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ON DECAY CENTRALITY IN GRAPHS

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Abstract

The decay centrality of a vertex v in a graph G with respect to a parameter $\delta \in (0, 1)$ is a polynomial in δ such that for fixed k the coefficient of δ^k is equal to the number of vertices of G at distance k from v. This invariant (introduced independently by Jackson and Wolinsky in 1996 and Dangalchev in 2011) is considered as a replacement for the closeness centrality for graphs, however its unstability was pointed out by Yang and Zhuhadar in 2011. We explore mathematical properties of decay centrality depending on the choice of parameter δ and the stability of vertex ranking based on this centrality index.

1. Introduction

Throughout this paper, we consider graphs without loops or multiple edges; we use the standard graph terminology as used in [4]. Given a graph G, the symbols $V(G)$, $E(G)$ stand for the vertex set and the edge set of G, respectively. The *distance* $d(x, y)$ between vertices $x, y \in V(G)$ is the length of a shortest $x{-}y$-path (if no $x{-}y$-path exists in G, we set $d(x, y) = +\infty$); the value $e(x) = \max_{y \in V(G)} d(x, y)$ is called the eccentricity of x, and the maximum of eccentricities of vertices of G is the diameter of G (denoted by $\operatorname{diam}(G)$).

Among fundamental questions discussed in the analysis of social networks of relations between actors, an important task is to identify the actors which play a key role within that network. The usual way to express a measure of their importance involves the *centrality index*, formally defined, for a graph G, as a function $c\colon V(G) \to \mathbb{R}$ which is invariant under graph isomorphism (for interpretation purposes, the vertices of G with higher centrality values correspond to more important actors of the network modelled by G). The most frequently used centrality indices are vertex degree, eccentricity, betweenness and, particularly, the closeness, which is defined as

$$C_C(x) = \frac{1}{\sum\limits_{y \in V(G)} d(x, y)}.$$

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For the detailed discussion on their properties and usage, see Chapters 3, 4, and 5 in [2].

In search for other approaches in the area of centrality research, several new centrality indices were defined. One of them is *reciprocal distances centrality* (see [9], [8]) defined as

$$C_R(x) = \sum_{y \in V(G)\setminus\{x\}} \frac{1}{d(x, y)}$$

and another one is the *decay centrality* defined in [6] (see also [7]) and independently in [3] (under the name "generalized closeness") as

$$C_\delta(x, G) = \sum_{y \in V(G)\setminus\{x\}} \delta^{d(x,y)}$$

(if G is known from context, we just write $C_\delta(x)$) where $\delta \in (0, 1)$ is a parameter (often set to $\frac{1}{2}$, see the Section 5 of [3] for a discussion on the choice of δ). These indices overcome the known deficiency of the closeness: the zero value for all vertices of disconnected graphs. On the other hand, [13] argue that C_R and C_δ cannot be considered as replacements of the closeness for disconnected graphs, because, for connected graphs, they lead to rankings of vertices which are not mutually consistent – they show that, for the complete binary tree T of height two, the closeness centrality ranks its root (that is, the vertex of degree 2 in T) as the unique most central vertex while the reciprocal distance centrality ranks, as the most central vertices, two vertices of T that have degree 3. Furthermore, using the generalized closeness for $\delta = \frac{1}{2}$, one obtains that all three non-pendant vertices of T are the most central.

Rather than advocating either opinion, we present an opinion that the ranking order anomaly reported in [13] is a new kind of phenomenon intrinsically connected with properties of decay centrality. In order to formally describe this phenomenon, for a graph G with $V(G) = \{v_1, \ldots, v_n\}$ and a general centrality index c: $V(G) \to \mathbb{R}$, define the *c-ranking* of G in the following way: let π a the permutation of $\{1, \ldots, n\}$ such that $(c(v_{\pi(1)}), \ldots, c(v_{\pi(n)}))$ is a non-increasing sequence (in other words, it sorts the vertices starting from the highest centrality) with ℓ blocks $\big(c(v_{\pi(1)}), \ldots, c(v_{\pi(k_1)})\big)$, $\big(c(v_{\pi(k_1+1)}), \ldots, c(v_{\pi(k_2)})\big), \ldots,$ $\big(c(v_{\pi(k_{\ell-1}+1)}), \ldots, c(v_{\pi(n)})\big)$; in this sequence, distinct blocks contain distinct values whereas the values within a block are the same. Then the c-ranking of G is the sequence

