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ON UNIQUENESS THEOREMS FOR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Introduction.

An interesting uniqueness theorem for solutions of non hyperbolic equations with constant coefficients has recently been given by F. John [5]. The main aim of this paper is to prove that part of his result remains valid in the hyperbolic case also. It then gives precise information concerning the support of a solution of the Cauchy problem when the data have compact support.

Questions concerning unique continuation of solutions of a differential equation $Pu=f$ in a variety V may be put in the following form: What are the conditions on a closed subset F of V in order that there exists a solution u of the equation $Pu=0$ having F as its support? We prove here that the theorem of John [5] and Holmgren's uniqueness theorem (see John [4]) together with our result give a rather complete answer to this question when $V=R^2$ and P has constant coefficients.

2. Preliminaries.

The symbol $P(D)$ will represent a differential operator

$$P(D) = \sum_{|\alpha| \leqq m} a_\alpha D^\alpha ,$$

where α is a multi-index, that is, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, the α_i are non negative integers, $|\alpha|=\sum\alpha_i$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{1^{\alpha_1}}\, \partial x^{2^{\alpha_2}} \ldots \partial x^{n^{\alpha_n}}} .$$

The coefficients a_α of P are assumed to be (complex) constants. The principal part of $P(D)$ is the homogeneous part $P_m(D)$ of order m, that is,

$$P(D) = P_m(D) + \text{terms of lower order} .$$

If $\zeta=(\zeta_1,\zeta_2,\ldots,\zeta_n) \in C^n$ then $P(\zeta)$ denotes the polynomial

$$P(\zeta) = \sum_{|\alpha| \leqq m} a_\alpha \zeta^\alpha ,$$

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where $\zeta^\alpha = \zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\dots\zeta_n^{\alpha_n}$. We call the operator irreducible (reducible) if $P(\zeta)$ is irreducible (reducible).

A vector $\xi \in R^n$ is called characteristic if it satisfies the equation $P_m(\xi) = 0$. A hypersurface in R^n is called characteristic (non characteristic) if its normal vector is everywhere (nowhere) characteristic.

According to Gårding [1] the operator $P(D)$ is called hyperbolic with respect to the plane $\langle x, \xi \rangle = 0$, $\xi \in R^n$, if $P_m(\xi) \neq 0$ and if there is a constant a such that $P(t\xi + i\eta) \neq 0$ when $|\mathrm{Re}\, t| > a$ and $\eta \in R^n$. The operator is called weakly hyperbolic with respect to the plane $\langle x, \xi \rangle = 0$ if the principal part of $P(D)$ is hyperbolic with respect to that plane, or, equivalently, if the equation $P_m(t\xi + \eta) = 0$ has only real roots for every $\eta \in R^n$. If P is hyperbolic with respect to a plane it is also weakly hyperbolic with respect to that plane. The converse is not always true (Gårding [1, Lemma 2.5 and page 19] or A. Lax [6]).

Following Schwartz [7] we denote by $D(\Omega)$ the class of infinitely differentiable functions which have compact support in the open set $\Omega \subset R^n$. We denote by $D'(\Omega)$ the class of distributions in Ω (that is, the class of linear forms $u : D(\Omega) \to C$ which are continuous for the pseudotopology of $D(\Omega)$.—A sequence $\varphi_n \to 0$ in $D(\Omega)$ if the supports of all φ_n are contained in a fixed compact subset K of Ω and if for every α the sequence $(D^\alpha \varphi_n) \to 0$ uniformly in K—.). The support of a distribution $u \in D'(\Omega)$ is the set of points $x \in \Omega$ such that to every neighbourhood $0 \subset \Omega$ of x there is a $\varphi \in D(0)$ with $u(\varphi) \neq 0$.

If $u \in D'(\Omega)$, then $D^\alpha u$ denotes the distribution defined by the equation

$$(D^\alpha u)(\varphi) = (-1)^{|\alpha|} u(D^\alpha \varphi), \qquad \varphi \in D(\Omega) .$$

By a solution u in Ω of the equation $P(D)u = 0$ we thus mean a distribution $u \in D'(\Omega)$ such that $u\big(P(-D)\varphi\big) = 0$ for all $\varphi \in D(\Omega)$.

In Section 3 we shall need an algebraic result which we state here without proof.

