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INFINITE WEIGHTED GRAPHS WITH BOUNDED RESISTANCE METRIC

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(To the memory of Ola Bratteli)

Abstract

We consider infinite weighted graphs G, i.e., sets of vertices V, and edges E assumed countably infinite. An assignment of weights is a positive symmetric function c on E (the edge-set), conductance. From this, one naturally defines a reversible Markov process, and a corresponding Laplace operator acting on functions on V, voltage distributions. The harmonic functions are of special importance. We establish explicit boundary representations for the harmonic functions on G of finite energy.

We compute a resistance metric d from a given conductance function. (The resistance distance $d(x, y)$ between two vertices x and y is the voltage drop from x to y, which is induced by the given assignment of resistors when 1 amp is inserted at the vertex x, and then extracted again at y.)

We study the class of models where this resistance metric is bounded. We show that then the finite-energy functions form an algebra of $1/2$-Lipschitz-continuous and bounded functions on V, relative to the metric d. We further show that, in this case, the metric completion M of (V, d) is automatically compact, and that the vertex-set V is open in M. We obtain a Poisson boundary-representation for the harmonic functions of finite energy, and an interpolation formula for every function on V of finite energy. We further compare M to other compactifications; e.g., to certain path-space models.

1. Introduction

Discrete analysis on infinite graphs (i.e., networks of resistors, (V, E, c), V for vertices, E for edges, and c for conductance function (see section 2)) is a subject where the applications and the examples (see the second half of the paper) are at least as important as the pure theorems. While the discrete setting is significant in its own right, it also makes intriguing connections to more classical results in continuous potential theory; see e.g., sections 7.1 and 7.2 below. For example, our present discrete graph Laplacians often serve as numerical approximations, e.g., finite differences, for classical (continuous) Laplacians. In section 7, we stress similarities and differences: for example when realized as densely defined operators in suitable L^2 spaces, the classical Laplacians are

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unbounded. By contrast, the (discrete) graph Laplacians may be bounded or not. This question, and a host of spectral theoretic properties, is decided by the properties of the conductance function c going into the definition of a particular graph Laplacian; see sections 5 and 6. We shall also make direct comparisons of the potential theoretic properties in the two contexts, discrete vs. continuous. Especially we offer new results for the associated Green's functions. The Green's function for the graph Laplacian is introduced first in Lemma 2.8 and Corollary 9.5, below; and it is then revisited at several instances inside the paper. In the classical case, a Green's function may be realized as a fundamental solution to a suitable Dirichlet problem. By contrast to the discrete case, if a graph Laplacian is realized in matrix form as an $\infty \times \infty$ matrix, with rows and columns indexed by the vertex set V, the corresponding Green's function is a matrix-inverse. With the use of our analysis in energy Hilbert space (section 2.1), we show that we get an explicit formula for this Green's function, and our results on resistance metrics, and path-space analysis (section 8) are a part of this.

We consider a certain class of infinite weighted graphs G. They are specified by prescribed sets of vertices V, and edges E; assumed countably infinite. An assignment of weights, is a positive symmetric function c of E (the edge-set). In electrical network models, the function c represents conductance, and its reciprocal resistance. So fixing a conductance function is then equivalent to an assignment of resistors on the edges of G. From this, one naturally defines a reversible Markov process, and a corresponding Laplace operator (called graph Laplacian) acting on functions on V, the vertex-set. Functions on V typically represent voltage distributions, and the harmonic functions are of special importance. For list of explicit details required on (V, E, c), we refer to the details in section 2.

We will be especially interested in boundary representations for harmonic functions of finite energy. From a given conductance function, we compute a resistance metric d (see Theorem 3.4). Intuitively, the resistance distance $d(x, y)$ between two vertices x and y is the voltage drop from x to y, which is induced by the given assignment of resistors when 1 amp is inserted at the vertex x, and then extracted again at y (see Figure 3.1). We study the realistic class of models when this resistance metric is assumed bounded. In this case the finite-energy functions form an algebra of continuous and bounded functions on V, relative to the metric d. We further show that, in this case, the metric completion M of (V, d) is automatically compact. The vertex-set V is open in M, and we obtain a Poisson boundary-representation for the harmonic functions of finite energy. A number of additional properties are established for M. In particular, we compare M to other compactifications in the literature; e.g., to path-space models.

There is a recent increased interest in analysis on large (infinite) networks, motivated by a host of applications; see e.g., [26], [27], [5], [29], [1]. We shall be citing standard facts from the general theory. In addition, we use facts from analysis, Hilbert space geometry, potential theory, boundaries, and Markov measures; see e.g., [35], [12], [34], [22], [16], [30], [8].

2. Basic setting

Let $G = (V, E, c)$ be a weighted graph, where c = conductance function (see Definition 2.1), V = vertex-set (countably infinite), and the edges $E \subset V \times V \setminus \{\text{diagonal}\}$ such that:

(G1) $(x, y) \in E \iff (y, x) \in E$; $x, y \in V$;

(G2) $0 < \#\{y \in V \mid (x, y) \in E\} < \infty$, for all $x \in V$;

(G3) The function c is strictly positive on E, and zero on its complement $(V \times V) \setminus E$;

(G4) Connectedness: $\exists\, o \in V$ s.t. for all $y \in V$ $\exists\, x_0, x_1, \ldots, x_n \in V$ with $x_0 = o$, $x_n = y$, $(x_{i-1}, x_i) \in E$, $\forall i = 1, \ldots, n$.

Notational convention

Pairs (x, y) with comma may refer to an edge linking two *neighbor* vertices. On occasion, we shall use the letter e to denote an edge. This choice is handy in cases when it is not important to identify the corresponding neighbor vertices of an edge. We further stress that our edges are not directed; and that neighbor vertices are distinct. These definitions are motivated in part by standard conventions from electrical network models. See Definition 2.1 and Remark 2.2 below.

When an arbitrary pair of two vertices w and z occurs, we shall use the notation wz; typically as a subscript notation. Because of our connectedness assumption (G4), any pair of vertices w and z may be "connected" with a finite set of edges, one starting at w, and the last edge ending at z. But we stress that, in general, there are many possible choices of finite edges accomplishing the linking from w to z (see Figure 2.1). In electric network models, current is traveling along paths between pairs of vertices. Following the accepted conventions in the subject, we shall often denote a function on the set of edges with a subscript, without the comma and parenthesis.

DEFINITION 2.1. A function $c\colon V \times V \to \mathbb{R}_+ \cup \{0\}$ is called a *conductance function* if

(1) $c(e) > 0$, $\forall e \in E$; and

(2) given $x \in V$, $c_{xy} = c_{yx}$, for all $(x, y) \in E$.

Also

(3) if $x \in V$, we set

$$c(x) := \sum_{y \sim x} c_{xy}, \quad \text{where} \quad x \sim y \overset{\text{Def}}{\iff} (x, y) \in E. \tag{2.1}$$

(We shall assume that (G2) holds, i.e., $\#\{y \in V \mid y \sim x\} < \infty$, for all $x \in V$.)

Examples of networks (V, E)*, vertices vs. edges*

1. Lattices. Fix $d \in \mathbb{N}$, and set $V_d := \mathbb{Z}^d$. Then every $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, $x_i \in \mathbb{Z}$, has $2d$ neighbors, i.e., $N(x)$ consists of the following points $y \in \mathbb{Z}^d$:

$y \in N(x)$ iff (Def.) $\exists\, i$ s.t. $y_i \in \{x_i \pm 1\}$, and $y_j = x_j$, when $j \neq i$.

Hence the corresponding set of edges $E_d \subset V_d \times V_d \setminus$ (diagonal) is the set of unordered pairs $(x, y) \in V_d \times V_d$ s.t. $\exists\, i$ with $|y_i - x_i| = 1$, and $y_j = x_j$, for $j \neq i$.

2. Binary trees. The set of vertices V is as follows:

If $d > 0$, then each $x = (x_1, x_2, \dots, x_d) \in \{0, 1\}^d$ has three neighbors:

$$N(x) = \big\{(x_1, x_2, \dots, x_d, y) \mid y \in \{0, 1\}\big\} \cup \{(x_1, x_2, \dots, x_{d-1})\}.$$

We denote the base-point of the binary tree to be $\emptyset$ (the empty word), and $N(\emptyset) =$ the set of two vertices, 0 and 1. Note that the binary tree is one of the simplest Bratteli diagrams; see section 7.3.

REMARK 2.2. After a reduction to the case of connected networks (V, E, c), we may assume that, for every vertex $x \in V$, there is a finite number of edges, connecting to what are called neighbors of x (see (G4)). So when x is fixed, its set of neighbors $N(x)$ is indexed by edges $e = (x, y)$, for y in $N(x)$. First we consider $c(\cdot, \cdot)$ to be a symmetric function on $V \times V$, but it is supported on the set E of all edges, so $c(x, y) = 0$ if $(x, y) \notin E$, see (G3). We may therefore also consider c as a *positive* function on E. Note that the total set E of all edges is the union of the sets $N(x)$, as x ranges over V. Pairs of sets $N(x)$ and $N(x')$ in general overlap of course.

In more detail: for every x, the conductance $c(x, y)$ is positive only on edges $e = (x, y)$ where y is one of the neighbors, so y in $N(x)$. The symmetry of c allows us to identify (x, y) and (y, x) for any pair of neighbors; the two represent the same edge, say e. In many computations, we also use $c(x) :=$ the sum over $c(x, y)$ for y in the finite set of neighbors $N(x)$; we write $y \sim x$ (see (3) in Definition 2.1). For some formulas it is useful for us to write $c(x, y)$ for all points in $V \times V$, but then it is understood that c is supported on the set E, so the union of neighbors. The finiteness assumption on $N(x)$ is realistic

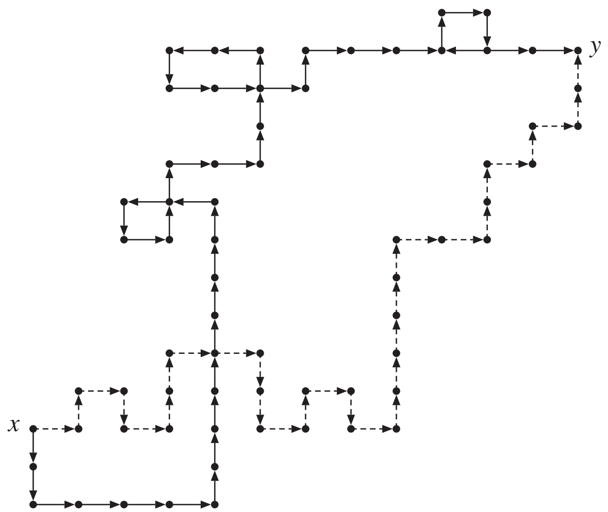


FIGURE 2.1. $V = \mathbb{Z}^2$ and neighbors: given distant pairs x and y in V, we sketch some examples of paths in $E(x, y)$. Note that when distant vertices x and y are picked, then each element in the set $E(x, y)$ (= paths from x to y) is made up of a finite set of edges (elements in E), linked together and forming a path from x to y. But the set $E(x, y)$ is generally infinite.

in electric network applications (see (G2)). It could be relaxed of course, but then we would have to assume instead that $c(x) :=$ the sum over $c(x, y)$ for y in the set $N(x)$ of neighbors be convergent; see (2.1). Here we stick with *finite* neighbors.

