MATH. SCAND. 9 (1961), 43–54

ON THE WEAK CONVERGENCE OF STOCHASTIC PROCESSES

MONROE D. DONSKER[1]

1. Introduction.

We say a sequence of processes $\{x_t^k,\ t \in T\}$, $k=1,2,\ldots$, converges weakly to a process $\{x_t,\ t \in T\}$ if for each n and each choice of points $t_1,t_2,\ldots,t_n$ in T we have at all continuity points of the distribution function on the right,

$$\lim_{k\to\infty} P\{x_{t_1}^k \leqq \lambda_1, x_{t_2}^k \leqq \lambda_2, \ldots, x_{t_n}^k \leqq \lambda_n\} = P\{x_{t_1} \leqq \lambda_1, x_{t_2} \leqq \lambda_2, \ldots, x_{t_n} \leqq \lambda_n\},$$

i.e., if the finite dimensional distributions converge. We consider the problem of under what further assumptions on the processes involved it follows that for a large class of functionals $G[x]$,

$$\lim_{k\to\infty} E_{x^k}\{G[x^k]\} = E_x\{G[x]\}.$$

What is needed then is a function space analogue of the Helly–Bray theorem. The most general theorem in this direction is that of Prohorov (Theorem 2.1 in [6]). In this paper we prove a theorem of Helly–Bray type in function space using characteristic functionals of processes and an explicit inversion formula for characteristic functionals recently obtained in [1]. The resulting theorem is not as general as that obtained by Prohorov [6] and indeed Theorem 1 in this paper follows from Prohorov's theorem referred to above. The proof of Theorem 1 below based on characteristic functionals is however quite straightforward and the uniform compactness condition (1.1), which is more restrictive on the processes than the corresponding condition 2.8 in [6], comes in very naturally as being necessary to justify the interchange in limits involved here.

Let $\alpha > \frac{1}{2}$ and $\{x_t, 0 \leqq t \leqq 1\}$ be a process almost all of whose sample functions $x(t)$ vanish at $t=0$ and satisfy a Hölder condition of order α, that is, for all $s,t \in [0,1]$,

$$|x(t)-x(s)| \leqq h\,|t-s|^\alpha,$$

Received May 26, 1960.

[1] This research was sponsored in part by the United States Air Forces under contract No. AF 18 (603)-30.

where h is a constant depending only on $x(.)$. Let $\{p_t, 0 \leqq t \leqq 1\}$ be the Wiener process, i.e., the Gaussian process with mean 0 and covariance function $\min(s,t)$ almost all of whose sample functions $p(t)$ also vanish at $t=0$. We denote the characteristic functional of the process $\{x_t, 0 \leqq t \leqq 1\}$ by

$$\Phi(p) = E_x\left\{\exp\left[i\int_0^1 x(t)\,dp(t)\right]\right\}.$$

For the same α as above, let $\{x_t^k, 0 \leqq t \leqq 1\}$, $k=1,2,\ldots$, be a sequence of processes each of which has the property that almost all its sample functions $x^k(t)$ vanish at $t=0$ and satisfy a Hölder condition of order α. Let $\Phi^k(p)$ be the associated sequence of characteristic functionals. In the following theorem let $S_{\mu,k}$ denote the set of sample functions $x^k(t)$ such that for all $s,t \in [0,1]$,

$$|x^k(t)-x^k(s)| \leqq \mu|t-s|^\alpha .$$

THEOREM 1. *If for each $\lambda > 0$, $\lim_{k\to\infty}\Phi^k(\lambda p)=\Phi(\lambda p)$ for almost all $p(t)$ and if for any $\varepsilon > 0$ there exists an H, independent of k, such that for all k*

$$P\{S_{H,k}\} \geqq 1-\varepsilon , \tag{1.1}$$

then, for any functional $G[x]$ defined on $C[0,1]$ which is bounded and almost everywhere continuous in the uniform topology on the sample functions of $\{x_t, 0 \leqq t \leqq 1\}$ as well as on the sample functions of $\{x_t^k, 0 \leqq t \leqq 1\}$, $k=1,2,\ldots$, we have