LEMMA 1. *Let $P(\zeta)$ be an irreducible polynomial such that $P(sN + \xi)$ is not independent of s for indeterminate s and ξ. Then the polynomial*

$$P(sN + t\xi + \eta)$$

is an irreducible polynomial of s and t except when ξ and η belong to an algebraic set. Hence there is an algebraic set $A \neq R^n$ such that when $\xi \notin A$ the polynomial is irreducible except when η belongs to an algebraic set $B \neq C^n$ depending on ξ.

A similar lemma is used by John [5]. In Hörmander [3] further references are to be found.

3. A uniqueness theorem.

The purpose of this section is to prove the following result.

THEOREM 1. *Let $P(D)$ be an irreducible differential operator with constant coefficients and $u \in D'(\Omega)$ a solution of the equation*

$$P(D)u = 0 \tag{3.1}$$

in the open set $\Omega \subset R^n$ defined by

$$a < \langle x, N \rangle < b , \tag{3.2}$$

where $a < b$ and

$$P_m(N) = 0 . \tag{3.3}$$

If the support of u is contained in the intersection of a compact set with the set (3.2), it then follows that $u = 0$ in Ω.

PROOF. We split the proof into two parts, I and II. In I we consider the case when $u \in C^m(\Omega)$ and the derivatives of order $\leqq m$ are bounded so that we have a classical solution. In II, finally, we use regularization to prove the theorem for $u \in D'(\Omega)$.

I. We first consider the case when $P(sN+\xi)$ is independent of s for indeterminate s and ξ. In that case $P(D)$ may be considered as an operator only in the variables of the orthogonal space of N. Hence, if u is a solution of $P(D)u=0$ in R^n then the restriction u_t of u to the plane $\langle x,N\rangle = t$ is a solution of $P(D)u=0$ in that plane. If, in particular, u satisfies the conditions of the theorem then u_t is a solution of $P(D)u=0$ with compact support when $a<t<b$. The only solution with that property is $u_t \equiv 0$. This follows, for instance, from Holmgren's uniqueness theorem, John [4].

We now assume that $P(sN+\xi)$ is not independent of s for indeterminate s and ξ. We also assume that the derivatives up to the order m of u are bounded. We have to show that

$$V(\zeta,t) = \int_{\langle x, N\rangle = t} u(x) e^{-\langle x, \zeta\rangle}\, dS_x , \tag{3.4}$$

where dS_x denotes the element of area in the plane $\langle x,N\rangle = t$, vanishes for $\zeta \in C^n$ and $t \in (a,b)$. We observe that V is a solution of the ordinary differential equation

$$P(D_t N + \zeta)\, V(\zeta,t) = 0 \tag{3.5}$$

in the interval $a<t<b$. (D_t denotes differentiation with respect to t.)

In order to make a more detailed study of V we perform the following construction. We define an operator $R(\zeta,\tau,D_t)$ by the euqation

(3.6) $$(D_t-\tau)R(\zeta,\tau,D_t) = P(D_tN+\zeta)-P(\tau N+\zeta)\,.$$

We set

(3.7) $$W(\zeta,\tau,t) = R(\zeta,\tau,D_t)\,V(\zeta,t)\,.$$

We have obviously

$$(D_t-\tau)\,W(\zeta,\tau,t) = -P(\tau N+\zeta)\,V(\zeta,t)\,.$$

If in particular τ and ζ are chosen such that

(3.8) $$P(\tau N+\zeta) = 0$$

we have

(3.9) $$(D_t-\tau)\,W(\zeta,\tau,t) = 0\,.$$

That is, for all $t\in(a,b)$

(3.10) $$W(\zeta,\tau,t_0) = e^{-(t-t_0)\tau}\,W(\zeta,\tau,t)\,,$$

where t_0 is fixed (but arbitrary) in the interval (a,b). From (3.4) and (3.7) it follows that we can find a constant C such that

$$|W(\zeta,\tau,t)| \leqq C(1+|\zeta|+|\tau|)^{m-1}e^{M|\operatorname{Re}\zeta|}\,,$$

where M is a constant such that $u(x)=0$ when $|x|>M$; $x\in\Omega$. Thus

(3.11) $$|W(\zeta,\tau,t_0)| \leqq C(1+|\zeta|+|\tau|)^{m-1}e^{-(t-t_0)\operatorname{Re}\tau+M|\operatorname{Re}\zeta|}$$

when ζ and τ satisfy (3.8).

