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PROJECTIONS OF MUKAI VARIETIES

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Abstract

This note is an answer to a problem proposed by Iliev and Ranestad. We prove that the projections of general nodal linear sections of suitable dimension of Mukai varieties M_g are linear sections of M_{g-1}.

1. Introduction

In [10] Mukai gave a description of general canonical curves, K3 surfaces and Fano threefolds of sectional genus $g \leq 10$ in terms of linear sections of appropriate varieties. For prime Fano threefolds of index 1 the description may be summarized in the Table 1. The table gives a classification of prime Fano threefolds of index 1 and genus $g \leq 10$ up to two exceptions. For $g = 6$ there exist also smooth prime Fano manifolds which are obtained as intersections of a cone over a linear section of the Grassmannian $G(2, 5)$ with a quadric not passing through the vertex. Whereas for $g = 3$ there exist double covers of $\mathbb{P}^3$ branched in quadric hypersurfaces. These are not included in the list, but are degenerations of the general cases below. Furthermore, there is only one more family of general prime Fano threefolds of index 1. It corresponds to the case $g = 12$.

In the table we use the notation $X_{i_1,\dots,i_n}$ for the generic complete intersection of given degree. The variety Q_2 is a generic quadric hypersurface. The notation $G(2, n)$ stands for the Grassmannians of lines in projective $n-1$-space in their Plücker embeddings. The variety $OG(5, 10)$ is the orthogonal Grassmannian. It is a component of the set of linear spaces of dimension 4 contained in a smooth eight dimensional quadric hypersurface in $\mathbb{P}^9$ in its spinor embedding. The variety $LG(3, 6)$ is the Lagrangian Grassmannian, it is a linear section of $G(3, 6)$ in its Plücker embedding parametrizing 3-dimensional vector spaces isotropic with respect to a chosen generic symplectic form. The variety G_2 is a linear section of $G(5, 7)$ in its Plücker embedding parametrizing 5-dimensional vector subspaces of a 7-dimensional vector space isotropic with respect to a chosen generic four-form. The notation M_g and the name Mukai varieties has

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TABLE 1. Anti-canonical model of prime Fano threefolds of index 1 and given genus.

Genus	Model
2	$X_6 \subset \mathbb{P}(1^4, 3)$
3	$X_4 \subset \mathbb{P}^4$
4	$X_{2,3} \subset \mathbb{P}^5$
5	$X_{2,2,2} \subset \mathbb{P}^6$
6	$X_{1,1} \subset Q_2 \cap G(2,5) =: M_6^5$
7	$X_{1,1,1,1,1,1,1} \subset OG(5,10) =: M_7^{10}$
8	$X_{1,1,1,1,1} \subset G(2,6) =: M_8^8$
9	$X_{1,1,1} \subset LG(3,6) =: M_9^6$
10	$X_{1,1} \subset G_2 =: M_{10}^5$

become common in this context. The upper index used in the table stands for the dimension of the variety and will be omitted from now on. We shall describe these varieties more precisely in Section 3.

It is now a natural problem to relate these Fano varieties by means of standard constructions such as, for example, projections. In particular the following problem was considered in [13], [4]. It concerns proper linear sections of Mukai varieties i.e. intersections of Mukai varieties with linear spaces whose codimension is equal to the codimension of the linear space.

PROBLEM 1.1. For given $7 \leq g \leq 10$, what is the highest n such that there exists a proper linear section H of dimension n of M_g admitting a single ordinary double point p as singularity and such that the projection of H from p is linearly isomorphic to a proper linear section of M_{g-1}?

The justification for proposing this problem is the observation that taking the projection of a nodal Fano manifold (K3 surface or canonical curve) of sectional genus g from the node we still get a Fano manifold (K3 surface or canonical curve) but with sectional genus reduced by 1, hence the result should appear as a section of M_{g-1}. The only problem arising is that the resulting variety might again be (and in fact, by Proposition 5.4, in dimension at least 3 will always be) singular and not prime (i.e. the Weil divisors class group is not generated by the canonical class) in which case Mukai's result does not work.

As evidence in [13] it was observed that the statement is true for $n = 1$ i.e. the projection of a nodal curve which is a linear section of M_g is always a proper linear section of M_{g-1}. Moreover an upper bound for n was given, by computing the maximal dimension of quadrics contained in M_{g-1} and observing that the result of the considered projection must contain a quadric divisor as the exceptional divisor of the projection.

Observe first that n can be arbitrarily large for an analogous problem formulated for $g \leq 5$. More precisely we have the following observation:

OBSERVATION 1.2. For $3 \leq g \leq 5$ and $n \in \mathbb{N}$ there exists a complete intersection M of type M_g (i.e. as in the Table) and dimension n in a corresponding weighted projective space such that M admits a single ordinary double point as singularity. Moreover for any such M the projection from the node is linearly isomorphic to a complete intersection of type M_{g-1}. Conversely a generic complete intersection of type M_{g-1} containing a smooth quadric as a codimension 1 subvariety can be obtained in such a way.

Similarly for $g = 6$.

OBSERVATION 1.3. There exists a quadric Q such that $G(2, 5) \cap Q$ has a single node. Moreover for any such intersection the projection from the node is linearly isomorphic to a complete intersection of type $X_{2,2,2}$. Conversely a generic complete intersection $X_{2,2,2}$ containing a smooth quadric as a codimension 1 subvariety can be obtained in such a way.

Indeed these are examples of standard Kustin-Miller unprojections, see [12, §4].

The case $g = 9$ was the main result of [4]. Before we state the theorem let us observe that the general singular hyperplane section of $LG(3, 6)$ has a single node as singularity. Let now L be any nodal hyperplane section of $LG(3, 6)$ and p its unique singularity.

THEOREM 1.4. *The projection of L from the node p is a proper codimension* 3 *linear section of* $G(2, 6)$*, containing a* 4*-dimensional quadric. Conversely a general* 5*-dimensional linear section of* $G(2, 6)$ *that contains a* 4*-dimensional quadric arises in this way.*

The proof proposed in [4] is based on the construction of an appropriate bundle on the resolution of a nodal hyperplane section of $LG(3, 6)$.

In this note we reprove Theorem 1.4 and solve the remaining cases in purely algebraic terms by analysis of equations of considered varieties in terms of natural representations appearing on the linear spaces they span. We understand that such arguments are unsatisfactory from the point of view of understanding the geometry of the relations involved. We believe however that in further investigations, in particular in applications, having an explicit form of the equations and the isomorphisms involved will be very helpful.

The original motivation of [13], [4] for studying Problem 1.1 was the construction of non-abelian Brill-Noether loci in moduli spaces of bundles over Mukai varieties.

Our main focus will be put on the understanding of the geometry of the constructions presented with a view toward future applications in the theory of Mirror Symmetry and Landau-Ginzburg models. For this reason in Section 5 we concentrate on the case of Fano 3-folds. We prove that for a Fano 3-fold of genus g admitting a single node its projection from the node is a Fano 3-fold of genus $g-1$ with also only nodes as singularities. We factorize the projection into a blow up of the node and a small contraction of lines and count the number of nodes obtained in each case. In this way we connect families of Fano 3-folds of genus g in the simplest way from the point of view of the theory of Landau-Ginzburg models.

The analogue of this in the case of Calabi-Yau threefolds is a cascade of geometric bitransitions connecting Calabi-Yau threefolds from the list of Borcea (see [6, §6]).