Since $c(\cdot, \cdot)$ represents conductance $= 1/\text{resistance}$, the symmetry condition is realistic for computation of the resistance distance; see section 3 below. The resistance distance (see Theorem 3.4) refers to a pair of distant vertices, i.e., points x and y from V that are not neighbors; and computation of the resistance distance will then involve path-space analysis (see section 8); in this case when distant vertices x and y are fixed, the set $E(x, y)$ consists of all finite paths from x to y (see Figure 3.1). Fix distant vertices x and y. A finite path connecting them is made up of a finite set of edges, so elements in $E(x, y)$, and of course each e in an element of $E(x, y)$ will link neighbors. The connectedness assumption states that for every pair of points from V, the set $E(x, y) := \{\text{all finite paths from } x \text{ to } y\}$ is non-empty (see Figure 2.1).

Now pick a path from the set $E(x, y)$: the first edge e in such a path will start at x, and the last e in it will end in y. Since for distant pairs of vertices x and y, the set $E(x, y)$ can be quite complicated, path space analysis is one of the useful tools (see Figure 2.1).

2.1. The energy Hilbert space

Let $G = (V, E, c)$ be a *connected* graph as above. Set

$$\mathscr{H}_E := \left\{ u \colon V \to \mathbb{C} \mid \|u\|_{\mathscr{H}_E} < \infty \right\},$$

where

$$\langle u, v\rangle_{\mathscr{H}_E} := \frac{1}{2} \sum_{(x,y)\in E} c_{xy}\big(\overline{u(x)} - \overline{u(y)}\big)\big(v(x) - v(y)\big), \tag{2.2}$$

$$\|u\|^2_{\mathscr{H}_E} := \frac{1}{2} \sum_{(x,y)\in E} c_{xy}|u(x) - u(y)|^2. \tag{2.3}$$

Then $\mathscr{H}_E$, modulo constants, is a Hilbert space of functions on V [26]. ($\mathscr{H}_E$ is known to be bigger than the $\mathscr{H}_E$-norm completion of the finitely supported functions on V. For electrical networks, the expression in (2.3) represents energy; see e.g. [26]. The non-constant harmonic functions on V are *not* in the $\mathscr{H}_E$-completion of the finitely supported functions.)

DEFINITION 2.3. Fix a weighted graph (connected), set the *graph Laplacian* $\Delta = \Delta_c$, where

$$(\Delta u)(x) = \sum_{y\sim x} c_{xy}(u(x) - u(y)) = c(x)u(x) - \sum_{y\sim x} c_{xy}u(y),$$

is defined for *all* functions u on V. It passes to the quotient modulo the constant functions.

LEMMA 2.4 ([26]). (i) *For every pair of vertices $x, y \in V$, there is a $v_{xy} \in \mathscr{H}_E$, unique up to an additive constant, such that*

$$f(x) - f(y) = \langle v_{xy}, f\rangle_{\mathscr{H}_E}, \quad \forall f \in \mathscr{H}_E. \tag{2.4}$$

(ii) *The vector v_{xy} in* (2.4) *satisfies*

$$\Delta v_{xy} = \delta_x - \delta_y, \tag{2.5}$$

where $(\Delta f)(u) := \sum_{y\sim u} c_{uy}(f(u) - f(y))$.

REMARK 2.5. The solution to (2.5) is not unique: if v_{xy} satisfies (2.5), and if $h \in \mathscr{H}_E$ satisfies $\Delta h = 0$ (harmonic), then $v_{xy} + h$ also satisfies (2.5); but generally *not* (2.4).

Let $V' := V \setminus \{o\}$, and set

$$v_x := v_{xo}, \quad \forall x \in V'. \tag{2.6}$$

COROLLARY 2.6. *For all $x, y \in V$, there is a unique real-valued* dipole *vector $v_{xy} \in \mathscr{H}_E$ such that*

$$\langle v_{xy}, u\rangle_{\mathscr{H}_E} = u(x) - u(y), \quad \forall u \in \mathscr{H}_E. \tag{2.7}$$

Moreover, $v_{xy} - v_{zy} = v_{xz}$, $\forall x, y, z \in V$.

DEFINITION 2.7. Let (V, E, c) and Δ be as outlined, and let $\mathscr{H}_E$ be the corresponding energy Hilbert space; see (2.3). Let $\ell^2 = \ell^2(V)$ denote the usual ℓ^2-space.

We shall need the subspace $\mathscr{D}_2 \subset \ell^2$ (dense in the ℓ^2-norm):

$$\mathscr{D}_2 := \text{span}\{\, \delta_x \mid x \in V \,\}.$$

If $\{v_x \mid x \in V'\}$ denotes a system of dipoles (see (2.6)), we set $\mathscr{D}_E \subset \mathscr{H}_E$ (dense in $\mathscr{H}_E$-norm):

$$\mathscr{D}_E := \text{span}\{\, v_x \mid x \in V' \,\}. \tag{2.8}$$

In both cases "span" means all finite linear combinations.

We show in section 8 that $\ell^2(V)$ contains *no* non-constant harmonic functions; but $\mathscr{H}_E$ generally does.

LEMMA 2.8. *The following hold:*

(1) $\langle \Delta u, v\rangle_{\ell^2} = \langle u, \Delta v\rangle_{\ell^2}$, $\forall u, v \in \mathscr{D}_2$;
(2) $\langle \Delta u, v\rangle_{\mathscr{H}_E} = \langle u, \Delta v\rangle_{\mathscr{H}_E}$, $\forall u, v \in \mathscr{D}_E$;
(3) $\langle u, \Delta u\rangle_{\ell^2} \geq 0$, $\forall u \in \mathscr{D}_2$; *and*
(4) $\langle u, \Delta u\rangle_{\mathscr{H}_E} \geq 0$, $\forall u \in \mathscr{D}_E$.

As a densely defined operator in $\ell^2(V)$, Δ is essentially selfadjoint; but, as an operator with dense domain in $\mathscr{H}_E$, Δ is generally not *essentially selfadjoint.*

Moreover, we have $\delta_x \in \mathscr{H}_E$, $x \in V$, where δ_x denotes Dirac's function; and

(5) $\langle \delta_x, u\rangle_{\mathscr{H}_E} = (\Delta u)(x)$, $\forall x \in V$, $\forall u \in \mathscr{H}_E$;
(6) $\Delta v_{xy} = \delta_x - \delta_y$, $\forall x, y \in V$, *where* $v_{xy} \in \mathscr{H}_E$; *in particular,* $\Delta v_x = \delta_x - \delta_o$, $x \in V' = V \setminus \{o\}$;
(7) $\delta_x(\cdot) = c(x)v_x(\cdot) - \sum_{y\sim x} c_{xy}v_y(\cdot)$, $\forall x \in V'$;
(8)

$$\begin{aligned}\Delta(\delta_x)(y) &= \Delta(\delta_y)(x)\\ &= \langle \delta_x, \delta_y\rangle_{\mathscr{H}_E} = \begin{cases} c(x) = \sum_{t\sim x} c_{xt}, & \text{if } y = x,\\ -c_{xy}, & \text{if } (x, y) \in E,\\ 0, & \text{if } (x, y) \notin E.\end{cases}\end{aligned} \tag{2.9}$$

PROOF. See [26], [27], [24]. For the selfadjointness of the graph Laplacian in $\ell^2(V)$, see Theorem 2.9 below.

THEOREM 2.9 ([24], [26], [27], [36], [28]). *Let $G = (E, V, c)$ be a weighted graph as specified above, so with a given conductance function c defined on the set of edges E of G, and let Δ be the corresponding Laplace operator. Then, as an operator in $\ell^2(V)$ with domain consisting of finitely supported functions, Δ is essentially selfadjoint.*

PROOF. Below we give a new proof of this essential selfadjointness. One advantage with the proof below is its use of different properties of the operator Δ than in earlier approaches. We also believe that the idea used here has wider use—that it is applicable to other operators in analysis and potential theory, both discrete and continuous.

Note the following are equivalent:

(i) $f \in \ell^2(V)$ is a Δ-defect vector;

(ii) $\langle \varphi + \Delta\varphi, f\rangle_{\ell^2} = 0$, $\forall \varphi \in \text{span}\{\delta_x\}$;

(iii) $(1 + c(x)) f(x) - \sum_{y\sim x} c_{xy} f(y) = 0$, $\forall x \in V$;

(iv) $(1 + c(x)) f(x) - c(x)(\mathbb{P} f)(x) = 0$, $\forall x \in V$, where $p_{xy} = c_{xy}/c(x)$, and $(\mathbb{P} f)(x) = \sum_{y\sim x} p_{xy} f(y)$;

(v) $(\mathbb{P} f)(x) = \left(1 + \frac{1}{c(x)}\right) f(x)$, $\forall x \in V$.

With the splitting $f = \text{Re}\{f\} + i\,\text{Im}\{f\}$, it is enough to consider the case when f is real valued.

Since $f \in \ell^2(V)$, it has a maximum, i.e., $\exists\, x_0 \in V$ s.t. $f(\cdot) \leq f(x_0)$ in V. Assume $f(x_0) > 0$ (otherwise replace f by $-f$). Now, if f is a defect vector, we have

$$\left(1 + c(x_0)^{-1}\right) f(x_0) \overset{\text{(by (v))}}{=} (\mathbb{P} f)(x_0) \leq f(x_0) \Longrightarrow c(x_0)^{-1} f(x_0) \leq 0,$$

which contradicts the assumption that $f(x_0) > 0$.

THEOREM 2.10. *Let $(V, E, c, \Delta, \mathscr{H}_E)$ be as above, and fix a base-point $o \in V$. Set $V' := V \setminus \{o\}$. Fix a dipole $v_x := v_{xo}$, $x \in V'$. Set*

$$(\Delta^{-1})_{xy} := \langle v_x, v_y\rangle_{\mathscr{H}_E}, \quad (x, y) \in V' \times V'.$$

Then Δ is not *essentially selfadjoint on $\mathscr{D}_E := \text{span}\{v_x \mid x \in V'\}$ if and only if there is a non-zero function $f \in \mathscr{H}_E$ such that*

$$h(x) := f(x) + \sum_{y\in V'} (\Delta^{-1})_{xy} f(y) \tag{2.10}$$

is harmonic.

Proof. By general operator theory (see [14]), the essential selfadjointness assertion holds if and only if the following implication holds:

$$\big[f \in \mathscr{H}_E, \text{ and } \langle \varphi + \Delta\varphi, f\rangle_{\mathscr{H}_E} = 0, \ \forall \varphi \in \mathscr{D}_E\big] \Longrightarrow \big[f = 0\big]. \tag{2.11}$$

Taking $\varphi = v_x$, and modulo an additive constant, we see that a possible solution $f \in \mathscr{H}_E$ to (2.11) will satisfy

$$(\mathbb{P}f)(x) = \big(1 + c(x)^{-1}\big) f(x), \quad \forall x \in V', \tag{2.12}$$

where $(\mathbb{P}f)(x) = \sum_{y\sim x} p_{xy} f(y)$, $p_{xy} = c_{xy}/c(x)$.