We now study the restriction of W to two dimensional planes in ζ,τ-space. For fixed $\xi\in R^n$ and $\eta\in C^n$ we set

(3.12) $$w(z,\tau) = W(z\xi+\eta,\tau,t_0)\,;$$

$w(z,\tau)$ is a polynomial in τ with coefficients which are entire analytic functions of z. We shall prove that $w(z,\tau)$ vanishes whenever z and τ satisfy the equation

(3.13) $$P(\tau N+z\xi+\eta) = 0\,,$$

where ξ and η are the same vectors as in (3.12). To do so we shall study $w(z,\tau)$ as an analytic function on the Riemann surface S defined by (3.13).

In view of (3.11) we have

(3.14) $$\big|w\big(z(\tau),\tau\big)\big| \leqq C(1+|z|+|\tau|)^{m-1}e^{-(t-t_0)\operatorname{Re}\tau+M|\xi|\,|\operatorname{Re}z|}$$

for some constant C, $(z,\tau)\in S$.

We now assume that ξ in (3.12) is chosen such that $P(\tau N+z\xi+\eta)$ is of degree m in z. This means that $P_m(\xi)\neq 0$. From the condition (3.3) it then follows (see [3, p. 258] for further references) that it is possible to find a branch $z = z(\tau)$ of the solution of (3.13) having a Puiseux series expansion

$$z(\tau) = \sum_{j=1}^{\infty} b_j \tau^{-j/q} \tag{3.15}$$

for some q with $1 \leqq q \leqq m$, convergent for $|\tau|^{1/q} \geqq M_0$. Here $\tau^{-j/q}$ is to be interpreted as $(\tau^{-1/q})^j$. We denote by S_1 the part of S where $|\tau| \geqq M_0{}^q$ and z is defined by (3.15).

From (3.15) we have that $|z(\tau)| = o(|\tau|)$ when $\tau \to \infty$, $(z, \tau) \in S_1$. Using this in (3.14) we get

$$\left|w(z(\tau), \tau)\right| \leqq C(1+|\tau|)^{m-1} e^{-(t-t_0)\operatorname{Re}\tau + o(|\tau|)} \tag{3.16}$$

for some constant C, $|\tau| \geqq M_0{}^q$ and $t \in (a,b)$. In the inequality (3.16) the left hand side is independent of t and thus we may freely choose t in the right hand side of (3.16) when estimating w. Taking $t = t_0$ we get

$$\left|w(z(\tau), \tau)\right| \leqq C e^{c|\tau|} \tag{3.17}$$

for some constants C and $c, (z, \tau) \in S_1$. Taking t with $t_0 < t < b$ we get

$$\left|w(z(\tau), \tau)\right| \leqq C(1+|\tau|)^{m-1} e^{-(t-t_0)|\tau|/2^{\frac{1}{2}} + o(|\tau|)} \tag{3.18}$$

when τ belongs to an angular domain

$$-\tfrac{1}{4}\pi \leqq \arg\tau \leqq \tfrac{1}{4}\pi \pmod{2\pi}\,. \tag{3.19}$$

Similarly, if $a < t < t_0$ in (3.16) we get

$$\left|w(z(\tau), \tau)\right| \leqq C(1+|\tau|)^{m-1} e^{-(t_0-t)|\tau|/2^{\frac{1}{2}} + o(|\tau|)} \tag{3.20}$$

when τ belongs to an angular domain

$$\tfrac{3}{4}\pi \leqq \arg\tau \leqq \tfrac{5}{4}\pi \pmod{2\pi}\,. \tag{3.21}$$

From (3.17) and the Phragmén–Lindelöf theorem it now follows immediately that w is bounded on the whole of S_1. Since $w(z(\tau), \tau)$ is a bounded analytic function in S_1 we have

$$w(z(\tau), \tau) = \sum_{j=1}^{\infty} c_j \tau^{-j/q}\,. \tag{3.22}$$

Assume that c_{j_0} is the first coefficient in (3.22) which is different from zero. Then $w(z(\tau), \tau)$ is asymptotically equal to $c_{j_0}\tau^{-j_0/q}$ when $\tau \to \infty$. From (3.18), however, it follows that

$$\left|w(z(\tau), \tau)\right| \leqq Ce^{-c'|\tau|} \qquad \text{when } \tau \to \infty$$

in the domain (3.19). Thus we have a contradiction and we conclude that all c_j in (3.22) vanish, that is, $w(z(\tau), \tau) \equiv 0$ in S_1.