$$\lim_{k\to\infty} E_{x^k}\{G[x^k]\} = E_k\{G[x]\} . \tag{1.2}$$

Before proving this theorem we make some remarks. The assumption that for each $\lambda > 0$, $\lim_{k\to\infty}\Phi^k(\lambda p)=\Phi(\lambda p)$ for almost all $p(t)$ is in essence the assumption that the sequence of processes $\{x_t^k, 0 \leqq t \leqq 1\}$ converges to the process $\{x_t, 0 \leqq t \leqq 1\}$ in the usual weak sense, i.e., the finite dimensional distributions converge. It is clear from the regularity assumptions that for each k and for any $\varepsilon > 0$ there exists an H_k such that $P\{S_{H_k,k}\} \geqq 1-\varepsilon$. What we ask in (1.1) is uniformity in k. This condition (1.1) is the function space analogue of the convergence at infinity condition in the one-dimensional Helly-Bray theorem. Prohorov in [6] uses such a uniform compactness condition to show there exists a convergent subsequence of the process measures, convergent in the sense that the left side of (1.2) has a limit for all continuous functionals $G[x]$. He then shows that the limit is unique and is given by the right side of (1.2). As will be seen below, using the inversion formula for characteristic functionals, the problem reduces in a line to the justification of the interchange of two limits which follows readily from (1.1). Our results are not as general as

Prohorov's because the inversion formula has as yet been proved only for processes whose sample functions satisfy certain regularity conditions.

For any functional $F[x]$ which is almost everywhere continuous in the uniform topology (not necessarily bounded) on both the sample functions of $\{x_t, 0 \leqq t \leqq 1\}$ and $\{x_t^k, 0 \leqq t \leqq 1\}$, $k=1,2,\ldots$, we have from (1.2)

$$\lim_{k\to\infty} P\{F[x^k] \leqq \lambda\} = P\{F[x] \leqq \lambda\} \tag{1.3}$$

at all continuity points of the distribution function on the right. This follows from (1.2) by letting $G[x]=e^{i\tau F[x]}$ and applying the ordinary continuity theorem for characteristic functions. In particular, because our assumptions involve only continuity in the uniform topology, we obtain as a special case of (1.3),

$$\lim_{k\to\infty} P\left\{\sup_{0\leqq t\leqq 1} x^k(t) \leqq \lambda\right\} = P\left\{\sup_{0\leqq t\leqq 1} x(t) \leqq \lambda\right\}, \tag{1.4}$$

again at all continuity points on the right. Even in very special cases (1.4) has proved difficult to demonstrate. In the last section of this paper we indicate possible extensions of Theorem 1 and also discuss certain special cases.

2. Proof of Theorem 1.

Let $E_p^w\{\ \}$ indicate expectation on the Wiener process $\{p_t, 0 \leqq t \leqq 1\}$. In [1] it is shown that for processes $\{x_t, 0 \leqq t \leqq 1\}$ almost all of whose sample functions vanish at $t=0$ and satisfy a Hölder condition of order $\alpha > \frac{1}{2}$ one can obtain an inversion for $\Phi(p)$, the characteristic functional of $\{x_t, 0 \leqq t \leqq 1\}$. To be specific, it is shown there that if $G[x]$ is a functional on the sample functions of such a process which is bounded and almost everywhere continuous in the uniform topology, we have for any positive increasing function $f(\lambda)$ such that $f(\lambda)=o\big(\lambda/(\log\lambda)^2\big)$ but $1/f(\lambda)= o(\lambda^{1-\alpha^{-1}})$,

$$\begin{aligned} &E_x\{G[x]\} \\ &= \lim_{\lambda\to\infty}\{D_w(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_a^w E_p^w \left\{\exp\left[-i\lambda f(\lambda)\int_0^1 a(t)\,dp(t)\right] G[f(\lambda)a]\,\Phi(\lambda p)\right\}, \end{aligned} \tag{2.1}$$

where $D_w(\mu)$ is the Fredholm determinant of $\min(s,t)$ $\big(D_w(-\mu^2)=\cosh\mu\big)$.