An iteration of (2.12) yields

$$(\mathbb{P}^{n+1} f)(x) = f(x) + \sum_{k=0}^{n} \mathbb{P}^k (f/c)(x). \tag{2.13}$$

But we have pointwise convergence on the right-hand side in (2.13), and $(1-\mathbb{P})^{-1} = (\Delta/c)^{-1}$, so $(1-\mathbb{P})^{-1}(f/c)(x) = \Delta^{-1}(\operatorname{diag}(c))(f/c)(x) = (\Delta^{-1} f)(x) = \sum_y (\Delta^{-1})_{xy} f(y)$. Hence the left-hand side in (2.13) must converge pointwise; but it is clear that $h = \lim_n \mathbb{P}^n f$ is harmonic.

Finally, it is clear that every solution $f \in \mathscr{H}_E$ to (2.10) will satisfy (2.11); which in turn is the equation which decides non-essential selfadjointness, by general theory.

Remark 2.11. We introduce the Markov measure $\mu^{(\text{Markov})}$ on the space Ω of all $G = (V, E)$-paths, and the Markov-walk process $\pi_n(\omega) := \omega_n$, $\forall \omega \in \Omega$, $n \in \mathbb{N}_0$, where $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$, $\omega_j \in V$, $(\omega_j, \omega_{j+1}) \in E$, $\forall j \in \mathbb{N}_0$. Then the matrix product $\mathbb{P}^k$ in (2.13) is $\operatorname{Prob}(\{\pi_{m+k} = y \mid \pi_m = x\}) = (\mathbb{P}^k)_{xy}$. We shall return to this Markov process in section 8 below.

3. From conductance to current flow

Let $G = (V, E, c)$ be an infinite weighted graph (connected, see (G4) before Definition 2.1). As before, V = vertex set, E = edges, and $c\colon E \to \mathbb{R}_+$ is a fixed conductance function, so that $c = (c_{xy})$, $(x, y) \in E$. Let $\mathscr{H}_E$ be the corresponding energy Hilbert space (see (2.2)–(2.3)).

Set the current flow $I_{(xy)} := \partial w$, where

$$I_{xy} = (\partial w)(x, y) = c_{xy}(w(x) - w(y)), \quad \forall (x, y) \in E, \ w \in \mathscr{H}_E, \tag{3.1}$$

and set

$$\text{Dissp} = \Big\{ \partial w \ \Big| \ w \in \mathscr{H}_E, \ \|\partial w\|^2_{\text{Diss}} := \tfrac{1}{2} \sum I^2_{xy}/c_{xy} < \infty \Big\}$$

as a weighted ℓ^2-space on E, where $1/c_{xy}$ = resistance.

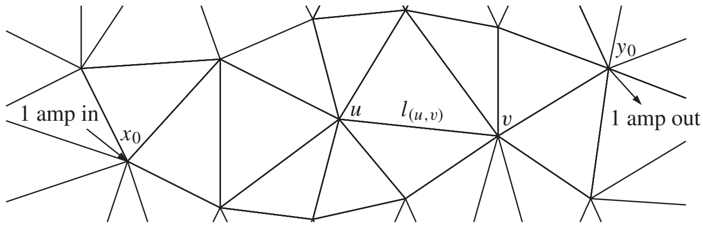


FIGURE 3.1. The convex set $W_{x_0 y_0}$. On edges $(u, v) \in E$ from x_0 to y_0 in V, the current is $I_{uv} = c_{uv}(f(u) - f(v))$, and f denotes a voltage-distribution.

As an illustration, Figure 3.1 shows a vertex set $W_{x_0 y_0}$, where current flows from vertex x_0 to vertex y_0; with a given conductance function c.

LEMMA 3.1. *The operator* $\partial\colon \mathscr{H}_E \to \text{Dissp}$ *is isometric; but generally not onto* Dissp.

PROOF. One checks that

$$\begin{aligned} \|w\|^2_{\mathscr{H}_E} &= \frac{1}{2}\sum c_{xy}|w(x) - w(y)|^2 \quad \text{(energy)} \\ &= \frac{1}{2}\sum I^2_{xy}/c_{xy} \quad \text{(dissipation)} \end{aligned}$$

where $I_{xy} = (\partial w)_{xy} = c_{xy}(w(x) - w(y))$, $1/c_{xy}$ = resistance on the edge (x, y), and where the summations are over the prescribed set E of edges; see (3.1) and the lemma follows.

DEFINITION 3.2. Set $d_{\text{res}}(x_0, y_0)$ = distance $x_0 \to y_0$ = voltage drop from x_0 to y_0 when current I satisfies $I = 1$ at x_0 "in" and current $I = -1$ at y_0 "out."

THEOREM 3.3. *There is a unique current flow such that*

$$d_{\text{res}}(x_0, y_0) = \inf\left\{\|I\|^2_{\text{Diss}} : I|_{\{x_0, y_0\}} = 1 \text{ amp in, and } 1 \text{ out}\right\} \tag{3.2}$$

PROOF. Recall that by Lemma 2.6, $\exists!\, v_{xy}$ s.t.

$$\langle v_{xy}, f\rangle_{\mathscr{H}_E} = f(x) - f(y), \quad \forall (x, y) \in V \times V,\ \forall f \in \mathscr{H}_E. \tag{3.3}$$

Set $I = \partial v_{xy}$, then

$$\begin{aligned} d_{\text{res}}(x_0, y_0) = \inf \|I\|^2_{\text{Diss}} &= \|\partial v_{x_0 y_0}\|^2_{\text{Diss}} \\ &= \|v_{x_0 y_0}\|^2_{\mathscr{H}_E} \quad (= \text{resistance distance}); \end{aligned} \tag{3.4}$$

i.e., the infimum in (3.2) is obtained at the flow $I = \partial v_{x_0 y_0}$, see (3.3)–(3.4). For a proof, see [26], [27].

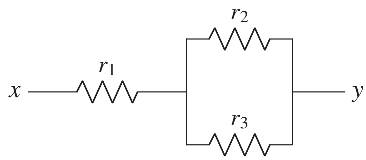


FIGURE 3.2. Example of resistor configuration in a network: configuration of three resistors, having values r_1, r_2, r_3 Ohm.

The infimum in (3.2) and (3.4) is justified with the following Hilbert space geometry applied to the energy Hilbert space $\mathscr{H}_E$: the infimum in (3.2) is attained when $I_0 = \partial v_{x_0 y_0}$. We use that I_0 is the vector in the convex set $W_{x_0 y_0}$ of minimum norm. Since ∂ from Lemma 3.1 is isometric, we see that $W_{x_0 y_0}$ is both closed and convex. From Hilbert space geometry, see e.g. [32], we know that $W_{x_0 y_0}$ contains a vector of smallest norm. From the definition of $W_{x_0 y_0}$ (see e.g., Figure 3.1), we conclude that the minimum must be $I_0 = \partial v_{x_0 y_0}$; see also [27].

Below, we offer five different, but equivalent, formulas for the resistance metric $d_{\text{res}}(x, y)$:

THEOREM 3.4 ([27]). *Let V, E, c, Δ, and d_{res} be as above; let $x, y \in V$, and let W_{xy} denote the set of all paths between a pair of vertices designated vertices, x and y (see Figure 3.1). Then*

$$\begin{aligned} d_{\text{res}}(x, y) = \|v_{xy}\|^2_{\mathscr{H}_E} &= \min\big\{\|I\|^2_{\text{Diss}} : I \in W_{xy}\big\} \\ &= \|w\|^2_{\mathscr{H}_E} \qquad \textit{when } \Delta w = \delta_x - \delta_y \\ &= 1/\min\big\{\|w\|^2_{\mathscr{H}_E} : w \in \mathscr{H}_E,\ |w(x) - w(y)| = 1\big\} \\ &= \sup\big\{|w(x) - w(y)|^2 : w \in \mathscr{H}_E,\ \|w\|_{\mathscr{H}_E} \le 1\big\}. \end{aligned}$$

EXAMPLE 3.5 (see Figure 3.2). $d_{\text{res}}(x, y) = r_1 + \left(r_2^{-1} + r_3^{-1}\right)^{-1}$.

4. The metric boundary

DEFINITION 4.1. By M we mean the set of equivalence classes of sequences $(x_i) \subset V$ of vertices such that $\lim_{i,j\to\infty} d(x_i, x_j) = 0$ (Cauchy) under the relation $(x_i) \sim (y_i)$ iff (Def.) $\lim_{i\to\infty} d(x_i, y_i) = 0$. Here, $d(x, y) = d_{\text{res}}(x, y)$ is the resistance metric in equation (3.2).

The vertex-set V is identified with a subset of M via the mapping $\gamma\colon V \to M$, $V \ni x \longmapsto \gamma(x) = \text{class}(x, x, x, \ldots)$. Hence $b \in M \setminus V$ (the boundary of V) iff $b = (y_i) \in M$ satisfies the following: $\forall x \in V, \exists\, \varepsilon \in \mathbb{R}_+, \exists\, (y_{i_k}) \subset (y_i)$, s.t. $d(x, y_{i_k}) \geq \varepsilon$, $\forall k \in \mathbb{N}$. Note that the assertion states that $d(\gamma(x), b) > 0$, $\forall x \in V$.

We now show that if $d := d_{\mathrm{res}}$ is *bounded*, then every function $f \in \mathscr{H}_E$ extends by closure to M: if $b \in M$, and $\{x_i\} \subset V$, are such that $\lim_{i\to\infty} d(x_i, b) = 0$, we set $\widetilde{f}(b) = \lim_{i\to\infty} f(x_i)$. It is then immediate that $\left|\widetilde{f}(b) - \widetilde{f}(b')\right|^2 \leq d(b, b')\|f\|^2_{\mathscr{H}_E}$. We set $\widetilde{\mathscr{H}}_E = \{\widetilde{f} \mid f \in \mathscr{H}_E\}$.

THEOREM 4.2. *If the resistance metric $d = d_{\mathrm{res}}$ is bounded on $V \times V$, then*

$$\mathscr{H}_E \subset \ell^\infty(V), \quad \textit{and} \quad \widetilde{\mathscr{H}}_E \subseteq C(M); \tag{4.1}$$

i.e., every energy function w on V is bounded, and $\mathscr{H}_E$ is an algebra under pointwise product.

PROOF. The containment in (4.1) follows from the estimate (5.1).

We proceed to show that $\mathscr{H}_E$ is an algebra when (V, d) is assumed bounded. Let $u, w \in \mathscr{H}_E$, then $(uw)(x) := u(x)w(x)$, $\forall x \in V$, satisfies

$$\|uw\|^2_{\mathscr{H}_E} \leq (\|u\|^2_\infty + \|w\|^2_\infty)(\|u\|^2_{\mathscr{H}_E} + \|w\|^2_{\mathscr{H}_E}). \tag{4.2}$$

Since $u, w \in \ell^\infty(V)$, it follows that $uw \in \mathscr{H}_E$, i.e., $\|uw\|_{\mathscr{H}_E} < \infty$. The proof of (4.2) is as follows:

$$\begin{aligned}
&\sum_E c_{xy}|(uw)(x) - (uw)(y)|^2 \\
&\quad = \sum_E c_{xy}|u(x)(w(x) - w(y)) + w(y)(u(x) - u(y))|^2 \\
&\quad \overset{\text{(Schwarz)}}{\leq} \sum_E c_{xy}(|u(x)|^2 + |w(y)|^2)(|u(x) - u(y)|^2 + |w(x) - w(y)|^2) \\
&\quad \leq (\|u\|^2_\infty + \|w\|^2_\infty)\left(\sum_E c_{xy}|u(x) - u(y)|^2 + \sum_E c_{xy}|w(x) - w(y)|^2\right),
\end{aligned}$$

which is the desired estimate.