$$\left(\overline{v_{\pi(1)}, \ldots, v_{\pi(k_1)}}, \overline{v_{\pi(k_1+1)}, \ldots, v_{\pi(k_2)}}, \ldots, \overline{v_{\pi(k_{\ell-1}+1)}, \ldots, v_{\pi(n)}}\right)$$

(the lines over vertices indicate the fact that centralities of vertices within the same group are equal). In the case of $c = C_\delta$, the c-ranking of a graph depends

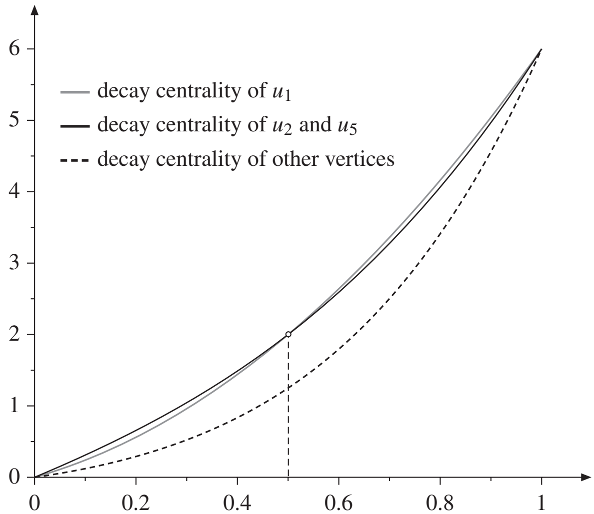

FIGURE 1. Graphs of decay centralities for the complete binary tree of height 2.

on δ; taking $\binom{n}{2}$ polynomials $C_\delta(v_i) - C_\delta(v_j)$ for all $i, j \in \{1, \dots, n\}$, their roots in $(0, 1)$ are called *decay thresholds* of G.

Now, if we denote the root of the above mentioned tree T as u_1, its neighbours as u_2, u_5 and the neighbours of u_2 and u_5 as u_3, u_4 and u_6, u_7, respectively, we can observe that T has the unique decay threshold at $\frac{1}{2}$ and C_δ-ranking of T is

$$\begin{aligned} &\left(\overline{u_2, u_5}, u_1, \overline{u_3, u_4, u_6, u_7}\right) \quad \text{for } 0 \le \delta < \tfrac{1}{2}, \\ &\left(\overline{u_1, u_2, u_5}, \overline{u_3, u_4, u_6, u_7}\right) \quad \text{for } \delta = \tfrac{1}{2}, \\ \text{and} \quad &\left(u_1, \overline{u_2, u_5}, \overline{u_3, u_4, u_6, u_7}\right) \quad \text{for } \tfrac{1}{2} < \delta < 1 \end{aligned}$$

(see Figure 1 for graphs of C_δ).

Hence, C_δ-ranking may vary, in general, with different values of δ; its behaviour and general properties are explored in a detail in Section 2.

On the other hand, there are graphs for which C_δ-ranking does not change; we will call these graphs as *decay-stable* and discuss their properties in Section 3. We show that the decay ranking of vertices is preserved within graphs of various graph products, and, also, within graphs of diameter two. However, we also exhibit several real-world networks whose decay rankings are highly unstable. These findings suggest that the decay centrality, although having some advantages over the closeness centrality, is perhaps not a good choice for analyzing real data.