From the first assumption of Theorem 1 we have immediately

$$\begin{aligned} &E_x\{G[x]\} \\ &= \lim_{\lambda\to\infty}\lim_{k\to\infty}\{D_w(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_a^w E_p^w \left\{\exp\left[-i\lambda f(\lambda)\int_0^1 a(t)\,dp(t)\right] G[f(\lambda)a]\;\Phi^{(k)}(\lambda p)\right\}. \end{aligned} \tag{2.2}$$

Since each process $\{x_t^{(k)}, 0 \leqq t \leqq 1\}$ satisfies the conditions of the inversion formula and $G[x]$ is assumed to have the same properties on the sample functions $x^{(k)}(t)$ as on the sample functions $x(t)$, we have that

$$\tag{2.3} \begin{aligned} &E_{x_k}\{G[x^{(k)}]\} \\ &= \lim_{\lambda\to\infty} \{D_w(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_a^w E_p^w \left\{\exp\left[-i\lambda f(\lambda)\int_0^1 a(t)\,dp(t)\right] G[f(\lambda)a]\Phi^{(k)}(\lambda p)\right\}. \end{aligned}$$

Thus, the proof of Theorem 1 consists in justifying the interchange of limits on the right of (2.2). To this end we show, using (1.1), that the limit in (2.3) is uniform in k.

Using the definition of the characteristic functional, the fact that

$$E_a^w \left\{\exp\left[-\tfrac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\} = \{D_w(-\lambda^2 f^2(\lambda))\}^{-\frac{1}{2}},$$

and formula (2.5) of [1], cf. Paley, Wiener and Zygmund [5, p. 653] we have

$$\begin{aligned} &\{D_w(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_a^w E_p^w \left\{\exp\left[-i\lambda f(\lambda)\int_0^1 a(t)\,dp(t)\right] G[f(\lambda)a]\Phi^{(k)}(\lambda p)\right\} \\ &= \{D_w(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_{x_k} E_a^w \left\{G[f(\lambda)a] E_p^w \left\{\exp\left[i\lambda\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]\,dp(t)\right]\right\}\right\} \\ &= \{D_w(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_{x_k} E_a^w \left\{G[f(\lambda)a] \exp\left[-\tfrac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right\} \\ &= \frac{E_{x_k} E_a^w \left\{G[f(\lambda)a] \exp\left[-\tfrac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right\}}{E_a^w \left\{\exp\left[-\tfrac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\}}. \end{aligned}$$

Hence, what we wish to show is that

$$\tag{2.4} \lim_{\lambda\to\infty} E_{x_k} \left| G[x^{(k)}] - \frac{E_a^w \left\{G[f(\lambda)a] \exp\left[-\tfrac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right\}}{E_a^w \left\{\exp\left[-\tfrac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\}} \right| = 0$$

uniformly in k.

Let M be the assumed bound on $G[x]$. From (1.1) we have that for any $\varepsilon > 0$ there exists an H independent of k such that, for all k, $P\{S_{H,k}\} \geqq 1 - \varepsilon/(4M)$. Letting $S'_{H,k}$ denote the complement of $S_{H,k}$ we write the limitand in (2.4) as

$$
\begin{aligned}
&E_{x_k \in S_{H,k}}\left\{ G[x^{(k)}] - \frac{E_a^w\left\{G[f(\lambda)a]\exp\left[-\frac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right\}}{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\}}\right\} + \\
(2.5)\qquad &+ E_{x_k \in S'_{H,k}}\left\{ G[x^{(k)}] - \frac{E_a^w\left\{G[f(\lambda)a]\exp\left[-\frac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right\}}{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\}}\right\}.
\end{aligned}
$$