COROLLARY 4.3. *Let $V, E, c, d = d_{\mathrm{res}}$ be as above, i.e., assume that d is bounded, and that M is compact. Then when the constant function $\mathbb{1}$ on M is adjoined $\widetilde{\mathscr{H}}_E$ is a dense subalgebra, dense in the uniform norm on $C(M)$.*

PROOF. We already proved that $\widetilde{\mathscr{H}}_E$ is an algebra of continuous functions on M (the metric completion of (V, d_{res})), so we only need to show that it is dense in the $\|\cdot\|_\infty$-norm on M. Since M is compact, $\|\widetilde{f}\|_\infty = \max\{|\widetilde{f}(b)| : b \in M\}$.

It is clear that $\widetilde{\mathscr{H}}_E$ is closed under complex conjugation; so, by the Stone-Weierstrass theorem, we only need to prove that it separates points. We will prove that if $b \neq b'$ in M then there is a vertex $x \in V$ such that $\widetilde{v}_x(b) \neq \widetilde{v}_x(b')$.

Since M is the metric completion of (V, d), it is enough to show that $\widetilde{\mathscr{H}}_E$ separates points in V. Assume the contrary: that there are vertices $y, z \in V$, $y \neq z$ such that $v_x(y) = v_x(z)$ holds for all $x \in V$; in other words, $\langle v_x, v_y - v_z\rangle_{\mathscr{H}_E} = 0$ holds for all $x \in V$. But $\operatorname{span}\{v_x \mid x \in V\}$ is dense in $\mathscr{H}_E$; and so $v_y - v_z = 0$, contradicting $d(y, z) = \|v_y - v_z\|^2_{\mathscr{H}_E} > 0$.

5. Discrete resistance metric-metric completions

Set $d := d_{\text{res}}$ the resistance metric, see (3.4). Let $(M, \widetilde{d})$ be the metric completion of (V, d), i.e., V consists of a metric space M with the metric $d_{\text{res}}(x, y) = \|v_{xy}\|^2_{\mathscr{H}_E}$, where v_{xy} is the *dipole* vector in (2.7). Now assume that $d = d_{\text{res}}$ is *bounded*.

Below we discuss compactness of the metric boundary. There are two main points. (i) We identify a setting where compactness does hold. (ii) In this setting, we prepare the ground for an application of the Arzelà-Ascoli Theorem. Caution, point (i) is subtle, as we illustrate in Example 5.6.

DEFINITION 5.1. We say that a system $(V, E, c, d_{\text{res}})$ is *type A* if whenever $\lim_j v_{x_j}$ exists in $C(V, d)$ then (x_j) is a Cauchy sequence in (V, d).

THEOREM 5.2. *If* d_{res} *is bounded on* $V \times V$, *and if the system* $(V, E, c, d_{\text{res}})$ *is of type A, then* $(M, \widetilde{d})$ *is a compact metric space.*

PROOF. Fix a base-point $o \in V$, and set $v_x = v_{x,o}$, $x \in V \setminus \{o\}$, then $v_{xy} = v_x - v_y$, see Lemma 2.6. By Schwarz, applied to the energy Hilbert space $(\mathscr{H}_E, \langle\cdot,\cdot\rangle_{\mathscr{H}_E})$, we get the following Lipschitz-estimate:

$$|f(x) - f(y)|^2 \leq d(x, y)\|f\|^2_{\mathscr{H}}, \quad \forall f \in \mathscr{H}_E,\ x, y \in V. \tag{5.1}$$

Consequences:

(1) Every $f \in \mathscr{H}_E$ extends to a uniformly continuous function $\widetilde{f}$ on M; extension by metric limits.

(2) If $x_i \in V$, and $d(x_i, x_j) \to 0$, for $i, j \to \infty$, then $f(x_i)$ has a limit in $\mathbb{C}$ (or $\mathbb{R}$). Set $\widetilde{x} \in M$, $\widetilde{x} = \lim_i x_i$. If $(x_i), (y_i) \subset V$ are Cauchy sequences, set $\widetilde{d}(\widetilde{x}, \widetilde{y}) = \lim_{i\to\infty} d(x_i, x_j)$, i.e., the extended metric; then by (5.1), we get

$$\left|\widetilde{f}(\widetilde{x}) - \widetilde{f}(\widetilde{y})\right|^2 \leq \widetilde{d}(\widetilde{x}, \widetilde{y})\|f\|^2_{\mathscr{H}_E}. \tag{5.2}$$

The assertion in the theorem follows from the considerations below.

LEMMA 5.3. *An application of Arzelà-Ascoli shows that*

$$\left\{\widetilde{f} \subset C(M) \;\middle|\; f \in \mathscr{H}_E, \|f\|_{\mathscr{H}_E} \leq 1\right\} \tag{5.3}$$

is relatively compact in $C(M)$, in the Montel topology of uniform convergence on compact sets.

PROOF. We refer to [31, Thm 11.28], combined with the results from section 4 above, especially Theorem 4.2. But if d is bounded on $V \times V$, then $\|v_{x_i}\|_{\mathscr{H}_E} \leq A \cdot K_d$, where A is a fixed global constant, since $d(x_i, x_j) = \|v_{x_i} - v_{x_j}\|^2_{\mathscr{H}_E}$.

Hence by (5.3) with K_d in place of 1, we get that:

COROLLARY 5.4. *Assume type A (see Definition 5.1). Then for every sequence $x_1, x_2, x_3, \ldots$ in V, the is a subsequence (x_{i_k}) such that*

(i) $\lim_{k\to\infty} x_{i_k} = b \in M$*; and*

(ii) *for $f_{\text{lim}} \in C(M)$ the limit of the subsequence $\{\widetilde{v}_{x_i}\} \subset C(M)$, we have*

$$\lim_{k\to\infty} \widetilde{v}_{x_{i_k}}(b) = f_{\text{lim}}(b).$$

PROOF. To see that $b \in M$, note that

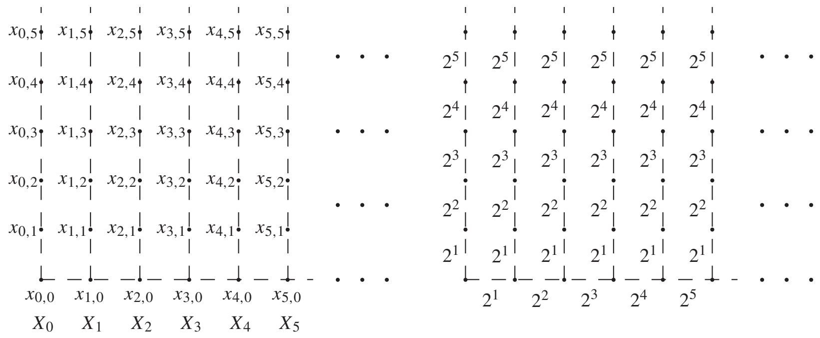
$$d(x_{i_k}, x_{i_\ell}) = \|v_{x_{i_k}} - v_{x_{i_\ell}}\|^2_{\mathscr{H}_E} = |\widetilde{v}_{i_k i_\ell}(x_{i_k}) - \widetilde{v}_{i_k i_\ell}(x_{i_\ell})| \xrightarrow{k,\ell\to\infty} 0;$$

since by (5.2), the functions $\widetilde{v}_{i_k i_\ell}(\cdot)$ are uniformly bounded, and equicontinuous on M. As we assume the system $(V, E, c, d_{\text{res}})$ is of type A, it follows that every sequence $x_1, x_2, \ldots$ in V has a convergence subsequence with limit in M. By the definition of M, the same is true for M, and so M is compact: Every sequence $b_1, b_2, \cdots \subset M$ contains a convergent subsequence.

REMARK 5.5. The following example from [18] shows that our assumed condition "type A" in Theorem 5.2 and Corollary 5.4 cannot be omitted. There are bounded resistance metrics (non-type A) for which the corresponding completions are non-compact. We learned from D. Lenz that the boundedness of the resistance metric does not imply the completion $(M, \widetilde{d})$ is compact [18]. Indeed, the type A assumption for the system $(V, E, c, d_{\text{res}})$ is required. (See Definition 5.1.)

EXAMPLE 5.6 (Example 8.6 in [18]). Figure 5.1a is a tree-like graph with many ends all of which have bounded distance to the root (making the resistance metric bounded) but at the same time being too far apart from each other to be covered by finitely many balls of an fixed but arbitrarily small size. Thus, the weighted graph in this case is bounded with respect to d_{res} metric and the completion is not compact with respect to the resistance metric.

The graph basically consists of a copy of the natural numbers with the property that each natural number has a ray emanating from it and this ray



(a) An infinitely long comb as an infinite array of teeth, each tooth infinitely long.

(b) The conductance function $c_{(xy)}$.

FIGURE 5.1. A doubly infinite planar graph, $V = \bigcup_{n=0}^{\infty} X_n, X_n = \{x_{nk} : k = 0, 1, 2, \ldots\}$ and its conductance function.

being again the natural numbers. There are weights (Figure 5.1b) on the graph making each of these copies of the natural numbers of bounded diameter in the resistance metric. This makes the resistance metric on this graph bounded. On the other hand, a point far out in one of the emanating rays has a uniform distance to any point far out in any other emanating ray. This makes the example non-totally bounded. Hence, the example has the mentioned properties.

LEMMA 5.7. *Let $G = (V, E, c)$ be the weighted graph in Example 5.6. Fix a base-point $o \in V$, and set $\mathscr{D}_E = \text{span}\{v_x \mid x \in V \setminus \{o\}\}$ (see (2.8)). Then $\Delta|_{\mathscr{D}_E}$, as a densely defined Hermitian operator in the energy Hilbert space $\mathscr{H}_E$, is not essentially selfadjoint. Moreover, the deficiency indices are (∞, ∞).*

PROOF. Let c be the conductance function as specified in Figuress 5.1a–5.1b. Suppose f is a defect vector for Δ. Since Δ is positive, it suffices to consider $\Delta f = -f$. Note that $\Delta f = -f \iff c(I - \mathbb{P})f = -f \iff \mathbb{P}f = (1 + c^{-1})f$. We proceed to show that f is in $\mathscr{H}_E$, i.e., $\|f\|_{\mathscr{H}_E} < \infty$.