2. Statements

The main results of the paper may be summarized as follows

THEOREM 2.1. *The subscheme of* $G(11, 16)$ *parametrizing codimension* 5 *singular linear sections of* M_7 *is irreducible. The general element of this subscheme corresponds to a* 5*-dimensional linear section* L *of* M_7 *admitting a single node. The projection of* L *from the node is isomorphic to a proper intersection* $G(2, 5) \cap Q$*, where* Q *is a quadric in* $\mathbb{P}^9$ *such that* $G(2, 5) \cap Q$ *contains a smooth* 4*-dimensional quadric. Moreover a generic variety* $G(2, 5) \cap Q'$ *containing a smooth* 4*-dimensional quadric arises in this way.*

THEOREM 2.2. *The general element of the projective dual variety of* $G(2, 6)$ *defines a hyperplane section* L *of* $G(2, 6)$ *of dimension* 7 *admitting a single node as singularity. The projection of* L *from the node is then a proper codimension* 3 *linear section of* $OG(5, 10)$*, containing a smooth* 6*-dimensional quadric. Moreover a generic codimension* 3 *linear section of* $OG(5, 10)$ *containing a smooth* 6*-dimensional quadric arises in this way.*

THEOREM 2.3. *The general element of the projective dual variety to* G_2 *defines a hyperplane section* L *of* G_2 *admitting a single node as singularity. Let* L *be any hyperplane section of* G_2 *admitting a single node. Then the projection of* L *from the node is a proper linear section of* $LG(3, 6)$*, containing a smooth* 3*-dimensional quadric. Moreover a generic linear section of* $LG(3, 6)$ *containing a smooth* 3 *dimensional quadric arises in this way.*

The proof of Theorems 2.1, 2.2, 1.4 and 2.3 is based on analyzing representations appearing on the linear sections involved and comparing the equations of the varieties involved.

3. Descriptions of Mukai varieties, their tangents and projective duals

In this section we recall the known descriptions of Mukai varieties and their projective duals. As reference for the descriptions contained in this section we suggest [10], [11], [14], [15].

We start with the general description of the Grassmannian $G(2,n)$.

3.1. The Grassmannian $G(2,n)$

Let V be a n-dimensional vector space with $n \geq 2$. The Grassmannian $G(2,V)$ is then the subvariety of $\mathbb{P}(\bigwedge^2 V)$ consisting of decomposable forms. It is scheme theoretically the zero locus of the quadratic form:

$$sq_V\colon \bigwedge^2 V \ni \omega \mapsto \omega \wedge \omega \in \bigwedge^4 V.$$

The Grassmannian is also a homogeneous space of $GL(V)$. In this language if V is the standard representation of $GL(V)$ then $G(2,V)$ is the unique closed orbit of the projectivized representation $\mathbb{P}(\bigwedge^2 V)$. Let us now fix a point p in $G(2,V)$ i.e. a two-dimensional subspace $V_2 \in V$. The stabilizer subgroup of p is the parabolic subgroup P of $GL(V)$ consisting of automorphisms preserving V_2. By standard Lie theory P has a decomposition into a semi-direct product of a reductive Lie group and a solvable ideal. Such a reductive Lie group is called a Levi subgroup of P. It is also known that all Levi subgroups of the same type are conjugate. In our case a choice of Levi subgroup of P corresponds to a choice of decomposition $V = V_2 \oplus V_{n-2}$, then the Levi subgroup is the direct product $GL(V_2) \times GL(V_{n-2})$. The representation $\bigwedge^2 V$ restricted to the Levi subgroup decomposes into

$$\bigwedge^2 V_2 \oplus (V_2 \otimes V_{n-2}) \oplus \bigwedge^2 V_{n-2}.$$

Let us consider the quadratic form sq_V with respect to the above decomposition. To do this we first observe that $\bigwedge^4 V$ restricted to our Levi subgroup also decomposes:

$$\bigwedge^4 V = \left(\bigwedge^2 V_2 \otimes \bigwedge^2 V_{n-2}\right) \oplus \left(V_2 \otimes \bigwedge^3 V_{n-2}\right) \oplus \bigwedge^4 (V_{n-2}).$$

Now

$$sq_V\colon (\omega_2, \varphi, \omega_{n-2}) \mapsto (\varphi \wedge \varphi + 2\omega_2 \otimes \omega_{n-2}, 2\varphi \wedge \omega_{n-2}, \omega_{n-2} \wedge \omega_{n-2}).$$

It follows that the invariant subspace $\mathbb{P}(\bigwedge^2 V_2 \oplus (V_2 \otimes V_{n-2}))$ is the embedded tangent space of the Grassmannian $G(2,V)$ in the point p.

The projective dual variety of the Grassmannian $G(2, V)$ (see [8]) is the closure of the maximal not open orbit of the representation $\bigwedge^2 V^*$. As such it is described by the condition on forms $\bigwedge^2 V^*$ to be of non-maximal rank. More precisely, the projective dual is the zero locus of the symmetric $[n/2]$-form

$$pf_V^n\colon \bigwedge^2 V^* \ni \omega \mapsto \omega \wedge \cdots \wedge \omega \in \bigwedge^{2[n/2]} V^*.$$

It follows that the projective dual variety of $G(2, V)$ is a hypersurface for n even or is of codimension 3 for n odd.

3.2. The orthogonal Grassmannian $OG(5, 10)$

(Cf. [14, §6] and references therein.) Let us start with some generalities about the variety $OG(n, 2n)$. To define this space we start with a $2n$-dimensional vector space V_{2n} endowed with a non-degenerate quadratic form q. Consider the variety S of n-subspaces of V_{2n} isotropic with respect to q. It is a subvariety of the Grassmannian $G(n, 2n)$ having two components S^{ev} and S^{odd}. They are called the even and odd orthogonal Grassmannians and are denoted by $OG(n, 2n)$. The orthogonal Grassmannian is a homogeneous variety of the group $SO_q(V_{2n})$. In this paper we are interested in the so-called spinor embeddings of these varieties. A convenient way to get the description of the image of these embeddings is to start with a point $p \in OG(n, 2n)$ i.e. a subspace V_n isotropic with respect to q and a decomposition $V_{2n} = V_n \oplus V_n^*$ in which q is given by the matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The parabolic subgroup of $SO_q(V_{2n})$ of elements preserving V_n has as Levi subgroup $GL(V_n)$. It is however more convenient to write the spinor embedding as an invariant variety in terms of the $SL(V_n)$ representation:

$$\mathbb{P}\Big(\bigwedge^{\mathrm{ev}} V_n\Big),$$

the projectivization of the even part of the exterior algebra of the standard representation V_n. The even orthogonal Grassmannian in its spinor embedding is then described in this space as the closure of the image of the exponential map:

$$\exp\colon \bigwedge^2 V_n \ni \omega \longmapsto \left[1 + \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{i!}\omega^{\wedge i}\right] \in \mathbb{P}\Big(\bigwedge^{\mathrm{ev}} V_n\Big).$$

In our case $n = 5$ and

$$\mathbb{P}\Big(\bigwedge^{\mathrm{ev}} V_5\Big) = \mathbb{P}\Big(\mathbb{C} \oplus \bigwedge^2 V_5 \oplus \bigwedge^4 V_5\Big).$$

To get a set of equations in an intrinsic way we use the identification of $SL(V_5)$ representations $\mathbb{C} = \det V_5$ and $\bigwedge^4 V_5 = V_5^*$. The orthogonal Grassmannian is now scheme theoretically the zero locus of the quadratic form:

$$\det V_5 \oplus \bigwedge^2 V_5 \oplus V_5^* \ni (x, A, v) \\ \longmapsto (x(v) + A \wedge A, A(v)) \in \bigwedge^4 V_5 \oplus V_5. \quad (3.1)$$

Finally observe that the above form is invariant with respect to the $GL(V_5)$ action on $\det V_5 \oplus \bigwedge^2 V_5 \oplus V_5^*$, hence $OG(5, 10)$ is a $GL(V_5)$ invariant subvariety in $\mathbb{P}\big(\det V_5 \oplus \bigwedge^2 V_5 \oplus V_5^*\big)$. In fact, one checks easily that it is the closure of one orbit.