It will be shown subsequently that for all k and almost all $x^{(k)}(t)$,

$$
(2.6)\qquad \frac{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right\}}{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right\}\right.} \leqq 1\,.
$$

Therefore the second term in (2.5) is in absolute value $\leqq 2MP\{S'_{H,k}\} \leqq \frac{1}{2}\varepsilon$. Thus, letting $S_H = \bigcup_{k=1}^{\infty} S_{H,k}$, and since $P\{S_{H,k}\} \leqq 1$, it will suffice, in order to prove (2.4), to show that

$$
(2.7)\qquad \lim_{\lambda \to \infty} \frac{E_a^w\left\{G[f(\lambda)a]\exp\left[-\frac{1}{2}\lambda^2\int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2\,dt\right]\right.}{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2 f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\}} = G[x^k]
$$

uniformly on S_H. Except for the uniformity needed here, the fact that (2.7) holds under the assumptions imposed on $x^{(k)}(t)$, $G[x]$, and $f(\lambda)$ is the crux of the inversion formula (2.1) itself and follows from theorems proved in [1]. However, in order to make this paper self-contained and to show the uniformity on S_H, we repeat the substance of the argument here referring to [1] only for certain calculations.

First of all a calculation (Lemma 2 in [1]) shows that

$$\frac{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2\int_0^1[x^{(k)}(t)-f(\lambda)a(t)]^2\,dt\right]\right\}}{E_a^w\left\{\exp\left[-\frac{1}{2}\lambda^2f^2(\lambda)\int_0^1 a^2(t)\,dt\right]\right\}}$$

$$=\exp\left[-\tfrac{1}{2}\lambda^2\int_0^1[x^{(k)}(t)]^2\,dt-\tfrac{1}{2}\lambda^4f^2(\lambda)\int_0^1\int_0^1 R_w\big(s,t;-\lambda^2f^2(\lambda)\big)\,x^{(k)}(t)\,x^{(k)}(s)\,ds\,dt\right],$$

where

$$R_w\big(s,t;-\lambda^2f^2(\lambda)\big)=\begin{cases}-\dfrac{\cosh\lambda f(\lambda)(1-t)\sinh\lambda f(\lambda)s}{\lambda f(\lambda)\cosh\lambda f(\lambda)}, & s\leqq t,\\[2ex] -\dfrac{\cosh\lambda f(\lambda)(1-s)\sinh\lambda f(\lambda)t}{\lambda f(\lambda)\cosh\lambda f(\lambda)}, & s\geqq t,\end{cases}$$

is the resolvent kernel of $\min(s,t)$. Now using the symmetry of $R_w\big(s,t;-\lambda^2f^2(\lambda)\big)$ and elementary calculation,

$$\lambda^2\int_0^1[x^{(k)}(t)]^2\,dt+\lambda^4f^2(\lambda)\int_0^1\int_0^1 R_w\big(s,t;-\lambda^2f^2(\lambda)\big)x^{(k)}(s)\,x^{(k)}(t)\,ds\,dt$$

$$=\lambda^2\int_0^1[x^{(k)}(t)]^2\left[[1+\lambda^2f^2(\lambda)\int_0^1 R_w\big(s,t;-\lambda^2f^2(\lambda)\big)\,ds\right]dt-$$

$$-\lambda^4f^2(\lambda)\int_0^1\int_0^t R_w\big(s,t;-\lambda^2f^2(\lambda)\big)\,[x^{(k)}(s)-x^{(k)}(t)]^2\,ds\,dt$$

$$=\frac{\lambda^2}{\cosh\lambda f(\lambda)}\int_0^1[x^{(k)}(t)]^2\cosh\lambda f(\lambda)(1-t)\,dt+$$

$$+\frac{\lambda^3f(\lambda)}{\cosh\lambda f(\lambda)}\int_0^1\int_0^t\cosh\lambda f(\lambda)(1-t)\sinh\lambda f(\lambda)s\,[x^{(k)}(t)-x^{(k)}(s)]^2\,ds\,dt\,.$$