Till now the only restriction on ξ and η in (3.12) is that $\xi \notin D = \{\xi:\ P_m(\xi) \neq 0\}$. We now further assume that

$$\xi \notin A \cup D \tag{3.23}$$

and also that

$$\eta \notin B\,, \tag{3.24}$$

where A and B are the algebraic sets mentioned in Lemma 1. Then $P(\tau N + z\xi + \eta)$ is irreducible and the Riemann surface S defined by (3.13) is hence connected. That means that we can continue w analytically from S_1 to the whole of S. As $w \equiv 0$ on S_1 we have that $w \equiv 0$ on S. That is, taking (3.10) into account, $W(z\xi + \eta, \tau, t) = 0$ for all $t \in (a,b)$ when z and τ satisfy (3.13).

Before proceeding we have to put still more restrictions on η in the definition (3.12) of w. Let $P(\tau N + \eta')$ be of degree k in τ for indeterminate η' and let $Q(\eta')$ be the coefficient of τ^k. In addition to (3.24) we now demand that η is chosen such that

$$Q(\eta) \neq 0\,. \tag{3.25}$$

Then it is clear that $P(\tau N + z\xi + \eta)$ will be of degree k in τ for all but finitely many z.

In view of the irreducibility of $P(\tau N + z\xi + \eta)$ we have that for every z except for finitely many $z = z_i$, $i = 1, 2, \ldots, n_0$ the equation (3.13) has exactly k different roots $\tau = \tau_l(z)$, $l = 1, 2, \ldots, k$. Thus we can decompose $\big(P(\tau N + z\xi + \eta)\big)^{-1}$ into partial fractions

$$\frac{1}{P(\tau N + z\xi + y)} = \sum \frac{C_l}{\tau - \tau_l}\,.$$

That is,

$$\begin{aligned} 1 &= \sum \frac{C_l P(\tau N + z\xi + \eta)}{\tau - \tau_l} \\ &= \sum \frac{C_l\big(P(\tau N + z\xi + \eta) - P(\tau_l N + z\xi + \eta)\big)}{\tau - \tau_l} \\ &= \sum C_l R(z\xi + \eta, \tau_l, \tau)\,. \end{aligned} \tag{3.26}$$

If we substitute D_t for τ in (3.26) and multiply from the right by $V(z\xi + \eta, t)$ we get

$$V(z\xi + \eta, t) = \sum C_l W(z\xi + \eta, \tau_l, t) = 0\,.$$

It is clear that we can approximate every vector $\zeta \in C^n$ with vectors of the form $z\xi + \eta$ where ξ and η are chosen such that (3.23), (3.24) and

(3.25) are fulfilled and $z \neq z_i$, $i=1,2,\ldots,n_0$. Thus it follows that $V(\zeta,t)=0$ for arbitrary $\zeta \in C^n$ and $t \in (a,b)$.

II. Let now $u \in D'(\Omega)$ satisfy the conditions of Theorem 1. Take $\varphi \in D(R^n)$ such that

$$\int \varphi\, dx = 1 , \tag{3.27}$$

$$\varphi(x) = 0 \quad \text{when} \quad |x| > 1 . \tag{3.28}$$

We set

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon) \tag{3.29}$$

with $\varepsilon > 0$ such that $a+2\varepsilon|N| < b-2\varepsilon|N|$. We further set

$$(3.30) \quad u_\varepsilon(x) = u\big(\varphi_\varepsilon(x-y)\big) ,$$
$$x \in \Omega_\varepsilon = \{x \in R^n\colon\ a+2\varepsilon|N| < \langle x,N\rangle < b-2\varepsilon|N|.\} .$$

In view of the definition of u, the support of u_ε in Ω_ε is contained in the intersection of a compact set with Ω_ε. Furthermore u_ε is infinitely differentiable and the derivatives of u_ε in Ω_ε are bounded and hence, from part I, we have $u_\varepsilon=0$ in Ω_ε. Letting $\varepsilon \to 0$ we get that $u=\lim u_\varepsilon=0$ in Ω. The proof is complete.

4. The support of a solution of a differential equation in two variables.

In the whole of this section we denote by $P(D)$ a differential operator in two variables and by Ω an open convex set in R^2. If u is a distribution in Ω we denote by $\operatorname{cosupp} u$ the complement relative to Ω of the support of u; it is clearly an open set.