The embedded tangent space to $OG(5, 10)$ in the point $p = \mathbb{P}(\det V_5)$ is clearly the space $\mathbb{P}\big(\det V_5 \oplus \bigwedge^2 V_5\big)$. Moreover, it is a well-known theorem (see for example [2, §4]) that the variety $OG(5, V_{10})$ in its spinor embedding is self dual. More precisely, its dual variety is $OG(5, V_{10}^*)$ embedded via its spinor embedding in $\mathbb{P}\big(\det V_5^* \oplus \bigwedge^2 V_5^* \oplus V_5\big) = \mathbb{P}\big(\det V_5 \oplus \bigwedge^2 V_5 \oplus V_5^*\big)^*$.

3.3. The Lagrangian Grassmannian $LG(3, 6)$

For a chosen vector space V_{2n} of dimension $2n$ and a non-degenerate 2-form $\omega \in \bigwedge^2 V_{2n}^*$ the variety $LG(n, V_{2n}) := LG_\omega(n, V_{2n})$ is the subvariety of the Grassmannian $G(n, V_{2n})$ parametrizing n-spaces isotropic with respect to the form ω. In this way $LG_\omega(n, V_{2n})$ is a non-proper linear section of the Grassmannian $G(n, V_{2n})$. The embedding that we consider is the one coming from the Plücker embedding of the Grassmannian. The variety $LG(n, V_{2n})$ is a homogeneous variety of the simple Lie group $Sp_\omega(V_{2n})$ of automorphisms of V_{2n} preserving the form ω.

From now on, to avoid technicalities we concentrate on the case $n = 3$. As in the previous case, to get a suitable description of our variety, it is convenient to fix a point $p \in LG(3, V_6)$ i.e. a subspace V_3 isotropic with respect to ω and a decomposition $V_6 = V_3 \oplus V_3^*$ such that ω is given by the matrix:

$$\begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}.$$

Then $\bigwedge^3 V_6 = \det V_3 \oplus \big(\bigwedge^2 V_3 \otimes V_3^*\big) \oplus \big(\bigwedge^2 V_3^* \otimes V_3\big) \oplus \det V_3^*$. Now, we observe that: $\bigwedge^2 V_3 \otimes V_3^* = (S^2 V_3^* \otimes \det V_3) \oplus V_3$ and the span of the

Lagrangian Grassmannian is the subspace:

$$\begin{aligned} \bigwedge^{\langle 3\rangle} V_3 &:= \left\{\alpha \in \bigwedge^3 V_3 \;\middle|\; \alpha(\omega) = 0\right\} \\ &= \det V_3 \oplus (S^2 V_3^* \otimes \det V_3) \oplus (S^2 V_3 \otimes \det V_3^*) \oplus \det V_3^*. \end{aligned} \tag{3.2}$$

Before we pass to the equations describing $LG(3, V_6)$, let us introduce some notation. As usual, the evaluation map will be denoted by

$$\det(V_3) \otimes \det(V_3)^* \ni a \otimes b \mapsto a(b) = b(a) \in \mathbb{C},$$

as well as any map based on this evaluation as for instance:

$$(S^2 V_3 \otimes \det(V_3)^*) \otimes \det(V_3) \ni B \otimes x \mapsto B(x) \in S^2 V_3,$$

and

$$(S^2 V_3^* \otimes \det(V_3)) \otimes \det(V_3^*) \ni A \otimes y \mapsto A(y) \in S^2 V_3^*.$$

We moreover have the natural projection

$$S^2(S^2 V_3 \otimes \det V_3^*) = (S^4 V_3 \otimes (\det V_3^*)^2) \oplus S^2 V_3^* \xrightarrow{\pi} S^2 V_3^*,$$

and on the dual space

$$S^2(S^2 V_3^* \otimes \det V_3) = (S^4 V_3^* \otimes (\det V_3)^2) \oplus S^2 V_3 \xrightarrow{\pi'} S^2 V_3.$$

Finally we have two projections from

$$(S^2 V_3 \otimes \det V_3^*) \otimes (S^2 V_3^* \otimes \det V_3) = S^2 V_3 \otimes S^2 V_3^* = \Sigma^{\lambda_{2,0,-2}} V_3 \oplus \Sigma^{\lambda_{1,0,-1}} V_3 \oplus \mathbb{C}$$

onto $\Sigma^{\lambda_{1,0,-1}} V_3$ and $\mathbb{C}$ which we shall denote by η_1 and η_0 respectively. Here the notation $\Sigma^{\lambda_i} V_3$ stands for the representation of $GL(3)$ with highest weight vector (i).

With the above notation the Lagrangian Grassmannian $LG(3, V_6)$ is defined as the zero locus of the form

$$\begin{gathered} \det V_3 \oplus (S^2 V_3^* \otimes \det V_3) \oplus (S^2 V_3 \otimes \det V_3^*) \oplus \det V_3^* \ni (x, A, B, y) \\ \mapsto (\eta_1(A \otimes B), y(x) - \eta_0(A \otimes B), \pi(A) - B(x), \pi'(B) - A(y)) \\ \in \Sigma^{\lambda_{1,0,-1}} V_3 \oplus \mathbb{C} \oplus S^2 V_3 \oplus S^2 V_3^*. \end{gathered}$$

For a more detailed analysis of these equations we refer to [4]. Let us however write down also an explicit version of the equations in appropriate coordinates which will be used in subsequent proofs. Let us choose coordinates (x, A, B, y) for

$$\det V_3 \oplus S^2 V_3 \otimes \det V_3^* \oplus S^2 V_3^* \otimes \det(V_3) \oplus \det V_3^*$$

such that A, B are interpreted as symmetric matrices:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,2} & a_{2,2} & a_{2,3} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{1,2} & b_{2,2} & b_{2,3} \\ b_{1,3} & b_{2,3} & b_{3,3} \end{pmatrix},$$

then the above equations defining $LG(3, 6)$ are:

$$A \cdot B = x \cdot y \cdot \mathrm{id}, \ \wedge^2 A = x \cdot B, \ \wedge^2 B = A \cdot y.$$

The embedded tangent space to $LG(3, V_6)$ in the point p is the space

$$\det V_3 \oplus (S^2 V_3^* \otimes \det V_3).$$

Finally, the projective dual variety to $LG(3, V_6)$ is an irreducible quartic hypersurface. For a more detailed description of the quartic and the type of singularities corresponding to orbits in its stratification we send the reader to [4, §2.5]. We shall use the fact that there is a unique orbit giving nodal sections, and it is the open orbit of the quartic.