Let $V = \{x_{n,k}\}$ be the vertex-set as specified in Figure 5.1a. Then, we have

$$c(x_{n,k}) = 2^k + 2^{k+1} \tag{5.4}$$

$$p_{x_{n,k},x_{n,k-1}} = \frac{2^k}{2^k + 2^{k+1}} = \frac{1}{3} \tag{5.5}$$

$$p_{x_{n,k},x_{n,k+1}} = \frac{2^{k+1}}{2^k + 2^{k+1}} = \frac{2}{3} \tag{5.6}$$

and so

$$(\mathbb{P}f)(x_{n,k}) = \frac{1}{3}f(x_{n,k-1}) + \frac{2}{3}f(x_{n,k+1}),$$

and

$$(1+c^{-1})f(x_{n,k}) = \left(1+\frac{1}{2^k\cdot 3}\right)f(x_{n,k}); \qquad \text{see (5.4)–(5.6).}$$

Thus, the defect vector f satisfies $\Delta f = -f \iff$

$$\frac{1}{3}f(x_{n,k-1}) + \frac{2}{3}f(x_{n,k+1}) = \left(1+\frac{1}{2^k\cdot 3}\right)f(x_{n,k}).$$

Set $\ell_k := \ell_{n,k} = f(x_{n,k})$; then we get the following recursive equation:

$$\frac{1}{3}\ell_{k-1} + \frac{2}{3}\ell_{k+1} = \left(1+\frac{1}{2^k\cdot 3}\right)\ell_k;$$

i.e.,

$$\ell_{k+1} = \frac{3}{2}\left[\left(1+\frac{1}{2^k\cdot 3}\right)\ell_k - \frac{1}{3}\ell_{k-1}\right] = \left(\frac{3}{2}+\frac{1}{2^{k+1}}\right)\ell_k - \frac{1}{2}\ell_{k-1}.$$

Or, using matrix notation, we have

$$\begin{pmatrix}\ell_{k+1}\\ \ell_k\end{pmatrix} = \begin{pmatrix}\frac{3}{2}+\frac{1}{2^{k+1}} & -\frac{1}{2}\\ 1 & 0\end{pmatrix}\begin{pmatrix}\ell_k\\ \ell_{k-1}\end{pmatrix}. \tag{5.7}$$

The asymptotic estimate of the sequence (ℓ_k) can be derived from the eigenvalues of the coefficient matrix in (5.7). Note the eigenvalues are given by

$$x_\pm = \frac{\frac{3}{2}-\frac{1}{2^{k+1}} \pm \sqrt{\left(\frac{3}{2}-\frac{1}{2^{k+1}}\right)^2 - 2}}{2} \sim \frac{\frac{3}{2}\pm\frac{1}{2}}{2}, \quad \text{asymptotically.}$$

Conclusion. The root $x_- = 1/2$ shows that $\ell_k \sim 1/2^k$ so $f(x_{n,k}) \sim 1/2^k$ asymptotically. Consequently,

$$\begin{aligned}\|f\|^2_{\mathscr{H}_E} &\sim \sum_k 2^k\left(\frac{1}{2^k}-\frac{1}{2^{k+1}}\right)^2 + \sum_n 2^n\left(\frac{1}{2^n}-\frac{1}{2^{n+1}}\right)^2\\ &\sim \sum_k \frac{1}{2^k} + \sum_n \frac{1}{2^n} < \infty.\end{aligned}$$

Therefore, the corresponding defect vector f is in $\mathscr{H}_E$, and so $\Delta\big|_{\mathscr{H}_E}$ is not essentially selfadjoint.

5.1. The Gelfand space

Set the Gelfand space G_E to be the set of $\beta\colon \mathscr{H}_E \to \mathbb{C}$ (or $\mathbb{R}$) such that

$$\beta(uw) = \beta(u)\beta(w), \quad \forall u, w \in \mathscr{H}_E; \tag{5.8}$$

i.e., *multiplicative functionals*. (See [32].)

DEFINITION 5.8. Let $M :=$ metric completion of (V, d_{res}). Set

$$(x_i) \sim (y_i) \overset{\text{Def}}{\iff} d_{\text{res}}(x_i, y_i) \to 0$$

for all Cauchy sequences (x_i), $(y_i) \subset V$.

THEOREM 5.9. *$M \subset G_E$, see* (5.8). (*The metric completion is contained in the Gelfand space.*)

PROOF. Note that every $w \in \mathscr{H}_E$ extends by closure to M, by $\widetilde{w}(\widetilde{x}) = \lim_{i\to\infty} w(x_i)$, where $d_{\text{res}}(x_i, x_j) \to 0$. To see this, use the estimate $|w(x) - w(y)|^2 \le d(x, y)\|w\|^2_{\mathscr{H}_E}$, $\forall w \in \mathscr{H}_E$; see (5.1).

Now, set $\beta_{\widetilde{x}}(w) = \widetilde{w}(\widetilde{x})$, and note that (5.8) is then immediate. (In fact, M is a compact metric space if d_{res} is bounded.)

REMARK 5.10. It was proved in [18] that the Gelfand space is the Royden compactification; see [18] for details.

THEOREM 5.11. *Assume that d_{res} is type A and bounded on $V \times V$ (thus $(M, \widetilde{d}_{\text{res}})$ is compact by Theorem 5.2), and that $\omega = (x_i)_{i\in\mathbb{Z}} \in \Omega$. Then there exists a subsequence $\{x_{i_1}, x_{i_2}, \ldots\} \subset \omega$, and an $\widetilde{x} \in M$ such that $\widetilde{d}_{\text{res}}(x_{i_k}, \widetilde{x}) \xrightarrow{k\to\infty} 0$.*

PROOF (Application of Arzelà-Ascoli). Recall that $v_i := v_{x_i,o}$, where $|v_i(z)|^2 = |\langle v_i, v_z\rangle|^2 \le d(i, o)d(z, o) \le K$; which implies that $|v_i(z) - v_i(z')|^2 \le K\, d(z, z')$. By Arzelà-Ascoli, $\exists$ a subsequence s.t. $v_{i_k} - v_{i_\ell} \to 0$ in $\mathscr{H}_E$, as $d(x_{i_k}, x_{i_\ell}) \xrightarrow{k,\ell\to\infty} 0$.

6. Poisson-representations

Let $G = (V, E)$ be as above, and let $c\colon E \to \mathbb{R}_+$ be a fixed conductance function. Let $d = d_{\text{res}}$ be the corresponding resistance metric.

Our standard assumptions on G and c are as outlined in section 2 above.

We assume in addition that

(1) d_{res} is bounded on $V \times V$,

(2) for all $x \in V$, there exists $\varepsilon = \varepsilon_x$ such that

$$\{y \in V \mid d(x, y) < \varepsilon_x\} = \{x\}, \text{ the singleton.} \tag{6.1}$$

We shall denote by M the metric completion of (V, d_{res}), and identify V as a subset of M in the usual way, where $x \in V \longleftrightarrow \text{class}(x, x, x, \ldots) \in M$ (∞ repetition of vertex x).

PROPOSITION 6.1. *For $n \in \mathbb{N}$, set $w = (z_1, \ldots, z_n)$ where $z_i \in V$ (vertices), a finite word, and denote by $(w\underline{x})$ the concatenation sequence $(z_1, z_2, \ldots, z_n, x, x, x, \ldots)$; we set $\underline{x} = (x, x, x, \ldots)$; then $\gamma(x) = \{\underline{x}\} \cup \{w\underline{x}\}$, as w ranges over all finite words.*

PROOF. If $(y_i)_{i=1}^{\infty}$ is a sequence of vertices such that $\lim_{i\to\infty} d(y_i, x) = 0$, then, since x is isolated by (2), see (6.1), there must be a $n \in \{0, 1, 2, \ldots\}$ such that $y_i = x$ for all $i \geq n$; and the desired conclusion follows.

THEOREM 6.2. *Let $G = (V, E)$, c, d_{res} satisfy the conditions above, including* (1)–(2) (*so d_{res} is bounded*). *Then*

$$B := M \setminus V \tag{6.2}$$

is closed in M; and for every $x \in V$, there is a Borel probability measure μ_x on B, i.e., $\mu_x \in M_1(B)$ such that, for all harmonic functions h on V with $\|h\|_{\mathscr{H}_E} < \infty$, we have

$$h(x) = \int_B \widetilde{h}(b) d\mu_x(b) \tag{6.3}$$

where $\widetilde{h}$ is the extension $\in C(M)$ of h, obtained by metric completion, and where the function on the right-hand side in (6.3) *is $\widetilde{h}|_B$.*

PROOF. By Corollary 4.3, every $f \in \mathscr{H}_E$ has a unique continuous extension $\widetilde{f}$ to M; and $|\widetilde{f}(b) - \widetilde{f}(b')|^2 \leq d(b, b')\|f\|_{\mathscr{H}_E}^2$ holds for $\forall b, b' \in M$. By (2), section 5, V identifies as an open subset in M, and so $B = M \setminus V$ is closed; and therefore compact. Recall M is compact by Theorem 5.2.

Recall from section 2, that a function h on V is harmonic if and only if $\mathbb{P}h = h$, where

$$(\mathbb{P}h)(x) = \sum_{y\sim x} p_{xy} h(y) \tag{6.4}$$

and $p_{xy} := c_{xy}/c(x)$, for $\forall (x, y) \in E$. Also recall, $(\Delta f)(x) = \sum_{y\sim x} c_{xy}(f(x) - f(y))$.

Hence the harmonic functions h in $\mathscr{H}_E (\subset C(M))$ satisfy

$$\sup_{x\in V} |h(x)| = \left\|\widetilde{h}|_B\right\|_{\infty}. \tag{6.5}$$

This is an application of (6.4) and a simple maximum principle.

Now set $\mathscr{A} \subset C(B)$ as follows: $\mathscr{A} = \{\widetilde{h}|_B : \mathbb{P}h = h,\ h \in \mathscr{H}_E\}$, where "$|_B$" denotes restriction; then, for every $x \in V$, the point-evaluation mapping:

$$\mathscr{A} \ni \widetilde{h}|_B \longmapsto h(x) \tag{6.6}$$

defines a positive linear functional. Since $\mathbb{P}(\mathbb{1}) = 1$ where $\mathbb{1}$ is the constant one function, it follows that $\mathbb{1} \in \mathscr{A}$, and that $\mathbb{1} \mapsto 1$ in (6.6) (i.e., the functional in (6.6) attains value 1 on the constant function "one").

By the extension theorem of Banach and Krein, there is a positive linear functional on all of $C(B)$ which extends (6.6) from $\mathscr{A}$. By Riesz' theorem, it is given by a unique probability measure $\mu_x \in M_1(B)$. Restricting to $\mathscr{A}$, and using (6.5), we get the desired formula (6.3); i.e., μ_x is the Poisson-kernel, and B is a Poisson-boundary, i.e., it reproduces the harmonic functions in $\mathscr{H}_E$.

7. Continuous vs. discrete: examples

Below we discuss examples which illustrate features of network models and the associated different energy spaces that arise.

7.1. Continuous models

Example 7.1. Consider the standard Sobolev space,

$$\mathscr{H}^1 = \{f : \mathbb{R} \to \mathbb{C} \mid \text{measurable},\ f \in L^2,\ f' \in L^2\}, \tag{7.1}$$

with

$$\|f\|^2_{\mathscr{H}^1} = \frac{1}{2}\left(\int_{\mathbb{R}} |f|^2 + \int_{\mathbb{R}} |f'|^2\right), \tag{7.2}$$

where f' in (7.1) denotes the weak-derivative of f.

(i) Then $\mathscr{H}^1$ is a reproducing kernel Hilbert space (RKHS) consisting of bounded continuous functions. The corresponding reproducing kernel is given by $K(x, y) = e^{-|x-y|}$.

(ii) Moreover, $\mathscr{H}^1$ is an algebra under pointwise product with $\|fg\|_{\mathscr{H}^1} \leq C\|f\|_{\mathscr{H}^1}\|g\|_{\mathscr{H}^1}$, $\forall f, g \in \mathscr{H}^1$, for some constant $C > 0$.

Proof. See, e.g., [23].