An operator in three or more variables may be hyperbolic with respect to some hyperplanes and non hyperbolic with respect to others; an example is the wave equation. In the case of two dimensions, however, an operator $P(D)$ which is hyperbolic (weakly hyperbolic) with respect to *one* non characteristic line is hyperbolic (weakly hyperbolic) with respect to *all* non characteristic lines.

In fact, the statement concerning the weakly hyperbolic case is obvious since $P_m(D)$ decomposes in real linear factors if $P(D)$ is weakly hyperbolic with respcet to one non characteristic line. The other statement follows from the results of A. Lax [6]. Thus in two dimensions we may speak about a hyperbolic, weakly hyperbolic or non weakly hyperbolic operator without referring to any special line.

We first examine the implications of Holmgren's uniqueness theorem (see John [4]) which for the case of two dimensions may be stated as follows: Let a solution u of $P(D)u=0$ vanish in a neighbourhood of a non characteristic line segment L. Then u vanishes in the parallelogram A which has L as a diagonal and sides which are parallel to the first characteristic lines we get when we rotate L in the positive or the negative direction. We call A the characteristic parallelogram belonging to L.

If it happens that there is just one characteristic direction of $P(D)$ we interprete the characteristic parallelogram belonging to a non characteristic line segment L as the parallel strip formed by the two characteristic lines which pass through the end points of L.

In the following we denote the line segment between two points x and y in R^2 simply by xy.

Lemma 2. *Let $u \in C^m(\Omega)$ be a solution of $P(D)u=0$ in Ω and let the closed line segments $x_i x_{i+1} \subset \operatorname{cosupp} u$, $i=1,2,\ldots,n$. Then the line segment $x_1 x_{n+1} \subset \operatorname{cosupp} u$.*

Proof. It suffices to prove the lemma for $n=2$. For if it is true for $n=2$ then we can successively conclude that the line segments x_1x_3, $x_1x_4,\ldots,x_1x_{n+1}$ belong to $\operatorname{cosupp} u$.

Thus we assume that x_1x_2 and $x_2x_3 \subset \operatorname{cosupp} u$. We first assume that x_1x_3 is non characteristic. Let $x_i(t)$ be the points $x_i(t)=tx_i+(1-t)x_2$, $i=1,3$. We set

$$M = \{t\colon\ 0<t<1,\ x_1(t)x_2(t) \subset \operatorname{cosupp} u\}.$$

This set is not empty, for all sufficiently small t belong to M.

Set $t_0=\sup t$ when $t \in M$. We have to show that $t_0=1$. Assume the converse, that is that $t_0<1$. Since $u \in C^m$ it follows that u must have zero Cauchy data on the line segment $L=x_1(t_0)x_3(t_0)$ which is non characteristic. Hence $u=0$ in the characteristic parallelogram A belonging to L. We can find neighbourhoods of $x_1(t_0)$ and $x_3(t_0)$ belonging to $\operatorname{cosupp} u$. The union of these neighbourhoods and A is a neighbourhood of L and thus we can find $t>t_0$ with $t \in M$.

Assume now that the line segment x_1x_3 is characteristic. Then for all t with $|1-t|$ sufficiently small but different from zero we have that $x_1x_3(t)$ is non characteristic. Hence $x_1x_3(t) \subset \operatorname{cosupp} u$ for $|1-t|$ sufficiently small and $\neq 0$. As $u \in C^m$ it follows that $x_1x_3(t) \subset \operatorname{cosupp} u$ when $t=1$, too; and thus the lemma is proved.

Theorem 2. *If $u \in D'(\Omega)$ satisfies the equation $P(D)u=0$ in Ω it follows that each connected component of* $\operatorname{cosupp} u$ *is a convex set whose boundary in Ω is a polygon with characteristic sides.*

Proof. We may assume in the proof that $u \in C^m(\Omega)$. In fact, if the theorem is proved for that case then it follws for $u \in D'(\Omega)$ by the regularization process used in part II of the proof of Theorem 1.

I. To any points x_1 and x_2 in the same component O of $\operatorname{cosupp} u$ we can find a polygon which connects x_1 and x_2 and which passes entirely in O. From Lemma 2 it follows that the line segment $x_1x_2 \subset O$ and thus O is convex.

II. Let again O be a component of $\operatorname{cosupp} u$. We have to prove that if x is a point on the boundary of O in Ω then there is a characteristic line of support of O at x. In view of the convexity of O there is a line of support l at x. If l is not characteristic we consider the two adjacent characteristic lines through x. If both intersect O we can find two points, x_1 and x_2, in O on these lines such that the line segment x_1x_2 is parallel to l and belongs to O. As O is open we can enlarge the line segment x_1x_2 a little such that it still belongs to O and such that the characteristic parallelogram belonging to this new line segment will contain x as an interior point. Thus x is not a boundary point of O and we have a contradiction. Hence there is a characteristic line of support of O at x.