3.4. The adjoint G_2 variety

Hereafter we describe the variety G_2; for more details on the subject we refer to [15, Ex. 30]. Let V_7 be a vector space of dimension 7 understood as a standard representation under the action of the group $GL(V_7)$. Then the representation $\bigwedge^4 V_7$ admits an open orbit (see [1, §5]). Choose a 4-form $\omega \in \bigwedge^4 V_7$ from this open orbit. The variety G_2 is the subvariety of the Grassmannian $G(2, V_7)$, consisting of those 2-spaces $U \subset V_7$ such that $\bigwedge^2 U \wedge \omega = 0$. To see it as a homogeneous variety observe that the stabilizer subgroup of ω in the representation $\bigwedge^4 V_7$ is a simple Lie group called $\mathbb{G}_2$. Let us also consider the group $\tilde{\mathbb{G}}_2 \subset GL(V_7)$ preserving $[\omega] \in \mathbb{P}(\bigwedge^4 V_7)$. The representation of $\mathbb{G}_2$ on V_7 is irreducible and called the standard representation of $\mathbb{G}_2$. By abuse of notation we shall denote it by V_7. Now $\bigwedge^2 V_7$ decomposes into $V_7 \oplus \mathrm{Ad}_{\mathbb{G}_2}$, where $\mathrm{Ad}_{\mathbb{G}_2}$ denotes the adjoint representation of the group $\mathbb{G}_2$. In this case the space $\mathrm{Ad}_{\mathbb{G}_2} = \{\alpha \in \bigwedge^2 V_7 \mid \alpha \wedge \omega = 0\}$. The variety G_2 is therefore obtained as the intersection $\mathbb{P}(\mathrm{Ad}_{\mathbb{G}_2}) \cap G(2, V_7)$ and thus is the unique closed orbit of the projectivized adjoint representation of the group $\mathbb{G}_2$. In particular, G_2 is a homogeneous variety under the action of the group $\mathbb{G}_2$ or $\tilde{\mathbb{G}}_2$.

For an intrinsic way to get the equations, let us fix a point $p \in G_2$ i.e. a subspace V_2 of dimension 2 such that $\bigwedge^2 V_2 \wedge \omega = 0$. The stabilizer subgroup in $\tilde{\mathbb{G}}_2$ of the point $p \in G_2$ contains $GL(2)$ embedded in such a way that $V_7 = V_2 \oplus V_2^* \oplus (S^2 V_2 \otimes \det V_2^*)$. After restriction we have

$$\begin{aligned}\bigwedge^2 V_7 = V_2 \oplus V_2^* \oplus (S^2 V_2 \otimes \det V_2^*) \oplus \det V_2 \oplus (S^3 V_2 \otimes \det V_2^*)\\ \oplus (S^2 V_2 \otimes \det V_2^*) \oplus \mathbb{C} \oplus (S^3 V_2^* \otimes \det V_2) \oplus \det V_2^*, \quad (3.3)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ad}_{\mathbb{G}_2} = \det V_2 \oplus (S^3 V_2 \otimes \det V_2^*) \oplus (S^2 V_2 \otimes \det V_2^*)\\ \oplus \mathbb{C} \oplus (S^3 V_2^* \otimes \det V_2) \oplus \det V_2^*.\end{aligned}$$

In fact, the decomposition can be read also directly from the root vectors of the group $\mathbb{G}_2$. By the description above, the variety G_2 being the intersection $\mathbb{P}(\mathrm{Ad}_{\mathbb{G}_2}) \cap G(2, V_7)$ is described as the zero locus of the form: $\mathrm{Ad}_{\mathbb{G}_2} \ni A \mapsto A \wedge A \in \bigwedge^4 V_7$.

Since our arguments related to G_2 are based on equations, we need to be very explicit. Let us write down the equations of G_2 in coordinates. According to the above, to get a description of G_2 we start with a 7-dimensional vector space V_7 with coordinates $v_1, \dots, v_7$ and a general 4-form $\omega \in \bigwedge^4 V_7$. From [1, Figure 1] by suitable change of coordinates we may assume:

$$\begin{aligned}\omega = v_1 \wedge v_2 \wedge v_3 \wedge v_7 + v_4 \wedge v_5 \wedge v_6 \wedge v_7 + v_1 \wedge v_2 \wedge v_4 \wedge v_5\\ + v_1 \wedge v_3 \wedge v_4 \wedge v_6 + v_2 \wedge v_3 \wedge v_5 \wedge v_6.\end{aligned}$$

Now, G_2 is obtained as a linear section of $G(2, V_7)$ by the linear space $\mathrm{Ad}_{\mathbb{G}_2,\omega} = \{\alpha \in \bigwedge^2 V_7 \mid \alpha \wedge \omega = 0\}$ which itself is defined by 7 linear equations of the form $\alpha \wedge \omega \wedge v_i = 0$. Putting the coordinates of the 2-form α in the shape of a skew-symmetric matrix the subspace $\mathrm{Ad}_{\mathbb{G}_2,\omega}$ is parametrized by coordinates $(a, \dots, n)$ of some $\mathbb{P}^{13}$ in the following way:

$$M_{G_2} = \begin{pmatrix} 0 & -f & e & g & h & i & a \\ f & 0 & -d & j & k & \ell & b \\ -e & d & 0 & m & n & -g-k & c \\ -g & -j & -m & 0 & c & -b & d \\ -h & -k & -n & -c & 0 & a & e \\ -i & -\ell & g+k & b & -a & 0 & f \\ -a & -b & -c & -d & -e & -f & 0 \end{pmatrix}.$$

The variety $G_2 = G(2, V_7) \cap \mathrm{Ad}_{\mathbb{G}_2,\omega}$ is then defined in $\mathrm{Ad}_{\mathbb{G}_2,\omega}$ by 4×4 Pfaffians of this matrix.

In our coordinates, one can also recover the decomposition (3.3) corresponding to a chosen subspace $V_2 \subset V_7$:

$$\mathrm{Ad}_{\mathbb{G}_2} = \det V_2 \oplus (S^3 V_2 \otimes \det V_2^*) \oplus (S^2 V_2 \otimes \det V_2^*) \\ \oplus \mathbb{C} \oplus (S^3 V_2^* \otimes \det V_2) \oplus \det V_2^*,$$

as given by h, (m, i, a, e), $(c, f, g+k)$, g, (b, d, n, ℓ), j.

The following lemma provides us a classification of orbits of hyperplanes in the projectivization of the adjoint representation $\mathrm{Ad}(\mathbb{G}_2)$ giving rise to singular sections of the variety G_2. Recall that in [7, Lemma 1] a classification of all orbits of the co-adjoint representation lying outside the dual variety of the subvariety G_2 is given in terms of a family of sextic hypersurfaces. In the lemma below we complete this classification with known results concerning orbits contained in the projective dual variety.

LEMMA 3.1. *The projective dual variety to the variety G_2 under the action of the simple Lie group $\mathbb{G}_2$ is a sextic hypersurface which admits a decomposition into the following orbits:*

- *an open orbit O_{12} of dimension* 12*,*
- *one orbit O_{11} of dimension* 11 *being an open subset of the base locus of sextic hypersurfaces,*
- *one orbit O_{10} of dimension* 10 *being an open subset of the singular locus of the projective dual sextic hypersurface,*
- *one orbit O_9 in dimension* 9 *being an open subset of the intersection of $\overline{O_{11}} \cap \overline{O_{10}}$,*
- *one orbit O_7 of dimension* 7*,*
- *one orbit O_5 of dimension* 5 *corresponding to the variety G_2 in the co-adjoint representation being isomorphic to the adjoint representation by the Killing form.*

PROOF. We follow the same argument as in [7, lem. 1] to get a classification of all orbits of $\mathbb{P}(\mathrm{Ad}(\mathbb{G}_2))$. Recall that the Jordan decomposition for Lie groups implies that there are three types of orbits of the adjoint representation: nilpotent orbits, semi-simple orbits and mixed orbits. In [7, lem. 1] all semi-simple orbits have been classified. In particular, the orbit O_{10} is the image in the projectivization of the semi-simple orbit corresponding to long root vectors of Cartan sub-algebras. Furthermore it was observed that there are only two types of mixed orbits: associated to the short root vectors, or to the long root vectors. Finally it was proven that there is a unique orbit of mixed type associated to short root vectors. Repeating the argument for long root vectors one proves the uniqueness of the mixed orbit associated to the long root vectors and that it's projectivization is O_{12}. To complete our classification of all orbits we need to recall the classical classification of nilpotent orbits of Bala-Carter applied to $\mathbb{G}_2$. Their list can be found for example in [3, app. A, table 2]. The projectivizations of the nilpotent orbits are O_{11}, O_9, O_7, O_5, which are distinguished by their dimensions. The geometric interpretation of O_9, O_7, O_5 can be found in [9, sec. 6], whereas the geometry of O_{11} follows from the fact that the base locus of sextics is clearly invariant and decomposes into a union of finitely many orbits; one of them must have dimension 11.