2. General properties

The decay centrality of a vertex v of a graph G is a polynomial of degree $k = e(v)$; its coefficients are equal to terms of the *distance degree sequence* $\mathrm{DDS}_G(v)$ of v which is the sequence $(1, d_1, \ldots, d_k)$ where d_i is the number of vertices of the distance i from v (see [1]). It is easy to see that, in graphs whose vertices have the same distance degree sequences (*distance degree regular graphs* or DDR graphs for short), all vertices have the same decay centrality. Conversely, if G is a graph such that, for any pair x, y of its vertices, $C_\delta(x) = C_\delta(y)$ for all $\delta \in (0, 1)$, then G is a DDR graph. Note also that the vertices of each DDR graph have the same closeness; the converse, however, is not true, as seen on the graph $C_{12}^{\times}$ constructed from a 12-cycle $v_1v_2 \ldots v_{12}$ by adding new edges v_1v_3, v_2v_4, v_5v_7, v_6v_8, v_9v_{11} and $v_{10}v_{12}$: its vertices have two different DDS sequences (hence, two different decay centralities), but the same closeness.

Lemma 2.1. *A vertex adjacent to every other vertex of a graph has, for any* $\delta \in (0, 1)$*, the maximum decay centrality among all vertices.*

Proposition 2.2. *Let G be a graph and x, y be its vertices such that* $\mathrm{DDS}_G(x) = (1, x_1, \ldots, x_k)$*,* $\mathrm{DDS}_G(y) = (1, y_1, \ldots, y_r)$ *and, for each* $j \in \{1, 2, \ldots, k\}$*,* $\sum_{i=1}^{j} x_i \geq \sum_{i=1}^{j} y_i$ *holds. Then, for all* $\delta \in (0, 1)$*,* $C_\delta(x) \geq C_\delta(y)$*. Moreover, if there exists* $j \in \{1, 2, \ldots, k\}$ *such that* $\sum_{i=1}^{j} x_i > \sum_{i=1}^{j} y_i$*, then the inequality is sharp.*

Proof. Since $\sum_{i=1}^{j} x_i \geq \sum_{i=1}^{j} y_i$ for all $j \in \{1, 2, \ldots, k\}$, we get $k \leq r$ and

$$\begin{aligned} C_\delta(x) - C_\delta(y) &= \sum_{i=1}^{k}(x_i - y_i)\delta^i - \sum_{i=k+1}^{r} y_i\delta^i \geq \sum_{i=1}^{k}(x_i - y_i)\delta^k - \sum_{i=k+1}^{r} y_i\delta^i \\ &= \left(\sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i\right)\delta^k - \sum_{i=k+1}^{r} y_i\delta^i = \sum_{i=k+1}^{r} y_i\delta^k - \sum_{i=k+1}^{r} y_i\delta^i \\ &= \delta^k \sum_{i=k+1}^{r} y_i(1 - \delta^{i-k}) \geq 0. \end{aligned}$$

Now if $\sum_{i=1}^{j} x_i > \sum_{i=1}^{j} y_i$ for some integer j, then we obtain $r > k$ and therefore $\delta^k \sum_{i=k+1}^{r} y_i(1 - \delta^{i-k}) > 0$ which means that $C_\delta(x) > C_\delta(y)$ for all $\delta \in (0, 1)$.

In the following, we explore the relation between decay thresholds and decay order of vertices. If u, v are vertices of a graph G and $\delta_1, \delta_2 \in (0, 1)$, $\delta_1 \neq$

δ_2 such that $C_{\delta_1}(u) > C_{\delta_1}(v)$ but $C_{\delta_2}(u) < C_{\delta_2}(v)$, then G has a decay threshold in (δ_1, δ_2) (this follows from the intermediate value theorem used on the function $C_\delta(u) - C_\delta(v)$). However, in general, the converse is not true. To illustrate this, we will use the following observation on decay centralities of end vertices of cut edges in graphs. Let G be a graph with a cut edge uv, and let G_1, G_2 be two components of $G - uv$; let $\mathrm{DDS}_{G_1}(u) = (1, d_1, \dots, d_k)$, $\mathrm{DDS}_{G_2}(v) = (1, t_1, \dots, t_\ell)$. Without loss of generality, let $k \geq \ell$ and $(1, h_1, \dots, h_k)$ be a sequence of length k such that, for each $i = 1, \dots, \ell$, $h_i = t_i$ and $h_j = 0$ for $j = \ell + 1, \dots, k$. Then