The resistance distance in this case is

$$d(x, y) = \|K_x - K_y\|^2_{\mathscr{H}^1} = 2\big(1 - e^{-|x-y|}\big), \tag{7.3}$$

and

$$\sup_{x,y\in\mathbb{R}} d(x, y) \leq 2.$$

Hence the resistance metric d in (7.3) is bounded on $\mathbb{R}$, and the completion of $\mathbb{R}$ with respect to d is the one-point compactification of $\mathbb{R}$, but for *discrete models*:

7.2. Discrete models

Let $G = (V, E, c)$ be a discrete weighted graph, with vertex-set V, edges E, and a fixed conductance function c. Let $d = d_{\text{res}}$ be the resistance metric, and we study the metric completion of G.

For functions on the $\mathbb{Z}$-lattice $L_d := \mathbb{Z}^d$, $d \geq 1$; see Figure 2.1. Set

$$\|f\|^2_{\mathscr{H}_E} = \frac{1}{2}\sum_{x\sim y} e^{-|x-y|}|f(x) - f(y)|^2,$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$. (See also Example 1 after Definition 2.1, i.e., the set of edges and the nearest neighbors, $\#N(x) = 2d$, $\forall x \in \mathbb{Z}^d$.) Let $\mathscr{H}_E = \left\{f \text{ on } \mathbb{Z}^d \mid \|f\|_{\mathscr{H}_E} < \infty\right\}$.

REMARK. In (7.5) and (7.6) below, we have two versions of the graph Laplacian with different conductance functions; see Definition 2.3.

LEMMA 7.2. *For all $x \in \mathbb{Z}^d$, we have the following: $\exists\, K = K_x < \infty$ s.t.*

$$|f(x) - f(y)|^2 \leq K_x\|f\|^2_{\mathscr{H}_E} \quad \text{(seeTheorem 3.4.)} \tag{7.4}$$

PROOF. Note that $\exists!\, i$ s.t. $|x_i - y_i| = 1$, so $x_j - y_j = 0$ for $j \neq i$. The proof of (7.4) is standard.

Set

$$(\Delta f)(x) = \sum_{y\sim x} e^{-|x-y|}(f(x) - f(y)), \quad \forall f \in \mathscr{H}_E. \tag{7.5}$$

EXAMPLE 7.3. $V = \mathbb{Z}$, $E =$ nearest neighbor edges, i.e., for $x \in \mathbb{Z}$, $N(x) = \{x \pm 1\}$. Set

$$(\Delta f)(x) = \sum_{y\sim x}(f(x) - f(y)) = 2f(x) - f(x-1) - f(x+1). \tag{7.6}$$

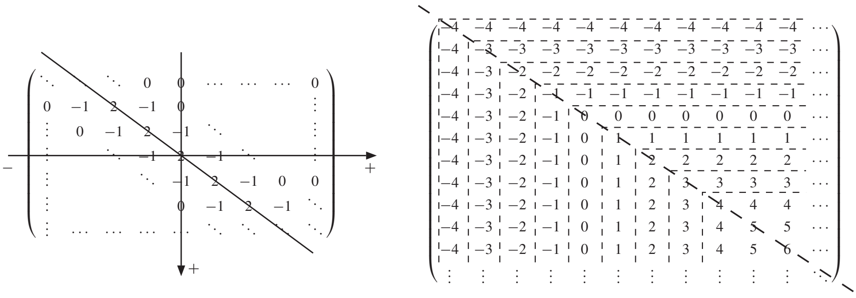
As an operator on $\ell^2(V)$ $(= \ell^2(\mathbb{Z}))$, one checks that the spectrum of Δ is continuous and equals the closed interval [0, 4]; so there is no gap in the bottom of the spectrum. As a result, the inverse matrix $\Delta^{-1} = K$ is unbounded. The two $\infty \times \infty$ matrices, Δ and K, are listed in Figure 7.1.

In detail, we have:

$$K = (K_{x,y}), \quad K_{x,y} = x \wedge y \ (= \text{minimum}), \quad x, y \in \mathbb{Z},$$

is the $\infty \times \infty$ matrix with $\mathbb{Z}$ as row and column indices. The matrix inversion formulas (see (9.7)–(9.9)) are sketched in Figure 7.1.

The matrix Δ The matrix K



(a) The $\infty \times \infty$ matrix Δ (see (7.6)). The number "2" in the diagonal.

(b) The $\infty \times \infty$ Green's function matrix K. The integers $\mathbb{Z}$ down the diagonal.

FIGURE 7.1. Illustration of the $\infty \times \infty$ matrices (7.7)–(7.10), $V = \mathbb{Z}$, nearest neighbors, unit conductance. Note that both matrices are positive definite (p.d.) So both the matrix-Laplacian Δ, and its inverse K, are p.d.; see also Lemma 2.8, and Corollary 9.5.

EXAMPLE 7.4. For $d = 1$, consider $\mathbb{Z}_+$ (see Figure 7.2), and

$$p_+ = \frac{e}{1+e}, \quad p_- = \frac{1}{1+e}.$$

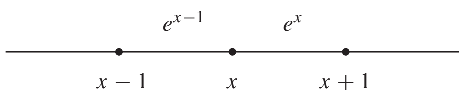


FIGURE 7.2. $\mathbb{Z}_+$ conductance function c, $c_{x,x+1} = e^x$, $x \in \mathbb{Z}_+$.

A function u on $\mathbb{Z}_+$ is harmonic if and only if $I_x := e^x(u_{x+1} - u_x)$ is constant; and

$$\|u\|^2_{\mathscr{H}_E} = \sum_x e^x(u_{x+1} - u_x)^2 = I_1^2 \sum_x e^{-x} = \frac{I_1^2}{e-1} < \infty.$$

Fix $0 < x < y$, then $v_{xy} = v_{yo}(t) - v_{xo}(t)$, where

$$v_{yo}(t) = \begin{cases} \sum_{i \le y} e^{-i}, & \text{if } t \le y, \\ \sum_{i=1}^{y} e^{-i}, & \text{if } t > y, \end{cases}, \quad v_{xy}(t) = \begin{cases} 0, & \text{if } 0 < t \le x, \\ \sum_{i=x+1}^{t} e^{-i}, & \text{if } x < t \le y, \\ \sum_{i=x+1}^{y} e^{-i}, & \text{if } y \le t, t \in \mathbb{Z}_+ \end{cases}$$

and

$$d_{\text{res}}(x, y) = \sum_{i=x+1}^{y} e^{-i} = \frac{e^{-x} - e^{-y}}{e-1};$$

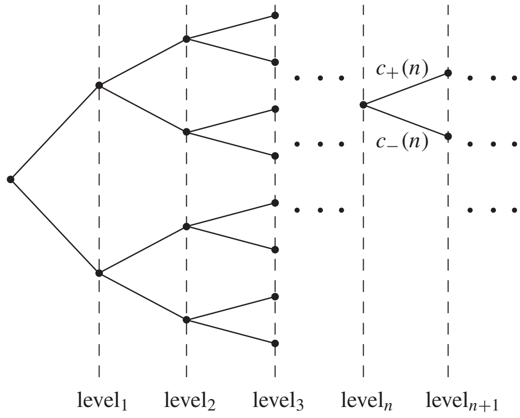


FIGURE 7.3. Binary tree with conductance.

and so d_{res} is clearly bounded.

But in this case the metric compactification is just the one-point compactification:

$$d_{\text{res}}(x, \infty) = \frac{e^{-x}}{e-1}; \quad x \in \mathbb{Z}_+.$$

It follows, in these examples, that $B = M \setminus V$ is a singleton; so M is the one-point compactification.

EXAMPLE 7.5. Let $V =$ the binary tree, see Figure 7.3. If a vertex x in the tree is at level n, set $c_{(x,x+)} = c_+(n)$, $c_{(x,x-)} = c_-(n)$. Then the arguments from above show that if $\sum_{n=1}^{\infty} \frac{1}{c_{\pm}(n)} < \infty$, then $B := M \setminus V$ is a Cantor-space.

7.3. Bratteli diagrams

In our present paper, we considered networks as weighted graphs $G = (V, E, c)$, vertices, edges and a weight (conductance) function. A Bratteli diagram is a special case of this, but the weighting usually doesn't refer to a conductance, but rather some kind of counting. In detail, if G is a Bratteli diagram, then its vertex set is stratified, by finite subsets V_n, called levels. While V is infinite, the sets V_n are finite. Then the requirement on G to be a Bratteli diagram is that the edges (lines in E) connect vertices from V_n to those at different levels; the nearest neighbor vertices are from level $n-1$, and level $n+1$. See [9].

Related to our present results are more recent applications to symbolic dynamics, see the papers in the bibliography, for example [21], and to measures on infinite path spaces obtained from "infinite strings of edges" from the given

Bratteli diagram. The papers [10] and [11] deal with the stationary case and classification up to order isomorphism. The questions we consider here are different as they do not restrict the focus to stationary diagrams; our present results even apply to graphs G which are not Bratteli diagrams.

We should add that compactifications of Bratteli diagrams (including binary trees) are studied in dynamics; see e.g., [1], [10], [12], [16], [21], [27], [28], [29], [34].

LEMMA 7.6. *If $\Delta = C - E$ as an $\infty \times \infty$ matrix, where $C = \operatorname{diag}(c(x))_{x\in V}$ and E consists of the off-diagonal terms, i.e., symmetric, $c_{xy} > 0$; then*

$$\Delta = (\Delta_{xy}) = C - E \tag{7.7}$$

where Δ_{xy} is as in (2.9)–(9.7)*, and we get the Green's function K as follows:*

$$K = \langle v_x, v_y \rangle_{\mathscr{H}_E}, \tag{7.8}$$

the Green's function of Δ satisfies

$$\sum_z \Delta_{xz} K_{zy} = \delta_{xy}, \tag{7.9}$$

and

$$\begin{aligned} \Delta^{-1} &= (C - E)^{-1} = \big(I - C^{-1}E\big)^{-1} C^{-1} \\ &= \sum_{n=0}^{\infty} (C^{-1}E)^n C^{-1} = G_P C^{-1}, \end{aligned} \tag{7.10}$$

where G_P is the Green's function of a Markov transition (*see Figure 7.4*)*. Note that C^{-1} is diagonal.*

An example is (see Figures 7.4–7.5)

$$c(n) = c_n + c_{n+1}, \quad c_n > 0. \tag{7.11}$$

LEMMA 7.7. *If (V, E, c) is constructed from a Bratteli diagram with levels $V_1, V_2, \ldots$, then the Green's function K for Δ satisfies*

$$K = G_P C^{-1},$$

where G_P is the random-walk Green's function associated with a $\pm$ Markov random walk, see Figure 7.4.

For Bratteli diagrams, see e.g., [11], [10], [9], [19], [21]; and for random walks, see e.g., [20].

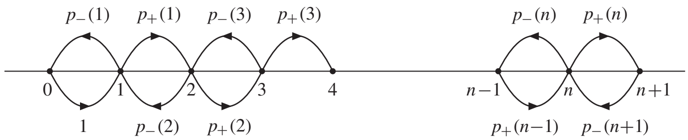


FIGURE 7.4. Transition probabilities $p_\pm(n)$, $n = 0, 1, 2, \ldots$.