3.5. Quadrics in Mukai varieties

Let us provide here a classification of maximal dimensional quadrics contained in above Mukai varieties.

PROPOSITION 3.2. *Let $Y_g \subset \mathbb{P}^{n_g}$, for $6 \le g \le 9$ be the homogeneous variety related to the Mukai variety M_g, so that $Y_6 := G(2,5)$ and $Y_g := M_g$ for $7 \le g \le 9$. Then the maximal dimension of a quadric contained in Y_g for $g = 6, 7, 8, 9$ is equal to 4, 6, 4, 3 respectively.*

PROOF. Let us call a non-special quadric in Y_g a quadric which is not contained in a linear space contained in Y_g. Let us first classify maximal non-special quadrics in Y_g. For this, consider the rational map $\phi_g \colon \mathbb{P}^{n_g} \dashrightarrow \mathbb{P}(H^0(I_{Y_g}(2)))$ defined by the system of quadrics through Y_g. Since Y_g is scheme theoretically defined by quadrics the map is well defined on $\mathbb{P}^{n_g} \setminus Y_g$ and factors through the blow up of Y_g in $\mathbb{P}^{n_g}$. Note that for two points $p_1, p_2 \in \mathbb{P}^{n_g} \setminus Y_g$ we have $\phi_g(p_1) = \phi_g(p_2)$ if and only if the line p_1p_2 intersects Y_g in a scheme of length 2. In particular, if two points $p_1, p_2 \in \mathbb{P}^{n_g} \setminus Y_g$ are in the same fiber then the line connecting them is in the closure of this fiber. It follows that the closures of fibers of ϕ_g are linear spaces. By restricting our linear system to such a fiber closure we conclude that the closure of a fiber of ϕ_g is either a point or is spanned by a maximal non-special quadric in Y_g. To see that, first note that by definition the fiber closure of ϕ_g is not contained in Y_g. Then observe that the space of quadrics defining Y_g restricted to a fiber closure is one-dimensional, hence the intersection of a fiber closure with Y_g is a quadric. Conversely, each linear space that meets Y_g in a quadric hypersurface is contracted by ϕ_g (or more precisely its intersection with $\mathbb{P}^{n_g} \setminus Y_g$ is contracted). Such linear space is hence contained in a fiber closure. It follows that maximal non-special quadrics appear exactly as intersections of fiber closures of ϕ_g. To classify maximal quadrics in Y_g we thus need only to understand the restriction of ϕ_g to $Sec(Y_g)$ the secant variety to Y_g. We have:

(1) For $g = 6$ we have $\phi_6(\omega) = \omega \wedge \omega$ and its image is $\mathbb{P}^5$ with all fibers being 5-dimensional linear spaces spanned by 4-dimensional quadrics.
(2) For $g = 7$ the map ϕ_7 was studied in [14] and maps $\mathbb{P}^{15} = Sec(OG(5, 10))$ to a quadric in $\mathbb{P}^9$. Its fibers are 7-dimensional linear spaces spanned by 6-dimensional quadrics.
(3) For $g = 8$ we again have $\phi_8(\omega) = \omega \wedge \omega$ and hence the image of $Sec(G(2, 6))$ is $G(4, 6)$ with all fibers being linear spaces of dimension 5 spanned by 4-dimensional quadrics.
(4) For $g = 9$ the situation is slightly more complicated. The secant variety of $LG(3, 6)$ is the whole $\mathbb{P}^{13}$. Furthermore, see [9, Prop 5.10 and 5.11], there are exactly 4 orbits of the representation of $Sp(6)$ on the $\mathbb{P}^{13}$. These are: the open orbit; a hypersurface discriminant type orbit; the variety $\sigma_+(LG(3, 6))$ of points lying on more than one secant; and $LG(3, 6)$. Through a point in $\mathbb{P}^{13} \setminus \sigma_+(LG(3, 6))$ there is a unique secant and hence the corresponding fiber of ϕ_9 is a line. However, for any point $p \in \sigma_+(LG(3, 6)) \setminus LG(3, 6)$ the fiber of the image of p is a linear space of dimension 4 spanned by a Lagrangian flag variety $LF(B, 3, B^\perp)$ for some one-dimensional subspace B of the vector 6-space. More precisely $LF(B, 3, B^\perp)$ denotes the subvariety of $LG(3, 6)$ parametrising those Lagrangian 3-spaces which contain B and are contained in $B^\perp$. The latter Lagrangian flag variety is a quadric of dimension 3.

It remains to consider maximal special quadrics, these are related to maximal dimensional linear spaces in Y_g. To classify such linear spaces we observe that in every Grassmannian $G(n, V_m)$ there are two types of maximal linear spaces, these are: $F(n, V_{n+1}, V_m)$ and $F(V_{n-1}, n, V_m)$ which are of dimensions n and $m - n$ respectively. Here our notation is $F(n, V, W) = G(n, V)$ and $F(V, n, W)$ is the flag variety of n-spaces containing V and contained in W, furthermore the indices denote the dimensions of the corresponding subspaces that we fix. By applying that to our Y_g we conclude that the highest dimensional quadrics in Y_g are non-special.

4. The proofs

The idea of the proofs is the following. We start with the fact that each of our Mukai varieties appears as an orbit of a representation of a suitable Lie Group on a projective space. Now, on one hand we consider the representation corresponding to M_g restricted to a suitable subgroup preserving a singular hyperplane section on the other we have the representation corresponding to M_{g-1} restricted to a subgroup preserving a linear space of suitable dimension containing a quadric. Finally, we identify these restricted representations which induces an isomorphism between studied varieties.

PROOF OF THEOREM 2.1. Since the condition for a linear space to intersect $OG(5, 10)$ in a singular variety is equivalent to intersecting a tangent space to $OG(5, 10)$ non-transversally we obtain the irreducibility of the family of singular hyperplane sections. For the remaining part of the proof we use the description and notation introduced in Section 3.2 replacing V_5 by V. We observe that a hyperplane containing the tangent space of $OG(5, 10)$ in the point p is given by choosing a hyperplane U corresponding to a point in the summand V of the decomposition $\det V^* \oplus \bigwedge^2 V^* \oplus V$ of the dual space $\bigwedge^{\mathrm{ev}} V^*$. A Levi subgroup of the stabilizer of such a hyperplane is isomorphic to $GL(1) \times GL(4)$ and corresponds to decompositions

$$V = U \oplus \mathbb{C}, \qquad V^* = U^* \oplus \mathbb{C}.$$

We consider only the $GL(4)$ representations, where by abuse of notation U is now the standard representation of $GL(4)$. We shall denote t and t^* the coordinates corresponding to the respective trivial summands above. The $GL(4)$ representation on the tangent hyperplane is thus $T_p = \mathbb{P}\big(\det U \oplus \bigwedge^2 U \oplus U \oplus U^*\big)$, hence on the projection from p we have the representation $\mathbb{P}\big(\bigwedge^2 U \oplus U \oplus U^*\big)$. Denote the coordinates corresponding to this decomposition by (B, u, u^*).