We next study the improvements of Theorem 2 given by the result of F. John [5] which for the case of two dimensions may be formulated as follows: Assume that $P(D)$ is irreducible and not weakly hyperbolic and let u be a solution of $P(D)u=0$ in a neighbourhood of a non characteristic line l. If l intersects the support of u in a compact set, it then follows that $u=0$ identically in the considered neighbourhood of l.

THEOREM 3. *If in addition to the hypotheses in Theorem 2 we assume that $P(D)$ is irreducible and not weakly hyperbolic, then* $\operatorname{cosupp} u$ *is a convex set (or, equivalently,* $\operatorname{cosupp} u$ *can have at most one connected component).*

PROOF. We shall prove that if the points x_1 and $x_2 \in \operatorname{cosupp} u$ then the line segment $x_1x_2 \subset \operatorname{cosupp} u$.

I. We first assume that the line segment x_1x_2 is non characteristic. We can find two open circular discs, O_1 and O_2, with equal radii and centres at x_1 and x_2 such that $O_1 \cup O_2 \subset \operatorname{cosupp} u$. Let l' and l'' be the tangents in common (of O_1 and O_2) which are parallel to the line segment x_1x_2. We define a solution u' of $P(D)u'=0$ in the parallel strip between l' and l'' by taking $u'=u$ between O_1 and O_2 and $u'=0$ elsewhere in the strip. The theorem of John gives that u' vanishes in the strip and hence $u=0$ between O_1 and O_2. Thus the line segment $x_1x_2 \subset \operatorname{cosupp} u$.

II. We now assume that the line segment x_1x_2 is characteristic. Let O_2 (for instance) be as above. We can find a point $x \in O_2$ such that the line segment x_1x is non characteristic. (I) gives that the line segment $x_1x \subset \operatorname{cosupp} u$, and since the line segment $xx_2 \subset \operatorname{cosupp} u$ we find that x_1 and x_2 belong to the same component of $\operatorname{cosupp} u$. Hence the line segment $x_1x_2 \subset \operatorname{cosupp} u$ and the theorem follows.

In the next theorem we shall use so called null solutions to show that Theorem 3 cannot be improved.

A null solution of the equation $P(D)u=0$ is a solution which vanishes in a half plane $\langle x,\xi\rangle < c$. From Holmgren's uniqueness theorem it follows that $\langle x,\xi\rangle = 0$ is necessarily characteristic if $u \not\equiv 0$. According to Hörmander [2, p. 216f] this is also a sufficient condition to assure the existence of non trivial null solutions.

LEMMA 3. *Let $P(D)$ be irreducible and let $u \not\equiv 0$ be a solution of $P(D)u=0$ with support in the half plane $\langle x,\xi\rangle \geqq c$. Then* cosupp u *has a component O which is exactly a half plane $\langle x,\xi\rangle < b$ with $b \geqq c$.*

PROOF. Cosupp u has obviously a component O containing the half plane $\langle x,\xi\rangle < c$. But O is convex and hence all lines of support for O are of the form $\langle x,\xi\rangle = d$ which proves the lemma.

THEOREM 4. *Let O be an open convex polygon in R^2 with characteristic sides. Let $P(D)$ be irreducible and not weakly hyperbolic. Then there exists a solution u of $P(D)u=0$ in R^2 such that* cosupp $u=O$.

PROOF. Let the boundary of O be composed of parts of the r characteristic lines l_i, $i=1,2,\ldots,r$ $(r \leqq 2m-2)$, and let F_i be the closed half plane bounded by l_i and not containing O.

We can find a null solution u_i with support in F_i. In view of Theorem 3 and lemma 3 we may assume that the support of u_i is equal to F_i.

Taking $u=\sum u_i$ we have $P(D)u=0$ and since cosupp u is a convex polygon containing O but no boundary point of O, we get cosupp $u=O$.

REMARK. If in Theorem 4 we permit the operator to be weakly hyperbolic, then, in view of Lemma 3, the construction still yields a solution of the equation $P(D)u=0$ such that one component of cosupp u is the given convex set O. However, this component will not always be the only one.