$$\begin{aligned}\mathrm{DDS}_G(u) &= (1, d_1 + 1, d_2 + h_1, \dots, d_i + h_{i-1}, \dots, d_k + h_{k-1}, h_k)\\ &= (1, d_1, \dots, d_k, 0) + (0, 1, h_1, \dots, h_k),\\ \mathrm{DDS}_G(v) &= (1, h_1 + 1, h_2 + d_1, \dots, h_i + d_{i-1}, \dots, h_k + d_{k-1}, d_k)\\ &= (1, h_1, \dots, h_k, 0) + (0, 1, d_1, \dots, d_k)\end{aligned}$$

and

$$\begin{aligned}&C_\delta(u, G) - C_\delta(v, G)\\ &\quad= \left[\sum_{i=1}^k d_i\delta^i + 1\cdot\delta + \sum_{i=1}^k h_i\delta^{i+1}\right] - \left[\sum_{i=1}^k h_i\delta^i + 1\cdot\delta + \sum_{i=1}^k d_i\delta^{i+1}\right]\\ &\quad= C_\delta(u, G_1) - C_\delta(v, G_2) - \delta\left(\sum_{i=1}^k d_i\delta^i - \sum_{i=1}^k h_i\delta^i\right)\\ &\quad= (1-\delta)(C_\delta(u, G_1) - C_\delta(v, G_2)).\end{aligned}$$

Thus, $C_\delta(u, G) - C_\delta(v, G)$ has a root $\alpha \in (0, 1)$ if and only if α is the root of $C_\delta(u, G_1) - C_\delta(v, G_2)$.

We use this observation to construct connected graphs containing a pair of adjacent vertices showing arbitrary behaviour of their decay centralities. Given non-negative integers n_1, n_2, choose $n_1 + n_2$ rational numbers $q_1, \dots, q_{n_1}$, $r_1, \dots, r_{n_2} \in (0, 1)$ and a positive integer A in such a way that the polynomial $P(x) = Ax\prod_{i=1}^{n_1}(x - q_i)^2\prod_{j=1}^{n_2}(x - r_j) = a_{2n_1+n_2}x^{2n_1+n_2} + \dots + a_1$ has integer coefficients. Next, for $i = 1, \dots, 2n_1 + n_2$, choose positive integers b_i, c_i such that $a_i = b_i - c_i$, and construct disjoint connected graphs G_1, G_2 such that there is a vertex u of G_1 with $\mathrm{DDS}_{G_1}(u) = (1, b_1, \dots, b_{2n_1+n_2})$ and a vertex v of G_2 with $\mathrm{DDS}_{G_2}(v) = (1, c_1, \dots, c_{2n_1+n_2})$ (G_1 and G_2 may be chosen as rooted trees of height $2n_1 + n_2$ with roots u and v having b_i and c_i vertices at i-th level). Now, let G be a graph obtained from G_1 and G_2 by adding a new edge uv. It follows that, in G, the difference of decay centralities of u and v is equal to $P(\delta)(1 - \delta)$, hence $q_1, \dots, q_{n_1}, r_1, \dots, r_{n_2}$ are decay

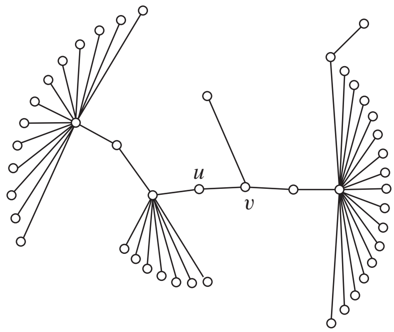

FIGURE 2. An example of graph with two thresholds of different nature.

thresholds of G. Observe that, for thresholds $q_i, \ldots, q_{n_1}$, a small local change of the parameter δ in their neighbourhoods preserves decay ordering of u, v whereas, for a local change of δ in neighbourhoods of thresholds $r_1, \ldots, r_{n_2}$, the order of u and v is always reversed.

