_1 = 2$ but $d_1 = 1$. Of course the same polynomial can be written, after a change of coordinates, as $\eta_2{}^2 + i\eta$, with $b_1 = d_1 = 2$. Then instead $b_2 = 1$ but $d_2 = \frac{1}{2}$.

Added in the proof. We know that every partially hypoelliptic polynomial satisfies the condition

$$(9) \qquad |\operatorname{Re}\zeta'| \leqq C(1+|\operatorname{Im}\zeta|+|\operatorname{Re}\zeta''|)^b$$

for some b and all $\zeta \in V(P)$. It can be proved that $P(\xi)$ satisfies (9) if and only if

$$P(\xi) = P_0(\xi') + \sum P_\gamma(\xi')\,\xi''_\gamma$$

with

$$\sum_{|\alpha+\gamma|\geqq 0} |P_\gamma^{(\alpha)}(\xi')|\,|\xi'|^{|\alpha+\gamma|/b} \leqq C(|P_0(\xi')|+1).$$

(Note that this is a strengthening of lemma 7, since every polynomial of type b in ξ' satisfies (9).) A polynomial $P(\xi)$ of this kind may well be called "conditionally hypoelliptic of type b in ξ'." Indeed, e.g. every solution $u \in \mathscr{E}$ of the equation $P(D)u=0$ belongs to the class A_b provided only that u and a number of its derivatives belongs to $A_{1(x'')}$. Details will probably follow in another paper.

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