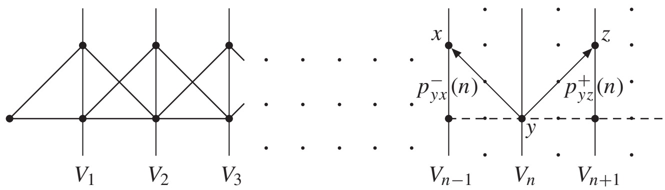


FIGURE 7.5. A Bratteli diagram, formula (7.11) with vertex-set $V = \{\emptyset\} \cup V_1 \cup V_2 \cup \cdots$ and transition between neighboring levels.

PROOF OF LEMMA 7.7 (sketch). Let $(p_-(n))$ and $(p_+(n))$ be the transition matrices:

$$(p_-(n))_{xy}\colon x \in V_n,\ y \in V_{n-1}, \text{ transition from vertex on } V_n \text{ to } V_{n-1},$$

$$(p_+(n))_{yz}\colon y \in V_n,\ z \in V_{n+1}, \text{ transition from vertex on } V_n \text{ to } V_{n+1},$$

see Figure 7.6, with row/column index picked from vertices in the respective levels.

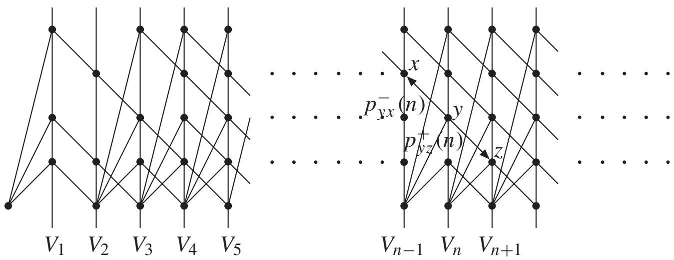


FIGURE 7.6

The product of $C^{-1}E$ in (7.10) is then

$$\left(C^{-1}E\right)^m_{xy} = \text{Prob(transition from vertex } x \text{ to vertex } y \text{ in time } m). \quad (7.12)$$

REMARK 7.8. Under the assumption in Theorem 5.11 and Theorem 6.2 one may show that in fact B (see (6.2)) is *Martin-boundary* (see [33], [17]) for the random walk on V defined by $p_{xy} := c_{xy}/c(x)$, $(x, y) \in E$.

PROOF (sketch). Let G_P be the random-walk Green's function from (7.10) and Lemma 7.7. Set

$$K_{\text{Martin}}(x, y) := G_P(x, y)/G_P(o, y).$$

Then the argument from Theorem 5.11 shows that $K_{\text{Martin}}(x, \cdot)$ extends to B, and that

$$h(x) = \int_B \widetilde{h}(b) K_{\text{Martin}}(x, b)\, d\mu^{(\text{Markov})}(b)$$

holds for all $h \in \text{Harm} = \mathscr{H}_E \cap \{h : \Delta h = 0\} = \mathscr{H}_E \cap \{h : \mathbb{P}h = h\}$.

EXAMPLE 7.9. For the transition matrix $C^{-1}E = P$, computed with the system in Figure 7.3 of transition probabilities, we get the following:

$$p_{i,i} = 0, \quad p_{i,i+1} = p_+(i) \quad \text{and} \quad p_{i,i-1} = p_-(i), \qquad \forall i \in \mathbb{Z},$$

with the remaining matrix-entries zero. For the computation of the matrix powers P^m, $m = 1, 2, \ldots$, we make the following simplification: $p_+(i) = p_+$, and $p_-(i) = p_-$.

Below we include a sample of matrix-entries for this binomial model.
Even powers of the transition-matrix P:

$$P^{2m}_{i,i+2k} = \binom{2m}{m-k} p_+^{m+k} p_-^{m-k} \quad \text{and} \quad P^{2m}_{i,i-2k} = \binom{2m}{m-k} p_+^{m-k} p_-^{m+k},$$

where $k = 0, 1, \ldots, m$.
Odd powers of the transition-matrix P:

$$P^{2m+1}_{i,i+1+2k} = \binom{2m+1}{m-k} p_+^{m+k+1} p_-^{m-k}$$

and

$$P^{2m+1}_{i,i-1-2k} = \binom{2m+1}{m-k} p_+^{m-k} p_-^{m+k+1}.$$

So for the $\infty \times \infty$ matrix G_P in (7.10) we get:

$$(G_P)_{i,i+2k} = \sum_{m=0}^{\infty} \binom{2m}{m-k} p_+^{m+k} p_-^{m-k}$$

and

$$(G_P)_{i,i+2k+1} = \sum_{m=0}^{\infty} \binom{2m+1}{m-k} p_+^{m+k+1} p_-^{m-k}.$$

As a result, (7.10) yields an explicit formula for $K_{i,j} = \langle v_i, v_j\rangle_{\mathscr{H}_E}$; see (7.10) and (7.8).

THEOREM 7.10. *The* Δ*-Green's function* K *in* (7.13) *has an explicit (and closed form) expression; for example, its diagonal entries are:*

$$K_{i,i} = \frac{1}{c(i)\sqrt{1-4p_+(1-p_+)}} \qquad \text{when } p_+ \neq \frac{1}{2}.$$

PROOF. The infinite sums used in computation of $(G_P)_{i,j}$, and therefore of

$$K_{i,j} = (G_P)_{i,j}/c(j), \tag{7.13}$$

can be computed with the use of *generating functions* for the associated binomial coefficients. For example,

$$\sum_{n=0}^{\infty} \lambda^m \binom{2m}{m} = \frac{1}{\sqrt{1-4\lambda}}, \qquad \text{setting } \lambda := p_+p_-;$$

and so we get

$$(G_P)_{i,i} = \frac{1}{\sqrt{1-4p_+p_-}}; \tag{7.14}$$

and therefore

$$K_{i,i} = \frac{1}{c(i)\sqrt{1-4p_+(1-p_+)}} = \langle v_i, v_i\rangle_{\mathscr{H}_E} = d_{\text{res}}(o,i), \tag{7.15}$$

which is the desired conclusion.

Note that to get absolute convergence in these series the requirement on p_+ is that $p_+ \in (0, 1/2) \cup (1/2, 1)$. (In this case, the resistance metric is bounded. We have $\sum_j 1/c(j) < \infty$.) The degenerate case is $p_+ = p_- = 1/2$. However the latter degenerate case can easily be computed by hand. It is the case of constant conductance function, $c_{i,i+1} = 1$.

For more details on this and related binomial models, see [2], [4], [7].

REMARK 7.11 (On general Bratteli diagrams). While the formulas (7.12)–(7.15) are derived subject to rather restricting assumptions, an inspection of the arguments shows that the ideas work for general Bratteli-diagrams; but then with modifications, as we now explain.

Given a Bratteli diagram with vertex-set $V = \{\emptyset\} \cup \left(\bigcup_{n=1}^{\infty} V_n\right)$, and vertices V_n corresponding to levels $n = 1, 2, \ldots$ (see Figure 7.6), we then have the following transition matrices:

$$\begin{cases} p^+(n)_{x,y}, & x \in V_n, y \in V_{n+1} \quad \text{and} \\ p^-(n)_{x,z}, & x \in V_n, z \in V_{n-1}. \end{cases} \tag{7.16}$$

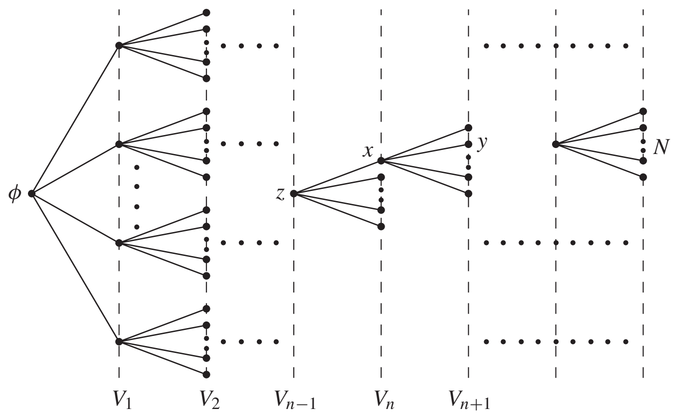


FIGURE 7.7. N-ary tree; the vertices at level n are denoted V_n, $n = 0, 1, \ldots, V_0 = \{\emptyset\}$, the empty word.

Therefore, in computing transition-probabilities, Prob($x \longrightarrow y$ in $2m$ iterations), we specialize to $x \in V_n$, and $y \in V_{n+2m}$. Rather than the easy formulas $\binom{2m}{m+k} p_+^{m+k} p_-^{m-k}$ from the proof in Example 7.9, we now instead get a sum of products of non-commutative matrices:

$$P_{w_1} P_{w_2} \cdots P_{w_{2m}}$$

where $w = (w_1, w_2, \ldots, w_{2m})$ is a finite word in the two-letter alphabet $\{\pm\}$, i.e., $w_i \in \{\pm\}$; but the estimates from before carry over; and we again arrive at an expression for the Green's function $(G_P)_{x,y}, x, y \in V$, analogous to (7.12)–(7.15).

EXAMPLE 7.12 (The N-ary tree). Fix $N > 1$. Let $b \in \mathbb{R}_+$, $b > 1$, be fixed, and set $c(n) := b^n$, $x \in V_n$, $y \in V_{n+1}$; then (see 7.16), we have (see Figure 7.7):

$$\begin{cases} p^+(n)_{xy} = \dfrac{b}{1+Nb}, \\ p^-(n)_{xz} = \dfrac{1}{1+Nb} \quad \text{and} \\ \quad c(n)_x = b^{n-1}(1+Nb) \end{cases}$$

where $x \in V_n$, $y \in V_{n+1}$, $z \in V_{n-1}$.

Generalizing (7.14), we get $(G_P)_{x,x'} = (Nb+1)(Nb-1)^{-1}$, for all $x, x' \in V_n$; and $d_{\text{res}}(\emptyset, x) = \left[(1+Nb)b^{n-1}\right]^{-1}$; and $d_{\text{res}}(x, B) < \infty$.

One can show that, if $\#V_1 < \#V_2 < \cdots$ (strictly increasing), then $\dim\{f : \Delta f = 0\} = \infty$.