CLAIM. The equations describing the projection are $B \wedge B = 0$, $B(u^*) = 0$ and $u(u^*) = 0$.

Indeed we have: $A = B + t \wedge u$, $v^* = u^* + t^*$, $v = u + t$ and $x = x' \wedge t$ for x' the coordinate representing the component $\det U$ of T_p. The equations of $OG(5, 10)$ in these coordinates are:

$$(x' \wedge t)(u^* + t^*) + (B + t \wedge u) \wedge (B + t \wedge u) = 0, \quad (B + t \wedge u)(u^* + t^*) = 0.$$

Expanding these we get:

$$(-x'(u^*) - 2B \wedge u) \wedge t + t(t^*)x' + B \wedge B = 0, \quad B(u^*) - t(t^*)u^* + t \wedge u(u^*) = 0.$$

Since the equations are in $\bigwedge^4 V = \bigwedge^4 U \oplus \big(\bigwedge^3 U \otimes \mathbb{C}\big)$ and $V = U \oplus \mathbb{C}$ we decompose them accordingly getting:

$$(-x'(u^*) - 2B \wedge u) = 0, \quad t(t^*)x' + B \wedge B = 0,$$
$$B(u^*) - t(t^*)u^* = 0, \quad u(u^*) = 0.$$

Now, the hyperplane section is given by $t^* = 0$ giving:

$$(-x'(u^*) - 2B \wedge u) = 0, \quad B \wedge B = 0, \quad B(u^*) = 0, \quad u(u^*) = 0. \tag{4.1}$$

The equations are now given by those elements in the ideal that do not involve x' (i.e. such elements which define hypersurfaces which are cones centered in the point with coordinates $x' = 1$ and the rest 0) which proves the claim.

In particular, the projection is the intersection of a cone over a Grassmannian $G(2,5)$ with vertex a $\mathbb{P}^3$ with a hyperplane and a quadric of rank 4. Furthermore, we deduce from equations (4.1) that a general codimension 4 linear section of the hyperplane section $t^* = 0$ containing p has a unique node as singularity. This proves the second assertion of the theorem.

Consider now on the other hand the following $GL(4)$ representation

$$\mathbb{P}\Big(\Big(\bigwedge^2(U^* \oplus \mathbb{C})\Big) \oplus U\Big) = \mathbb{P}\Big(\bigwedge^2 U^* \oplus U^* \oplus U\Big).$$

Denote the coordinates corresponding to the above decomposition by (B', w'^*, w'). Clearly, the cone G with vertex the linear space $\mathbb{P}(U)$ spanned over the Grassmannian $G(2, U^* \oplus \mathbb{C})$ and the quadric Q' of rank 8 given by $w'^*(w') = 0$ are invariant under the $GL(4)$ action. The variety $G \cap Q'$ is then defined by the equations

$$B' \wedge B' = 0, B' \wedge w'^* = 0, w'^*(w') = 0.$$

We now clearly see that by choosing an element in $\det U$ giving us an isomorphism $\bigwedge^2 U^* \to \bigwedge^2 U$ we get the desired isomorphism between the projection of a general singular hyperplane section of $OG(5, 10)$ and the intersection of the cone spanned over $G(2, 5)$ (with vertex the linear space $\mathbb{P}(U)$) with the quadric Q' defined above. We now need only to observe that the projection of a general one-nodal codimension 5 section from its node is a general section of the variety obtained above. It follows that it is isomorphic to the intersection of $G(2, V)$ with a quadric Q containing a linear space L_5 (given by $w'^* = 0$ on our linear section) isomorphic to $\mathbb{P}^5$ and meeting the Grassmannian $G(2, V)$ in some four-dimensional quadric corresponding to some $G(2, V_4)$ for $V_4 \subset V_5$ a 4-dimensional vector subspace of V.

For the converse, let $G(2, V) \cap Q$ be an intersection containing a four-dimensional quadric Q_4. Then, by our classification of quadrics in $G(2, V)$ (cf. 3.5), we know that Q_4 must be equal to $G(2, V_4) \subset G(2, V)$ for $V_4 \subset V$ a 4-dimensional vector subspace of V. Let $L_5 \simeq \mathbb{P}^5$ be the span of Q_4 then $L_5 \cap G(2, V) = Q_4 \subset Q$. It follows that there exists a Plücker quadric Q_{Pl} containing $G(2, V) \subset \mathbb{P}(\bigwedge^2 V)$ such that $Q_{\mathrm{Pl}} \cap L_5 = Q \cap L_5$. Hence, there exists a quadric $\tilde{Q}$ such that $\tilde{Q} \cap G(2, V) = Q \cap G(2, V)$ and $\tilde{Q} \supset L_5$. It is now easy to see that such an intersection can be embedded as a linear section of $G \cap Q'$. Thus $\tilde{Q} \cap G(2, V) = G(2, V) \cap Q$ is a projection of a singular section of $OG(5, 10)$. Taking a general intersection $G(2, V_5) \cap Q$ containing a four-dimensional quadric Q_4 ensures us that the latter projection will be performed from a node.

We proceed similarly with Theorem 2.2. The argument is due to L. Manivel.

PROOF. Consider a point $p \in G(2,6)$. A Levi subgroup of its stabilizer is $GL(2)\times GL(4)$ and the representation involved is $\bigwedge^2 V_2 \oplus (V_2\otimes V_4)\oplus \bigwedge^2 V_4$. Denote the corresponding coordinates $(p, v, \tilde{A})$. The tangent space to $G(2,6)$ in the fixed point is given by $\tilde{A}=0$. Hence choosing a hyperplane tangent to $G(2,6)$ at p relies on choosing an element $\omega \in \bigwedge^2 V_4^*$. The subgroup of $GL(4)$ fixing ω is the symplectic group $Sp(4)$ and we have a representation of $Sp(4)$ on the ambient space of the image of the projection given by $V_4 \oplus V_4 \oplus \bigwedge^{\langle 2\rangle} V_4$, where $\bigwedge^{\langle 2\rangle} V_4$ is the representation of $Sp(4)$ on the invariant hyperplane in $\bigwedge^2 V_4$ corresponding to $\omega \in \bigwedge^2 V_4^*$ (cf. (3.2)). We shall denote coordinates on this space by (v_1, v_2, A).

Observe that we recover the same representation on the space spanned by the intersection of $OG(5,10)$ with a linear space as follows. As in the previous theorem consider $OG(5,10)$ as invariant under the action of $GL(5)$ in the representation $\mathbb{P}\big(\det(V_5^*)\oplus \bigwedge^2 V_5^* \oplus V_5\big)$, take a decomposition of $V_5 = V_4 \oplus \mathbb{C}$. The corresponding representation of $GL(4)$ is $\mathbb{P}\big(\det(V_4^*)\oplus \bigwedge^2 V_4^* \oplus V_4^* \oplus V_4 \oplus \mathbb{C}\big)$. If we now fix a symplectic form ω' on V_4 we get a representation of $Sp(4)$ given by $V_4 \oplus V_4^* \oplus \bigwedge^{\langle 2\rangle} V_4 \oplus \det V_4 \oplus 2\mathbb{C}$. Consider the component $V_4 \oplus V_4^* \oplus \bigwedge^{\langle 2\rangle} V_4$ and denote the corresponding coordinates by (w_1, w_2, B). Note that $V_4 \simeq V_4^*$ canonically via ω' and we recognize the same representation as above.