At this point one should notice that both terms are positive which proves (2.6). For any $x^{(k)}(t)\in S_H$, the last expression is less than or equal to

$$\frac{\lambda^2H^2}{\cosh\lambda f(\lambda)}\int_0^1 t^{2\alpha}\cosh\lambda f(\lambda)(1-t)\,dt+$$

$$+\frac{\lambda^3f(\lambda)H^2}{\cosh\lambda f(\lambda)}\int_0^1\int_0^t(t-s)^{2\alpha}\cosh\lambda f(\lambda)(1-t)\sinh\lambda f(\lambda)s\,ds\,dt\,.$$

By the Hölder inequality this is less than

$$\lambda^2 H^2 \left[\int_0^1 t^2 \frac{\cosh \lambda f(\lambda)(1-t)}{\cosh \lambda f(\lambda)} dt\right]^{\alpha} \left[\int_0^1 \frac{\cosh \lambda f(\lambda)(1-t)}{\cosh \lambda f(\lambda)} dt\right]^{1-\alpha} +$$

$$+ \lambda^3 f(\lambda) H^2 \left[\int_0^1 \int_0^t (t-s)^2 \frac{\cosh \lambda f(\lambda)(1-t) \sinh \lambda f(\lambda) s}{\cosh \lambda f(\lambda)} ds\, dt\right]^{\alpha} \cdot$$

$$\cdot \left[\int_0^1 \int_0^t \frac{\cosh \lambda f(\lambda)(1-t) \sinh \lambda f(\lambda) s}{\cosh \lambda f(\lambda)} ds\, dt\right]^{1-\alpha}$$

$$= \lambda^2 H^2 \left[\frac{1}{\cosh \lambda f(\lambda)} \left(\frac{2 \sinh \lambda f(\lambda)}{\lambda^3 f^3(\lambda)} - \frac{2}{\lambda^2 f^2(\lambda)}\right)\right]^{\alpha} \left[\frac{\sinh \lambda f(\lambda)}{\lambda f(\lambda) \cosh \lambda f(\lambda)}\right]^{1-\alpha} +$$

$$+ \lambda^3 f(\lambda) H^2 \left[\frac{1}{\lambda^3 f^3(\lambda)} \left(1 + \frac{2}{\cosh \lambda f(\lambda)}\right)\right]^{\alpha} \left[\frac{1}{2\lambda f(\lambda)} - \frac{\sinh \lambda f(\lambda)}{2\lambda^2 f^2(\lambda) \cosh \lambda f(\lambda)}\right]^{1-\alpha}.$$

The first term here is

$$\sim 2^{\alpha} H^2 \frac{\lambda^{1-2\alpha}}{[f(\lambda)]^{1+2\alpha}}$$

and, since by assumption $1/f(\lambda) = o(\lambda^{1-1/\alpha})$, this goes to zero as $\lambda \to \infty$. The second term in the last displayed formula is

$$\sim \frac{H^2}{2^{1-\alpha}} \frac{\lambda^{2-2\alpha}}{[f(\lambda)]^{2\alpha}}$$

and this also goes to zero as $\lambda \to \infty$. Since H is independent of k we have shown that

$$\lim_{\lambda \to \infty} \frac{E_a^w \left\{\exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda) a(t)]^2 dt\right]\right\}}{E_a^w \left\{\exp\left[-\frac{1}{2}\lambda^2 f^2(\lambda) \int_0^1 a^2(t)\, dt\right]\right\}} = 1 \tag{2.8}$$

uniformly on S_H.

Finally we show that

$$\lim_{\lambda \to \infty} \frac{E_a^w \left\{G[f(\lambda) a] \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda) a(t)]^2 dt\right]\right\}}{E_a^w \left\{\exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda) a(t)]^2 dt\right]\right\}} = G[x^{(k)}] \tag{2.9}$$

uniformly on S_H. Then (2.9) and (2.8) imply (2.7) and thus Theorem 1.