EXAMPLE 2.3. Consider $q_1 = \frac{1}{2}$, $r_1 = \frac{1}{3}$. Then one can set $P(x)$ to be equal to $12x\left(x - \frac{1}{2}\right)^2\left(x - \frac{1}{3}\right) = 12x^4 - 16x^3 + 7x^2 - x$, hence, $a_4 = 12$, $a_3 = -16$, $a_2 = 7$, $a_1 = -1$ and we can choose $b_1 = 1$, $c_1 = 2$, $b_2 = 8$, $c_2 = 1$, $b_3 = 1$, $c_3 = 17$, $b_4 = 13$, $c_4 = 1$. An example of a graph realizing these parameters is on Figure 2; it is easy to see that the decay centralities of vertices u and v are $2\delta + 10\delta^2 + 2\delta^3 + 30\delta^4 + \delta^5$ and $3\delta + 2\delta^2 + 25\delta^3 + 2\delta^4 + 13\delta^5$, respectively, and their difference is $\delta - 8\delta^2 + 23\delta^3 - 28\delta^4 + 12\delta^5 = \delta(\delta - 1)(3\delta - 1)(2\delta - 1)^2$ yielding the roots 0, 1, $\frac{1}{3}$ (single root) and $\frac{1}{2}$ (double root).

From the above examples, one can conclude that the decay centrality, although being well defined for disconnected graphs, may sometimes lead, in these and other graphs, to "unpleasant" issues involving relation between decay order of vertices and decay thresholds.

3. Decay-stable graphs

We start our search for decay-stable graphs with several examples of graphs from simple yet nontrivial classes (note that since each vertex-transitive graph is also DDR graph and therefore decay-stable, we concentrate on classes of nontransitive graphs).

PROPOSITION 3.1. *All paths are decay-stable.*

PROOF. Let $P_n = v_1 v_2 \ldots v_n$ be an n-vertex path. For integer k, $1 \leq k \leq$

$\lfloor \frac{n}{2} \rfloor$, we have

$$C_\delta(v_k) = C_\delta(v_{n-k+1}) = 2\sum_{i=1}^{k-1}\delta^i + \sum_{i=k}^{n-k}\delta^i = \frac{\delta^k + \delta^{n-k+1} - 2\delta}{\delta - 1}.$$

Now, for fixed n and $\delta \in (0,1)$, the function $c(x) = (\delta^x + \delta^{n-x+1} - 2\delta)/(\delta - 1)$ is increasing on $\left(1, \frac{n}{2}\right)$ because $c'(x) = (\ln\delta(\delta^x - \delta^{n-x+1}))/(\delta - 1) > 0$; thus, for $1 \le p < q \le \lfloor \frac{n}{2} \rfloor$, we have $C_\delta(v_p) < C_\delta(v_q)$ which proves the claim.

In search for decay-stable trees, we checked, with the help of the Maple computer algebra system, all trees up to 20 vertices. The numbers of decay-stable trees are given in Table 1.

TABLE 1. Numbers of decay-stable trees with a given number of vertices.

Vertices	4	5	6	7	8	9	10	11	12	13	14
Decay-stable trees	2	3	6	9	19	20	40	49	88	102	191
All trees	2	3	6	11	23	47	106	235	551	1301	3159

Vertices	15	16	17	18	19	20
Decay-stable trees	207	356	391	678	731	1265
All trees	7741	19320	48629	123867	317955	823065

These numbers suggest that, from the asymptotic point of view, the following conjecture is true:

CONJECTURE 3.2. *Almost every tree is decay-unstable.*

It is not easy to find a particular graph constructions producing decay-stable graphs from smaller graphs, as the most common graph operations yield, in general, negative results even for decay-stable operands. We illustrate this for several graph products, namely, the Cartesian, tensor, strong and the lexicographic product (the definition and properties of these graph operations can be found in [5]). For example, the graphs $K_{1,3}$, $K_{1,3}^+$ (that is, $K_{1,3}$ with an extra edge between its nonadjacent vertices), P_3, P_4 and P_5 are decay-stable, but the Cartesian product $K_{1,5} \,\square\, P_5$ has unique threshold

$$\frac{1}{15} \cdot \sqrt[3]{918 + 30\sqrt{921}} + \frac{8}{5\sqrt[3]{918 + 30\sqrt{921}}} - \frac{1}{5} \doteq 0.7460547439,$$

the strong product $K_{1,3} \boxtimes P_4$ and the tensor product $K_{1,3}^{+} \times P_4$ have unique threshold $\frac{1}{2}$, and the lexicographic product $P_5[P_3]$ has unique threshold $\frac{1}{3}$ (see Figure 3 for detailed visualization of decay curves and their intersections). Nevertheless, for particular factors of these graph products, one can obtain decay stability of the result; we show this by several different examples:

THEOREM 3.3. *If G is decay-stable, then, for any positive integer n, $G \,\square\, K_n$ is also decay-stable.*

PROOF. Let u be a vertex of G, $\mathrm{DDS}(u) = (1, d_1, \ldots, d_k)$ where $k = e_G(u)$, and let $[u, v]$ be any vertex of $G \,\square\, K_n$ which lies in a copy of the factor K_n that covers u in a copy of G in the product. Note that for each vertex $[w, z]$ of $G \,\square\, K_n$ (with $w \in V(G)$, $z \in V(K_n)$), the distance of $[u, v]$, $[w, z]$ is equal to $d(u, w) + 1$; this implies that $\mathrm{DDS}([u, v]) = (1, d_1 + n - 1, d_2 + (n - 1)d_1, \ldots, d_k + (n-1)d_{k-1}, (n-1)d_k)$. Hence, for any vertex $[w, z]$ of $G \,\square\, K_n$ with $\mathrm{DDS}_G(w) = (1, d'_1, \ldots, d'_\ell)$, $\ell = ec_G(w)$ (without loss of generality, let $k \leq \ell$), we obtain

$$\begin{aligned}
C_\delta([w, z]) - C_\delta([u, v]) &= \sum_{i=0}^{k-1} (d'_{i+1} + (n-1)d'_i - (d_{i+1} + (n-1)d_i))\delta^{i+1} \\
&\quad + (d'_{k+1} + (n-1)d'_k - (n-1)d_k)\delta^{k+1} \\
&\quad + \sum_{i=k+2}^{\ell-1} (d'_{i+1} + (n-1)d'_i)\delta^i + (n-1)d'_\ell \delta^\ell \\
&= (C_\delta(u) - C_\delta(w))(\delta(n-1) + 1).
\end{aligned}$$

Since G is decay-stable, $C_\delta(u) - C_\delta(w)$ has no real root in $(0, 1)$, thus $C_\delta([u, v]) - C_\delta([w, z])$ has no real root in $(0, 1)$ either, which proves the claim.

THEOREM 3.4. *If G with $\delta(G) \geq 1$ is decay-stable and H is a regular graph, then $G[H]$ is decay-stable.*

PROOF. Let H be an r-regular graph on s vertices, v be a vertex of H and let u be a vertex of G with $ec(u) = k$, $\mathrm{DDS}_G(u) = (1, d_1, \ldots, d_k)$. Then $\mathrm{DDS}_{G[H]}([u, v]) = (1, d_1 s + r, d_2 s + s - r - 1, d_3 s, \ldots, d_k s) = s \cdot \mathrm{DDS}_G(u) + (1 - s - r, r, s - r - 1, 0, \ldots, 0)$. Hence, for any two vertices $[u, v]$, $[w, z]$ of $G[H]$, the difference of their decay centralities is

$$\begin{aligned}
&C_\delta([u, v]) - C_\delta([w, z]) \\
&\quad = s \cdot (C_\delta(u) - C_\delta(w)) + (r - r)\delta + (s - r - 1 - (s - r - 1))\delta^2 \\
&\quad = s \cdot (C_\delta(u) - C_\delta(w)).
\end{aligned}$$