Let us now compare the equations defining the corresponding varieties. To determine the equations of the projection of the considered singular section of $G(2,6)$ from the singular point corresponding to the coordinate p we start with the equations of the Grassmannian in the coordinates $(p, v_1, v_2, \tilde{A})$. We get

$$
\begin{aligned}
sq_{(V_2\oplus V_4)}(p, v_1, v_2, \tilde{A}) &= \big(\tilde{A}\wedge\tilde{A}, \tilde{A}\wedge(v_1, v_2), p\wedge\tilde{A} - v_1\wedge v_2\big)\\
&\in \bigwedge^4 V_4 \oplus \Big(V_2 \otimes \bigwedge^3 V_4\Big) \oplus \Big(\bigwedge^2 V_2 \otimes \bigwedge^2 V_4\Big) = \bigwedge^4 (V_2\oplus V_4).
\end{aligned}
$$

The hyperplane given by ω is $\omega(\tilde{A}) = 0$. This means that applying ω to $p\wedge\tilde{A} - v_1\wedge v_2 = 0$ we get the equation $\omega(v_1\wedge v_2) = 0$ not involving p. Furthermore, the equations of the projection are exactly $A\wedge A = 0$, $A\wedge v_1 = 0$, $A\wedge v_2 = 0$, $\omega(v_1\wedge v_2) = 0$.

For the section of $OG(5,10)$ restricting equations (3.1) to our subspace we have $B\wedge w_1 = 0$, $B(w_2) = 0$, $B\wedge B = 0$, $w_1(w_2) = 0$. It is clear that these varieties are isomorphic after taking into account the isomorphism $V_4 \simeq V_4^*$ induced by ω'. Furthermore, they both contain a quadric of dimension 6 (given by $w_1(w_2) = 0$ in the space corresponding to $B = 0$).

For the converse, observe that in the $GL(4)$ representation $\mathbb{P}\big(\det(V_4^*) \oplus \bigwedge^2 V_4^* \oplus V_4^* \oplus V_4 \oplus \mathbb{C}\big)$ the space $\mathbb{P}(V_4 \oplus V_4^*)$ is a $\mathbb{P}^7$ meeting $OG(5, 10)$ in a quadric of dimension 6, but all such quadrics are equivalent under the action of $SO_q(V_{10})$ (see Section 3.5). Hence, we may assume that the quadric is the considered quadric of dimension 6 in $OG(5, 10)$. Restricting to the $SL(4)$ action we get a representation: $\mathbb{P}\big(\mathbb{C} \oplus \bigwedge^2 V_4 \oplus V_4 \oplus V_4 \oplus \mathbb{C}\big)$. We take a general codimension 3 linear section containing $\mathbb{P}(V_4 \oplus V_4^*)$. Such a codimension 3 linear space is a graph of a linear map from $V_4 \oplus V_4 \oplus H \to \mathbb{C}^3$ for some hyperplane $H \subset \bigwedge^2 V_4$ corresponding to some $\omega_H \in \bigwedge^2 V_4^*$. Restricting to the subgroup $Sp(4) \subset SL(4)$ preserving ω_H we get a representation $\mathbb{P}\big(\bigwedge^{\langle 2\rangle} V_4 \oplus V_4 \oplus V_4 \oplus 3\mathbb{C}\big)$. By suitable change of coordinates the codimension 3 linear space is $\mathbb{P}\big(V_4 \oplus V_4 \oplus \bigwedge^{\langle 2\rangle} V_4\big)$ and we conclude as above.

Let us approach similarly Theorem 1.4 looking for an alternative to the proof given in [4].

Proof. We use notation and description from Section 3.3. We have seen that the projective tangent space T_p is the subspace given by $B = 0$, $y = 0$. Next, we observe that a choice of hyperplane containing T_p is equivalent to choosing an element Q of $\mathbb{P}((S^2 V_3 \otimes \det V_3^* \oplus \det V_3^*)^*)$. Consider the isotropy subgroup of $GL(3)$ fixing Q. For a generic choice of Q it contains $SL(2)$ in such a way that the corresponding $SL(2)$ representation on V_3 is $S^2 V_2$. The $SL(2)$ representation on the ambient space of $LG(3, 6)$ is then

$$\det(S^2 V_2) + S^2(S^2 V_2) \otimes \det(S^2 V_2^*) + S^2(S^2 V_2^*) \otimes \det(S^2 V_2) + \det(S^2 V_2^*).$$

Now using

$$S^2(S^2 V_2) = S^4 V_2 \oplus \mathbb{C}$$

and

$$\det(S^2 V_2) = \mathbb{C},$$

and $V_2 = V_2^*$, we get the representation on the ambient space of $LG(3, 6)$ as

$$\mathbb{C}_1 \oplus (S^4 V_2 \oplus \mathbb{C}_2) \oplus (S^4 V_2 \oplus \mathbb{C}_3) \oplus \mathbb{C}_4.$$

Here, we added indices to the one-dimensional representations to have a way distinguish their corresponding coordinates. The hyperplane section is given by a linear form on the subspace generated by $\mathbb{C}_3$, $\mathbb{C}_4$. In fact, since all nodal hyperplane section are in the same orbit (see [4]) on the dual space, we may assume that the hyperplane is given by the vanishing of the coordinate corresponding to $\mathbb{C}_3$. We hence get the following representation on the hyperplane section:

$$\mathbb{C}_1 \oplus (S^4 V_2 \oplus \mathbb{C}_2) \oplus S^4 V_2 \oplus \mathbb{C}_4.$$

Now the representation on the projection of the hyperplane from p is

$$S^4 V_2 \oplus \mathbb{C}_2 \oplus S^4 V_2 \oplus \mathbb{C}_4.$$

On the other hand, consider a $G(2, V_4)$ in $G(2, V_6)$ together with a decomposition $V_6 = V_4 \oplus V_2$ and the associated $GL(2) \times GL(4)$ representation: $\bigwedge^2 V_2 \oplus (V_2 \otimes V_4) \oplus \bigwedge^2 V_4$. Next, consider a general codimension 3 linear space $\tilde{H}_3$ in $V_2 \otimes V_4$. Note that the choice of V_2 and $\tilde{H}_3$ determines a general codimension three section containing $G(2, V_4)$. The latter is the projectivization of:

$$H_3 := \bigwedge^2 V_2 \oplus H_3 \oplus \bigwedge^2 V_4 \subset \bigwedge^2 V_6.$$

We now observe that there is a subgroup of $GL(2)\times GL(4)$ which is isomorphic to $SL(2)$ which fixes H_3. Indeed, geometrically, $\mathbb{P}(H_3)$ on gives a general codimension 3 section of the closed Segre orbit $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}(V_2 \otimes V_4)$ i.e. a rational normal quartic. The latter is a graph of the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ and hence is preserved by a subgroup $SL(2) \subset GL(2) \times GL(4)$ which must hence also preserve H_3. Moreover, it follows that the associated $SL(2)$ representation on V_6 is $V_2 \oplus S^3 V_2$ and hence on $\bigwedge^2 V_6$ is

$$\mathbb{C} \oplus V_2 \otimes S^3 V_2 \oplus \bigwedge^2 (S^3 V_2).$$

We now observe that

$$V_2 \otimes S^3 V_2 = S^4 V_2 \oplus S^2 V_2$$

and

$$\bigwedge^2 (S^3 V_2) = S^4 V_2 \oplus \mathbb{C}.$$

Hence, we get

$$\bigwedge^2 V_6 = \mathbb{C} \oplus S^4 V_2 \oplus S^2 V_2 \oplus S^4 V_2 \oplus \mathbb{C}$$