Let $x^{(k)}(t) \in S_H$ be a point where $G[x]$ is continuous in the uniform topology: Let $\eta > 0$ and let $\delta > 0$ be such that

$$\sup_{0 \leq t \leq 1} |x^{(k)}(t) - f(\lambda)a(t)| < \delta$$

implies

$$|G[f(\lambda)a] - G[x^{(k)}]| < \eta .$$

Let

$$A = \left\{ a(t) \colon \sup_{0 \leq t \leq 1} |x^{(k)}(t) - f(\lambda)a(t)| \leq \delta \right\}.$$

To show (2.9) consider

$$(2.10) \quad \frac{E_a^w \left\{ G[f(\lambda)a] \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}}{E_a^w \left\{ \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}} - G[x^{(k)}]$$

$$= \frac{E_{a \in A}^w \left\{ \left[G[f(\lambda)a] - G[x^k] \right] \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}}{E_a^w \left\{ \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}} +$$

$$+ \frac{E_{a \in A'}^w \left\{ \left[G[f(\lambda)a] - G[x^{(k)}] \right] \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}}{E_a^w \left\{ \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}},$$

where A' denotes the complement of A. The first term on the right of equation (2.10) is in absolute value $\leq \eta$ and the second term is in absolute value

$$\leq 2M \frac{E_{a \in A'}^w \left\{ \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}}{E_a^w \left\{ \exp\left[-\frac{1}{2}\lambda^2 \int_0^1 [x^{(k)}(t) - f(\lambda)a(t)]^2 \, dt \right] \right\}}.$$

The latter expression will approach zero uniformly on S_H if we can show that

$$(2.11)\qquad \lim_{\lambda\to\infty}\frac{E^w_{a\in A}\left\{\exp\left[-\frac{1}{2}\lambda^2\int_0^1[x^{(k)}(t)-f(\lambda)a(t)]^2\,dt\right]\right\}}{E^w_a\left\{\exp\left[-\frac{1}{2}\lambda^2\int_0^1[x^{(k)}(t)-f(\lambda)a(t)]^2\,dt\right]\right\}}=1$$

uniformly on S_H.

To show (2.11), let $\varphi(u)=1$ if $|u|\leqq\delta$ and 0 otherwise. It follows from Lemma 11 in [1] that

$$\frac{E^w_{a\in A}\left\{\exp\left[-\frac{1}{2}\lambda^2\int_0^1[x^{(k)}(t)-f(\lambda)a(t)]^2\,dt\right]\right\}}{E^w_a\left\{\exp\left[-\frac{1}{2}\lambda^2\int_0^1[x^{(k)}(t)-f(\lambda)a(t)]^2\,dt\right]\right\}}=E^w_a\left\{\varphi\left(\sup_{0\leqq t\leqq 1}|\theta(t)-x^{(k)}(t)|\right)\right\},$$

where

$$\begin{aligned}\theta(t) = x^{(k)}(t) &+ f(\lambda)\cosh\lambda f(\lambda)(1-t)\int_0^t \operatorname{sech}\lambda f(\lambda)(1-s)\,da(s) +\\ &+\frac{\sinh\lambda f(\lambda)t}{\cosh\lambda f(\lambda)}\int_t^1\sinh\lambda f(\lambda)(1-s)\,dx^{(k)}(s)\\ &-\frac{\cosh\lambda f(\lambda)(1-t)}{\cosh\lambda f(\lambda)}\int_0^t\cosh\lambda f(\lambda)s\,dx^{(k)}(s)\,.\end{aligned}$$