In studying the remaining weakly hyperbolic case we use the theorem proved in Section 3 instead of John's theorem. The result so obtained is of course weaker than Theorem 3.

THEOREM 5. *If in addition to the hypotheses in Theorem 2 we assume that the operator $P(D)$ is irreducible, then it follows that no characteristic line can intersect more than one component of* cosupp u.

The proof is identical with part I of the proof of Theorem 3, except that we now assume that the line segment considered there is characteristic and use Theorem 1 instead of John's theorem.

From Theorem 5 we obtain the following conditions on the support of a solution of the Cauchy problem when the data have compact support on a non characteristic line.

THEOREM 6. *Let $P(D)$ be irreducible and weakly hyperbolic, and let u be a solution $\not\equiv 0$ of $P(D)u = 0$ in R^2. Let a non characteristic line l have a compact intersection with the support of u. It then follows that* cosupp u *has two unbounded components O_1 and O_2 whose infinite sides are characteristics adjacent to l. When extended to infinity in both directions these lines intersect neither O_1 nor O_2. All other components of* cosupp u *are situated in the parallelogram formed by these lines (compare fig. 2 below).*

PROOF. Let x_1 and x_2 be points in $l \cap \operatorname{supp} u$ such that the complement relative to l of the line segment x_1x_2 belongs to cosupp u. Let $x_1x_3x_2x_4$ be the characteristic parallelogram belonging to this segment and define S_1 and S_2 as the shadowed domains in fig. 1. Then $S_i \subset \operatorname{supp} u$, $i = 1,2$. For if $x \in S_1$ and ξ_1 and ξ_2 are defined as in the figure it follows from Theorem 5 that the segment $\xi_1 x$ and $x\xi_2$ belong to cosupp u, hence Theorem 2 shows that the whole line l would belong to cosupp u. But this contradicts the assumption that $u \not\equiv 0$. Hence the components O_1 and O_2 having x_1 and x_2 respectively as boundary points are convex polygons with the infinite sides parallel to the segments x_1x_3 and x_2x_3. The rest of the theorem follows at once from Theorem 5.

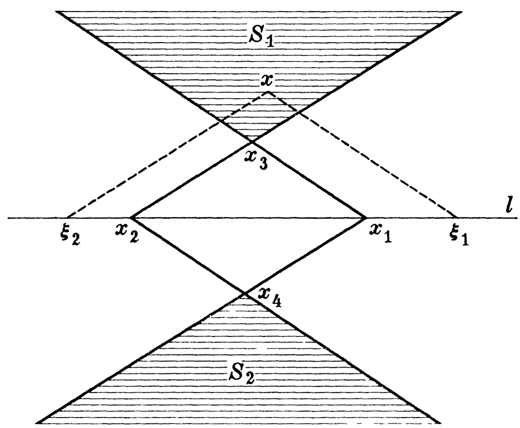

Fig. 1.

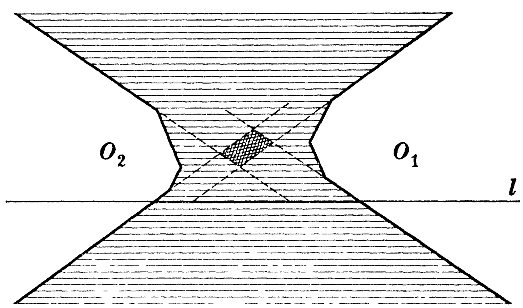

Fig. 2.

Fig. 2 demonstrates the result of Theorem 6 (and Theorem 7, below). The shadowed domain in the figure is the support of a solution u of $P(D)u = 0$ such that the line l intersects supp u in a compact. The parallelogram formed by the infinite sides of the boundaries of O_1 and O_2 (cf. Theorem 6) is the doubly crossed domain in the centre; compare also Theorem 2.

Next we will show a converse of Theorem 6. Again we shall use null solutions to construct suitable solutions of the equation $P(D)u=0$. To be able to construct the desired solutions we shall rely on an existence theorem concerning Cauchy's problem and a lemma concerning null solutions.

The existence theorem is the following (see remark at p. 392 in [5]): Let $P(D)$ be weakly hyperbolic and of degree m. Let Cauchy data be given on a non characteristic line l. Then Cauchy's problem has an infinitely differentiable solution in the whole of R^2 if the data given are of class $(1+\delta)$ where $0<\delta<1/m$.—A function φ is said to be of class β if it is infinitely differentiable and if for every compact set K there is a constant C such that $|D^\alpha \varphi(x)| \leqq C^{|\alpha|+1}(|\alpha|!)^\beta$ for all α and $x \in K$. It is easy to see that the class β is closed under addition and multiplication.