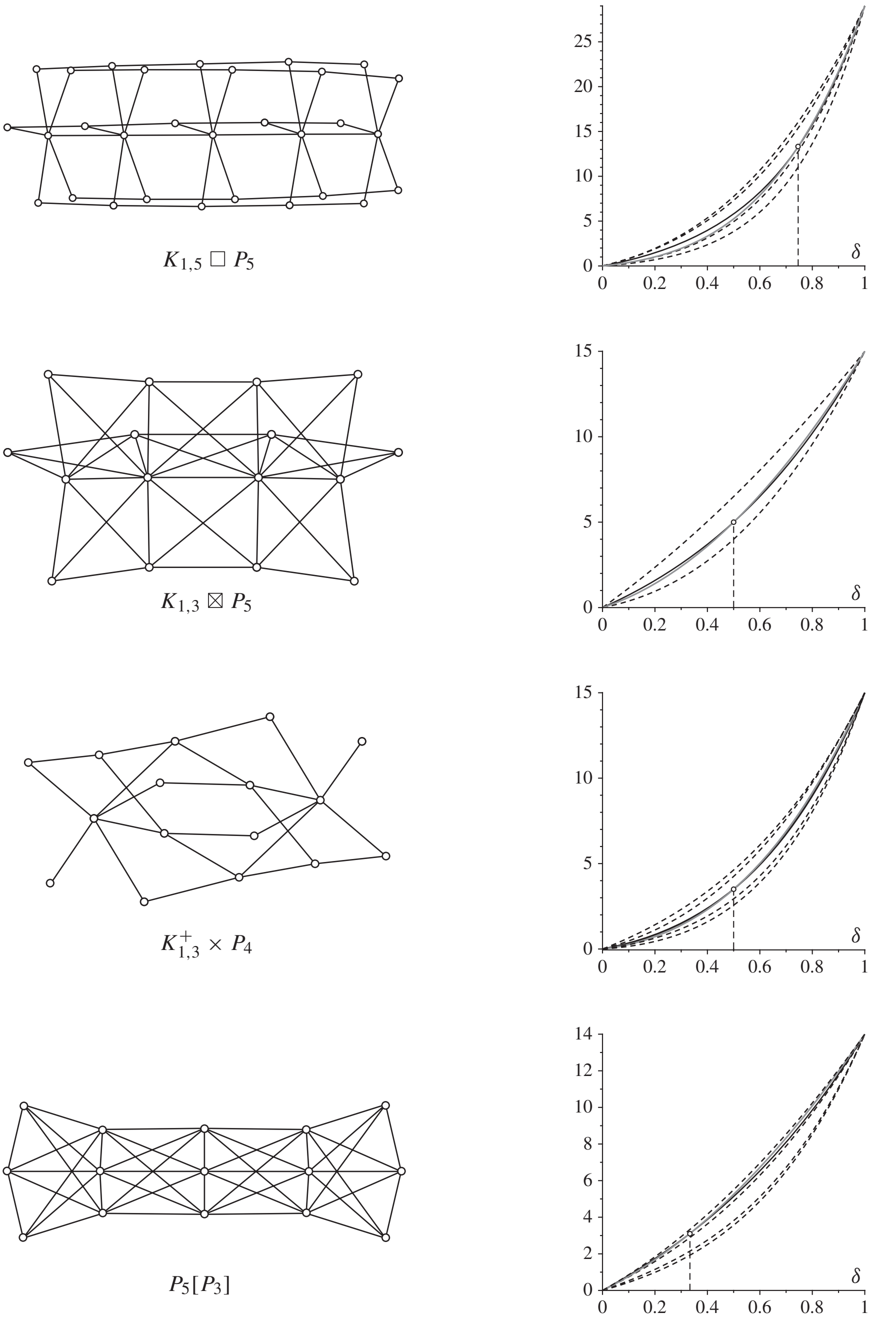

FIGURE 3. Examples of decay-unstable graph products.

Again, G is decay-stable, thus, $C_\delta(u) - C_\delta(w)$ has no real root in $(0, 1)$ as well as $C_\delta([u, v]) - C_\delta([w, z])$; this proves the claim.