We must show therefore that

$$(2.12)\qquad \lim_{\lambda\to\infty}\theta(t) = x^{(k)}(t)$$

uniformly for $t\in[0,1]$ and uniformly on S_H. Consider the second term of $\theta(t)$ which does not involve $x^{(k)}(t)$. Almost all sample functions $a(t)$ of the Wiener process satisfy for some h depending on $a(t)$ the modified Hölder condition

$$|a(t)-a(s)| \leqq h\{|t-s|\log[|t-s|/e]\}^{\frac{1}{2}}\,.$$

Therefore, letting $\mu=(\log\lambda)/[\lambda f(\lambda)]$, we have

$$\begin{aligned}&\left|f(\lambda)\cosh\lambda f(\lambda)(1-t)\int_0^{t-\mu}\operatorname{sech}\lambda f(\lambda)(1-s)\,da(s)\right|\\ &\qquad\leqq f(\lambda)\cosh\lambda f(\lambda)(1-t)\,\{\operatorname{sech}[\lambda f(\lambda)(1-t)+\log\lambda]\}h\\ &\qquad< 2hf(\lambda)\exp[\lambda f(\lambda)(1-t)]\exp[-\lambda f(\lambda)(1-t)-\log\lambda] = 2hf(\lambda)/\lambda\,.\end{aligned}$$

This last expression approaches zero uniformly in $t \in [0,1]$ as $\lambda \to \infty$ since by assumption $f(\lambda) = o(\lambda/(\log\lambda)^2)$. Also for sufficiently large λ,

$$\begin{aligned}
&\left| f(\lambda) \cosh \lambda f(\lambda)(1-t) \int_{t-\mu}^{t} \operatorname{sech} \lambda f(\lambda)(1-s)\, da(s) \right| \\
&\qquad \leqq [f(\lambda) \cosh \lambda f(\lambda)(1-t) \operatorname{sech} \lambda f(\lambda)(1-t)] h\{\mu |\log(\mu/e)|\}^{\frac{1}{2}} \\
&\qquad = hf(\lambda)\{\log\lambda/\lambda f(\lambda)\}^{\frac{1}{2}} \{\log(\lambda f(\lambda)) - \log\log\lambda + 1\}^{\frac{1}{2}} \\
&\qquad \leqq h\{f(\lambda)\log\lambda/\lambda\}^{\frac{1}{2}} (2\log\lambda)^{\frac{1}{2}} = h\{2f(\lambda)(\log\lambda)^2/\lambda\}^{\frac{1}{2}},
\end{aligned}$$

and again, since $f(\lambda) = o(\lambda/(\log\lambda)^2)$, this last expression approaches zero uniformly in $t \in [0,1]$ as $\lambda \to \infty$. The third and fourth terms of $\theta(t)$ do involve $x^{(k)}(t)$. The argument to show that these terms approach zero uniformly in $t \in [0,1]$ and uniformly on S_H is almost exactly the same for each term and therefore we consider only the third. Taking $\mu = (\log\lambda)/[\lambda f(\lambda)]$ again, we have, since $x^{(k)}(t) \in S_H$,

$$\begin{aligned}
&\left| \frac{\sinh \lambda f(\lambda) t}{\cosh \lambda f(\lambda)} \int_{t}^{t+\mu} \sinh \lambda f(\lambda)(1-s)\, dx^{(k)}(s) \right| \\
&\qquad \leqq H \frac{\sinh \lambda f(\lambda) t \sinh \lambda f(\lambda)(1-t)}{\cosh \lambda f(\lambda)} \mu^{\alpha} \\
&\qquad \leqq H \left(\frac{\log\lambda}{\lambda f(\lambda)} \right)^{\alpha},
\end{aligned}$$

and this last expression goes to zero as $\lambda \to \infty$ uniformly for $t \in [0,1]$ and uniformly on S_H. Also

$$\begin{aligned}
&\left| \frac{\sinh \lambda f(\lambda) t}{\cosh \lambda f(\lambda)} \int_{t+\mu}^{1} \sinh \lambda f(\lambda)(1-s)\, dx^{(k)}(s) \right| \\
&\qquad \leqq H \frac{\sinh \lambda f(\lambda) t \ \sinh \lambda f(\lambda)(1-t-\mu)}{\cosh \lambda f(\lambda)} \sim \tfrac{1}{2} H/\lambda,
\end{aligned}$$

and this last expression goes to zero uniformly in $t \in [0,1]$ and uniformly on S_H.