LEMMA 4. *Let $P(D)$ be irreducible and of degree m. For every characteristic $\langle x,\xi\rangle = c$ there exists a non trivial solution of the equation $P(D)u=0$ with support in the half plane $\langle x,\xi\rangle \geqq c$ such that u is of class $(1+\delta)$ for some δ with $0<\delta<1/m$.*

The statement may be seen directly from the construction given by Hörmander [2]. Alternatively, let $u \not\equiv 0$ be an arbitrary null solution vanishing in the half plane $\langle x,\xi\rangle < c+1$ and let φ be of class $(1+\delta)$ with $0<\delta<1/m$ and satisfying the conditions (3.27) and (3.28). Define φ_ε as in (3.29.) and u_ε as in (3.30). Then u_ε is of class $(1+\delta)$ if $\varepsilon>0$ and the support of u_ε is contained in the half plane $\langle x,\xi\rangle \geqq c$ if $\varepsilon < 1/|\xi|$. Finally u_ε is a solution of $P(D)u=0$ and $u_\varepsilon \not\equiv 0$ if ε is sufficiently small.

THEOREM 7. *Let O_1 and O_2 be two open unbounded polygons which fulfill the conclusions of Theorem* 6. *Assume that the parallelogram mentioned there has interior points. (This assumption is not necessary in the hyperbolic case.) Then there is a solution u of the weakly hyperbolic equation $P(D)u=0$ for which the unbounded components of* cosupp u *are exactly O_1 and O_2.*

PROOF. In view of Lemma 4 and the remark following Theorem 4 we can find solutions u_i, $i=1,2$, of the equation $P(D)u=0$ so that u_i is of class $(1+\delta)$, $0<\delta<1/m$, and O_i is one component of cosupp u_i. Assume that the diagonal of the parallelogram formed by the infinite sides of O_1 and O_2 is a part of the x^1-axis and assume that the diagonal is defined by $a_1 \leqq x^1 \leqq a_2$. Let θ_1 be a function of class $(1+\delta)$ such that $\theta_1(x^1)=1$ for

$x^1 \geqq (2a_1+a_2)/3$ and $\theta_1(x^1)=0$ for $x^1 \leqq (a_1+2a_2)/3$, and set $\theta_2=1-\theta_1$. The function

$$\theta_1(x^1)\,u_1(x)+\theta_2(x^1)\,u_2(x)$$

induces now Cauchy data of class $(1+\delta)$ and with compact support on the x^1-axis. Let u be the solution of Cauchy's problem for the equation $P(D)u=0$ with these Cauchy data. Then $u-u_1$ vanishes in a neighbourhood of the x^1-axis for $x^1>(2a_1+a_2)/3$, hence $u-u_1$ vanishes in a neighbourhood of $\bar{O}_1$. It follows that O_1 is a component of $\operatorname{cosupp} u$. Similarly O_2 is shown to be a component of $\operatorname{cosupp} u$. The proof is complete.

We finally use Theorems 2, 5 and 7 to describe the unbounded components of the complement of the support of solutions of weakly hyperbolic equations in some different cases.

Let $P(D)$ be irreducible and weakly hyperbolic and let u be a solution in R^2 of the equation $P(D)u=0$.

1. If there is only one characteristic of P (that is, $P_m(D)=a\langle y,D\rangle^m$, y real) then there may be arbitrarily (even infinitely) many unbounded components of $\operatorname{cosupp} u$. These are all parallel strips with the direction of the characteristic.

2. If there are two characteristics of P, then there are at most four unbounded components of $\operatorname{cosupp} u$. If there actually are four, then they are half strips with the directions of the characteristics.

3. If there are three characteristcs of P, then there are at most three unbounded components of $\operatorname{cosupp} u$. If there are three, then they are half strips with the directions of the characteristics.

4. If there are four or more characteristics of P, then there are at most two unbounded components of $\operatorname{cosupp} u$ (cf. Theorem 6).

5. If there are two or more characteristics of P and if $\operatorname{cosupp} u$ has an unbounded component O for which the infinite sides are not adjacent characteristics, then O is equal to $\operatorname{cosupp} u$. In particular, if u is a null solution, then the support of u is exactly a half plane with characteristic boundary.

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