It seems that also the following is true:

CONJECTURE 3.5. *For all positive integers $\ell, k_1, \ldots, k_\ell$, the ℓ-dimensional grid $P_{k_1} \square \cdots \square P_{k_\ell}$ is decay-stable.*

CONJECTURE 3.6. *The strong product $P_k \boxtimes P_\ell$ is decay-stable for all positive integers k, ℓ.*

Before exploring another large set of decay-stable graphs, we prove auxiliary result concerning relative stability of two vertices whose distance profiles do not differ much:

LEMMA 3.7. *Let G be a graph and $x, y \in V(G)$. If $\mathrm{DDS}_G(x)$ and $\mathrm{DDS}_G(y)$ differ in exactly two terms, then, for all $\delta \in (0, 1)$, $C_\delta(x) \neq C_\delta(y)$.*

PROOF. By contradiction. Let G be a graph with two vertices x, y with eccentricities $e(x), e(y)$ such that $\mathrm{DDS}_G(x) = (1, x_1, \ldots, x_{e(x)})$ and $\mathrm{DDS}_G(y) = (1, y_1, \ldots, y_{e(y)})$ differ exactly in i-th and j-th terms, $i < j$. Put $k = \max\{e(x), e(y)\}$. Then $\sum_{\ell=1}^{k} x_\ell = \sum_{\ell=1}^{k} y_\ell$, which gives $x_i - y_i = y_j - x_j$. Assume that there exists a $\delta \in (0, 1)$ such that $C_\delta(x) = C_\delta(y)$. This implies that $x_i\delta^i + x_j\delta^j = y_i\delta^i + y_j\delta^j$, hence $x_i - y_i = (y_j - x_j)\delta^{j-i}$. Thus $\delta^{j-i} = 1$, a contradiction.

Considering now a graph of diameter 2, the degree distance sequences of every two vertices are the same or differ in two terms; thus, we obtain the following

COROLLARY 3.8. *All graphs of diameter 2 are decay-stable.*

This also shows that – in probabilistic sense involving the concept of random graphs – almost every graph is decay-stable, as well as all joins of graphs. Furthermore, each strongly regular graph (that is, a regular graph with the property that every two adjacent vertices have λ common neighbours and every two nonadjacent vertices have μ common neighbours) has diameter 2, thus, it is decay-stable. By [12], every finite group A can serve as an automorphism group of some strongly regular graph, and hence of a decay-stable graph.

COROLLARY 3.9. *All regular graphs of diameter 3 are decay-stable.*

Note that there are nonregular graphs of diameter 3 which are not decay-stable, for example, the graph on Figure 4. Also, there are regular graphs of diameter at least 4 which are not decay stable – for example, the cubic graph on Figure 5 has decay threshold $\frac{1}{2}$. Hence, the assumptions of both above corollaries are best possible.

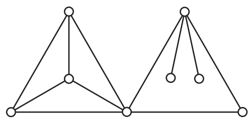

FIGURE 4. An example of decay-unstable graph of diameter 3.

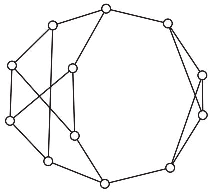

FIGURE 5. An example of decay-unstable regular graph of diameter 4.

On the other hand, it seems that most of real-world networks are decay-unstable. We have checked this property for several well-known networks (the data files are found at `http://www-personal.umich.edu/~mejn/netdata/`): the network of Zachary karate club (see [14]; describing friendships between 34 members of a karate club at a US university in the 1970s), the dolphin social network (see [10]) of frequent associations between 62 dolphins in a community living off Doubtful Sound, New Zealand, and the Padgett's network (see [11]) of marriages between Florentine medieval families. These networks have high numbers of decay thresholds – using Maple computer algebra system, we have determined that the Padgett's network has seven thresholds while Zachary karate club network and the dolphin social network have 35 and 372(!) thresholds, respectively. It would be interesting to compare these numbers with statistical characteristics of threshold numbers of random graphs with main parameters (number of vertices and edges, diameter) being the same as for real-world networks.

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