We have thus shown (2.12) which implies (2.11) and hence (2.9) and Theorem 1.

3. Possible extension of Theorem 1.

The reason that Theorem 1 is restricted to processes whose sample functions satisfy a Hölder condition of order $\alpha > \frac{1}{2}$ is, of course, that the inversion formula for characteristic functionals employed here is thus restricted. Now it is clear that any extension of the inversion formula itself implies a corresponding extension of Theorem 1. In particular it is pointed out in [1] that if $0<\alpha\leqq 1$ and if $\varrho(s,t)$ is a covariance function continuous in both variables with reciprocal kernel $R_\varrho(s,t;\mu)$ and Fredholm determinant $D_\varrho(\mu)$ satisfying, as $\mu\to\infty$, the two conditions

$$(3.1)\qquad \int_0^1\left|1+\mu\int_0^1 R_\varrho(s,t;-\mu)\,ds\right|t^{2\alpha}\,dt = o\big([\log D_\varrho(-\mu)]^{-1}\big)$$

and

$$(3.2)\qquad \mu\int_0^1\int_0^t |R_\varrho(s,t;-\mu)|\,(t-s)^{2\alpha}\,ds\,dt = o\big([\log D_\varrho(-\mu)]^{-1}\big),$$

then one can invert characteristic functionals of processes almost all of whose sample functions satisfy a Hölder condition of corresponding order α where now α is not restricted to be $>\frac{1}{2}$. Although $\min(s,t)$ satisfies (3.1) and (3.2) for any $\alpha>\frac{1}{2}$ it is not clear whether there exist covariance functions $\varrho(s,t)$ satisfying these two conditions for any $\alpha\leqq\frac{1}{2}$. However, it is shown in [1] that if $\{x_t, 0\leqq t\leqq 1\}$ is a process almost all of whose sample functions vanish at $t=0$ and satisfy a Hölder condition of order $\alpha>0$ and if there exists a $\varrho(s,t)$ satisfying (3.1) and (3.2) for this α, then there exists a positive increasing $f(\lambda)$ such that

$$(3.3)\qquad E_x\{G[x]\}$$

$$= \lim_{\lambda\to\infty}\{D_\varrho(-\lambda^2 f^2(\lambda))\}^{\frac{1}{2}} E_y^\varrho E_p^w\left\{\exp\left\{-i\lambda f(\lambda)\int_0^1 y(t)\,dp(t)\right\}G[f(\lambda)y]\,\Phi(\lambda p)\right\},$$

where $\Phi(p)$ is the characteristic functional of $\{x_t, 0\leqq t\leqq 1\}$, $G[x]$ is a functional bounded and almost everywhere continuous in the Hilbert topology on the sample functions of $\{x_t, 0\leqq t\leqq 1\}$, and where $E_y^\varrho\{\ \}$ denotes expectation on the Gaussian process $\{y_t, 0\leqq t\leqq 1\}$ having mean function zero and covariance function $\varrho(s,t)$. Using (3.3) instead of (2.1) it is clear that one can prove a theorem like Theorem 1, but for processes whose sample functions satisfy a Hölder condition of any positive order α. Unfortunately we do not know whether a $\varrho(s,t)$ having the requisite properties exists for $\alpha\leqq\frac{1}{2}$. Using inversion measures other than Gaussian it is possible that one could obtain an inversion formula like (2.1) or (3.3) which applies to characteristic functionals of processes whose sample functions are not even continuous. To do this one apparently needs to prove the corresponding generalization of (2.7) which we have not been able to do.

It should be pointed out that condition (1.1) in special cases may be difficult to demonstrate. Indeed, in the classical invariance principle [2], [4] as well as in the Kolmogorov–Smirnov theorems of mathematical statistics [3], to demonstrate (1.1) requires the same sort of technique which was used in those cases while demonstrating (1.4).

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UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINN., U. S. A.,
AND
UNIVERSITY OF AARHUS, DENMARK