8. The path-space Markov measure vs. the Poisson-measure on B

Here, we consider a class of models (V, E, c):

(i) $B = M \setminus V$, where M is the metric completion;

(ii) path space $\Omega = \{\omega = (\omega_i) \mid \omega_i \in V,\ (\omega_i, \omega_{i+1}) \in E,\ \forall i \in \mathbb{N}\}$;

(iii) set $\pi_i(\omega) = \omega_i \in V$ (vertex at time i), $i = 0, 1, 2, \ldots$, and

$$\Omega_x = \{\omega \in \Omega \mid \pi_0(\omega) = x\};$$

(iv) set $p_{xy} = c_{xy}/c(x)$, $(x, y) \in E$;

(v) $\mu_x^{(M)}$: Markov measure on Ω_x, $x \in V$ with transition

$$\mu_x^{(M)}(\text{cylinder}) = p_{x\omega_1} p_{\omega_1\omega_2} \cdots. \tag{8.1}$$

In more detail, a cylinder set $\subset \Omega$ is specified by a finite word $(xx_1x_2 \ldots x_n)$ of vertices such that $(x, x_1), (x_1, x_2), \ldots$ are edges (i.e., in E). Then set

$$C_{xx_1\ldots x_n} = \{\omega \in \Omega \mid \pi_0(\omega) = x,\ \pi_i(\omega) = x_i\ 1 \le i \le n\}.$$

Formula (8.1) then reads as follows:

$$\mu_x^{(M)}(C_{xx_1x_2\ldots x_n}) = p_{xx_1} p_{x_1x_2} \cdots p_{x_{n-1}} p_{x_n}$$

The following is known, see e.g., [13], [15]:

LEMMA 8.1. *There is a one-to-one correspondence between harmonic functions h on V, on the one hand, and shift-invariant L^1-functions F on Ω, on the other. It is given as follows:*

Let $\mathbb{E}$ denote the expectation computed with respect to the Markov-measure on Ω. Then

$$h(x) = \mathbb{E}(F \mid \pi_0 = x), \quad x \in V, \tag{8.2}$$

is harmonic of finite energy iff there is a shift-invariant L^1-function F on Ω such that (8.2) *holds.* (*In* (8.2)*, the symbol $\mathbb{E}(\cdot \mid \pi_0 = x)$ refers to conditional expectation.*)

PROOF. We use the formula $(\Delta h)(x) = c(x)\big(h(x) - (\mathbb{P}h)(x)\big)$, $x \in V$. Also see [15].

DEFINITION 8.2. $(V, E, c, d_{\text{res}})$ is of *class A* if

$$\lim_{k,\ell\to\infty} d_{\text{res}}(\pi_k(\omega), \pi_\ell(\omega)) = 0 \tag{8.3}$$

for all $\omega \in \Omega$, or in a subset of Ω.

REMARK 8.3. A large subset of Bratteli diagram will be of *class A*, i.e., that (8.3) holds; for example, if

$$\sum_n r(n) < \infty \tag{8.4}$$

where $r(n)$ denotes the resistance $V_n \to V_{n+1}$ between vertices of level n and level $n+1$. So (8.4)$\Longrightarrow$(8.3); but (8.3) holds much more generally.

PROPOSITION 8.4. *Assume* (8.3). *Then there is a well defined mapping:* $\Omega \xrightarrow{\Phi} B$, *given by* $\Omega \to$ (Cauchy − sequences) $\to$ (Cauchy − sequences/$\sim$),

$$\omega \longmapsto \Psi(\omega) = \text{class}(\pi_0(\omega), \pi_1(\omega), \pi_2(\omega), \ldots) \tag{8.5}$$

where $\sim$ *on Cauchy-sequences* $(\widetilde{x}) \sim (\widetilde{y}) \overset{\text{Def}}{\Longleftrightarrow} \lim_{i\to\infty} d_{\text{res}}(x_i, y_i) = 0$.

THEOREM 8.5. *Let* (V, E, c), *with* $p_{xy} = c_{xy}/c(x)$ *and Markov measure* $\mu_x^{(M)}$, *and let* $\Psi\colon \Omega \to B$ *be the mapping in* (8.5) *of Proposition 8.4. Then*

$$\{\mu_x^{(M)} \circ \Psi^{-1}\}_{x\in V} \tag{8.6}$$

constitutes the Poisson-measure on B in Theorem 6.2; i.e., if $S \in \mathscr{B}(B)$, $S \subset B$ *is a given Borel subset, then the measure in* (8.6) *is* $\mu_x^{(M)}(\Psi^{-1}(S))$.

PROOF (sketch). Set $\mu_x := \mu_x^{(M)} \circ \Psi^{-1}$, we then need to prove that

$$h(x) = \int_B \widetilde{h}\, d\mu_x \tag{8.7}$$

holds for all harmonic function $h \in \mathscr{H}_E$, i.e., $\|h\|_{\mathscr{H}_E} < \infty$, $\Delta h = 0$ ($\Longleftrightarrow$ $\mathbb{P}h = h$), and where $\widetilde{h} \in C(B)$ is the restriction to B of the extension from

$$\underset{h}{V} \longrightarrow \underset{\widetilde{h}}{M} \longrightarrow \underset{\widetilde{h}|_B}{B}$$

With this, we can check directly that μ_x satisfies (8.7), and so μ_x must be the Poisson-measure by uniqueness.

9. Boundary and interpolation

THEOREM 9.1. *Let* $V, E, c, \Delta, d_{\text{res}}, \mathscr{H}_E$, *and* B *be as above. We pick a base-point* $o \in V$, *and dipoles* $v_x = v_{(xo)}$ *such that* $v_x(o) = 0$, *and we set*

$$K(x, y) = \langle v_x, v_y \rangle_{\mathscr{H}_E} = v_x(y) = v_y(x), \tag{9.1}$$

the Green's function for Δ. *Finally, set* $Q := Q_{\text{Harm}}$ *denote the projection of* $\mathscr{H}_E$ *onto the subspace* $\text{Harm} = \{h \in \mathscr{H}_E \mid \Delta h = 0\}$. *For* $x \in V$, *let* μ_x *denote the Poisson-measure.*

Then we have the following interpolation/boundary formula:

$$f(x) = \sum_{y \in V \setminus \{o\}} K(x, y)(\Delta f)(y) + \int_B (\widetilde{Qf})(b)\, d\mu_x(b), \tag{9.2}$$

valid for all $f \in \mathscr{H}_E$, *and all* $x \in V$.

Proof. From [3], [25], we have that the projection $Q^\perp = I_{\mathscr{H}_E} - Q$ is given by

$$(Q^\perp f) = \sum_{y \in V} (\Delta f)(y) v_y = \sum_{y \in V} \underbrace{|v_y\rangle\langle\delta_y|}_{\text{Dirac-notation}} (f); \tag{9.3}$$

or equivalently,

$$(Q^\perp f)(x) = \sum_{y \in V \setminus \{o\}} K(x, y)(\Delta f)(y), \quad \forall x \in V.$$

Since $f = (Q^\perp f) + (Qf)$ with $Qf \in \text{Harm}(\subset \mathscr{H}_E)$, the desired formula (9.2) follows from the Poisson-representation:

$$(Qf)(x) = \int_B (\widetilde{Qf})(b)\, d\mu_x(b).$$

We have used the following:

Lemma 9.2. *The operator* $A = Q^\perp$ *in* (9.3) *indeed is a projection in* $\mathscr{H}_E$, *i.e.,* $A^2 = A = A^*$ *where the adjoint* $*$ *is computed with respect to the* $\mathscr{H}_E$*-inner product.*

Proof. We have $A = \sum_x |v_x\rangle\langle\delta_x|$, and so

$$\begin{aligned} A^2 &= \sum_{x,y} (|v_x\rangle\langle\delta_x|)(|v_y\rangle\langle\delta_y|) = \sum_{x,y} \langle\delta_x, v_y\rangle_{\mathscr{H}_E} |v_x\rangle\langle\delta_y| \\ &= \sum_{x,y} \delta_{xy} |v_x\rangle\langle\delta_y| = \sum_x |v_x\rangle\langle\delta_x| = A. \end{aligned}$$

But we also have for $f, g \in \mathscr{H}_E$, that

$$\langle f, Ag\rangle_{\mathscr{H}_E} = \sum_x \overline{f(x)} (\Delta g)(x) = \sum_x \overline{(\Delta f)(x)}\, g(x) = \langle Af, g\rangle_{\mathscr{H}_E},$$

where we use Lemma 2.8(1), so $A = A^*$.

From this, we get the operator-norm $\|A\|_{\mathscr{H}_E \to \mathscr{H}_E} = 1$. It is immediate from (9.3) that $Ah = 0$ for all $h \in$ Harm, and further that A is projection onto $\mathscr{H}_E \ominus$ Harm. Recall $\mathscr{H}_E \ominus$ Harm is the $\mathscr{H}_E$-norm closure of $\{\delta_x \mid x \in V\}$.

REMARK 9.3. Note that the function $K(\cdot, \cdot)$ from (9.1)–(9.2) is a Green's function of the Laplacian Δ. Recall Δ from Lemma 2.8 has the following $\infty \times \infty$ matrix-representation; see (2.9) & (9.7).

One checks from Lemma 2.8, that Green's inversion then holds:

$$\sum_{z \in V'} \Delta_{xz} K(z, y) = \delta_{x,y}, \quad \forall (x, y) \in V' \times V', \tag{9.4}$$

where $K(\cdot, \cdot)$ in (9.4) is the $\infty \times \infty$ matrix introduced in (9.1). So information about the resistance metric results from an inversion of the matrix (Δ_{xy}) in (2.9) above.

COROLLARY 9.4. *For every $f \in \mathscr{H}_E$ with $f(o) = 0$, we have the following representation:*

$$\|f\|^2_{\mathscr{H}_E} = \langle f, \Delta f \rangle_{\ell^2} + \int_{B_{\text{Markov}}} \left|\widetilde{Qf}\right|^2 d\mu^{(\text{Markov})}, \tag{9.5}$$

where $\langle f, \Delta f \rangle_{\ell^2} = \sum_{x \in V} \overline{f(x)} (\Delta f)(x)$, and where $\mu^{(\text{Markov})}$ is the Markov measure from Theorem 8.5.

PROOF. First, by Theorem 9.1 we have $f = Q^\perp f + Qf$ as an orthogonal splitting, relative to the $\mathscr{H}_E$-inner product. Hence

$$\|f\|^2_{\mathscr{H}_E} = \|Q^\perp f\|^2_{\mathscr{H}_E} + \|Qf\|^2_{\mathscr{H}_E}. \tag{9.6}$$

For the first term in (9.6), we have

$$\begin{aligned} \|Q^\perp f\|^2_{\mathscr{H}_E} &= \left\langle f, Q^\perp f \right\rangle_{\mathscr{H}_E} \\ &= \sum_x (\Delta f)(x) \langle f, v_x \rangle_{\mathscr{H}_E} \quad \text{by (9.3)} \\ &= \sum_x \overline{f(x)}\, (\Delta f)(x) = \langle f, \Delta f \rangle_{\ell^2}. \end{aligned}$$

For the second term in (9.6), we get, using Proposition 8.4 and Theorem 8.5,

$$\|Qf\|^2_{\mathscr{H}_E} = \int_{B_{\text{Markov}}} \left|\widetilde{Qf}\right|^2 d\mu^{(\text{Markov})};$$

see also [6]. The desired conclusion (9.5) now follows.

COROLLARY 9.5. *The two $\infty \times \infty$ matrices*

$$\Delta_{xy} := \langle \delta_x, \delta_y \rangle_{\mathscr{H}_E}, \quad \text{(see (8))}; \tag{9.7}$$

and

$$K_{xy} := \langle v_x, v_y \rangle_{\mathscr{H}_E}$$

are formal inverses; more precisely, for any $x, y \in V$, the following $\infty \times \infty$ matrix-products, ΔK and $K \Delta$ are well defined; and

$$\sum_{z \in V'} \Delta_{xz} K_{zy} = \delta_{x,y}, \tag{9.8}$$

and

$$\sum_{z \in V'} K_{xz} \Delta_{zy} = \delta_{x,y} \tag{9.9}$$

both hold. However, the operator theoretic interpretation of the two, (9.8) *vs.* (9.9)*, is different.*

PROOF. See Lemma 7.6 and the discussion above. (Explicit formulas are illustrated in Example 7.3 and Figure 7.1.)

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