then $H_3 = \mathrm{Ker}\big(\bigwedge^2 V_6 \to S^2 V_2\big)$. Then the representation on H_3 is

$$H_3 = \mathbb{C} \oplus S^4 V_2 \oplus S^4 V_2 \oplus \mathbb{C}.$$

The coordinates corresponding to the decompositions above give us a hint on the isomorphism between any nodal hyperplane section of $LG(3, V_6)$ and a generic codimension 3 section of $G(2, V_6)$ containing some $G(2, V_4)$ i.e. a generic quadric $Q \subset G(2, V_6)$ of dimension 4. Note however that the decomposition does not provide any uniquely determined isomorphism. In fact, our group is too small to determine an isomorphism. To provide the correct isomorphism in this case, we will investigate explicit descriptions and use the above decomposition only as a hint. Let us hence write the equations explicitly: Let v and u be coordinates corresponding to the two one-dimensional components of H_3 and let $x_0, \dots, x_4$ and $y_0, \dots, y_4$ be the natural coordinates of the two $S^4 V_2$ components of H_3. In these coordinates, by the above discussion, the section is defined by 4×4 Pfaffians of the matrix:

$$\begin{pmatrix} 0 & v & x_0 & x_1 & x_2 & x_3 \\ -v & 0 & x_1 & x_2 & x_3 & x_4 \\ -x_0 & -x_1 & 0 & y_0 & y_1 & u - y_2 \\ -x_1 & -x_2 & -y_0 & 0 & y_2 & y_3 \\ -x_2 & -x_3 & -y_1 & -y_2 - u & 0 & y_4 \\ -x_3 & -x_4 & -y_2 & -y_3 & -y_4 & 0 \end{pmatrix}.$$

For the projection of the section of the Lagrangian Grassmannian, let v' and u' be coordinates corresponding to the two one-dimensional representations and let $x'_0, \dots, x'_4$ and $y'_0, \dots, y'_4$ be the coordinates of the two $S^4 V_2$ in the projection of the hyperplane section of $LG(3, 6)$, such that the section of $LG(3, 6)$ is given by:

$$\left(t, \begin{pmatrix} y'_0 & y'_1 & y'_2 \\ y'_1 & u' & y'_3 \\ y'_2 & y'_3 & y'_4 \end{pmatrix}, \begin{pmatrix} x'_0 & x'_1 & x'_2 \\ x'_1 & x'_2 & x'_3 \\ x'_2 & x'_3 & x'_4 \end{pmatrix}, v'\right),$$

where t is the coordinate corresponding to the projection. The isomorphism is then given by:

$$(v', x'_0, x'_1, x'_2, x'_3, x'_4, y'_0, y'_1, y'_2, y'_3, y'_4, u') \\ = (v, x_4, -x_3, -x_2, x_1, x_0, y_0, y_1, y_2, -y_3, y_4, u).$$

The fact that this is indeed an isomorphism between our sections is easily but tediously checked by comparing equations. The presentation of such an

LISTING 1. Macaulay 2 script confirming isomorphism.

```
R=QQ[u,v,x_0..x_4,y_0..y_4,t]
W=matrix{
  {0,v,x_0,x_1,x_2,x_3},
  {-v,0,x_1,x_2,x_3,x_4},
  {-x_0,-x_1,0,y_0,y_1,u-y_2},
  {-x_1,-x_2,-y_0,0,y_2,y_3},
  {-x_2,-x_3,-y_1,-y_2,0,y_4},
  {-x_3,-x_4,y_2-u,-y_3,-y_4,0}}
G26=pfaffians(4,W);
A=matrix{{y_0,y_1,-y_2},{y_1,u,-y_3},{-y_2,-y_3,y_4}};
B=matrix{{x_4,-x_3,-x_2},{-x_3,x_2,x_1},{-x_2,x_1,x_0}};
adjugate=MM -> matrix( for ii from 0 to 2 list
  (for jj from 0 to 2 list
  (-1)^(ii+jj)*det(submatrix'(MM,{ii},{jj})) ));
LG=(xx,AA,BB,zz) -> ideal(AA*BB-xx*zz,
  adjugate(AA)-xx*BB,adjugate(BB)-zz*AA);
PLG=eliminate(LG(t,A,B,v), t)
PLG==G26
```

argument is impossible so we provide a simple Macaulay 2 script, Listing 1, that permits us to quickly confirm our computations.

The situation in Theorem 2.3 is even more complicated. Indeed, as the involved representation of $\mathbb{G}_2$ is the adjoint one and the general singular section appears on an orbit of codimension 1 we have a one-dimensional subgroup of $\mathbb{G}_2$ acting on the general singular hyperplane section. Hence we are left with the comparison of representations of $\mathbb{C}^*$ which does not give any hint on the isomorphism between the varieties.

Theorem 2.3 is however still true and will be proved by guessing the isomorphism for one representative of each orbit of the dual variety giving a nodal section of G_2. The correctness of the guessed isomorphism has tediously been checked by hand by the author, however for ease of presentation we provide a simple script in Macaulay 2 performing the check. We are aware that this does not shed light on the geometry of the construction but we believe that the theorem itself has interesting geometric consequences.

We shall compare, using Macaulay 2, nodal hyperplane sections of G_2 with codimension 2 sections of $LG(3, 6)$ containing a 3-dimensional quadric.

PROOF OF THEOREM 2.3. Passing to the proof we check that we have exactly two orbits of nodal hyperplane sections of G_2. Using the description from Lemma 3.1 and the Killing form (see [7, proof of lem 1]) given by:

$$Q = 48(ad + be + cf) + 16(g^2 + k^2 + (g + k)^2 + jh + im + n\ell)$$

to identify the space $\mathbb{P}(\mathrm{Ad}_{G_2,\omega})$ with its dual, we can choose the following singular hyperplane sections as representatives of the orbits from Lemma 3.1:

(1) $j = c + f$ in the 12-dimensional orbit;

(2) $f = m$ in the 11-dimensional orbit;

(3) $g + k = 0$ in the 10-dimensional orbit;

(4) $h = f$ in the 9-dimensional orbit;

(5) $f = 0$ in the 7-dimensional orbit;

(6) $h = 0$ in the 5-dimensional orbit.

Only the following two among these sections are nodal hyperplane sections:

- The section given by $j = c + f$ is a representative of the open orbit O_{12} of the projective dual variety to G_2. It is singular at the point with only nonzero coordinate $h = 1$.
- The section given by $f = m$ is also nodal at the point with only nonzero coordinate $h = 1$ but corresponds to a hyperplane represented in the dual space by a point which lies in the intersection of the dual variety with the quadric defined by the Killing form i.e. is an element of the 11-dimensional orbit O_{11}.

The first part of the theorem now amounts to finding an embedding of the projection of the two above sections of G_2 from the point with only nonzero coordinate $h = 1$ (being their node) into $LG(3, 6)$ as a proper linear section.

We start by considering both cases at once using the pencil $f = tm + (1 - t)(j - c)$ parametrized by t. Then the image of the projection in the space with natural coordinates $(a, b, c, d, e, g, i, j, k, \ell, m, n)$ is given by all Pfaffians of the matrix M_{G_2} not involving h (with the substitution $f = tm + (1 - t)(j - c)$ made) and the quadric Q_t being the difference of the Pfaffian Pf_{hf} involving hf and the combination $t\,\mathrm{Pf}_{hm} + (1 - t)(\mathrm{Pf}_{hj} - \mathrm{Pf}_{hc})$ of Pfaffians Pf_{hm},Pf_{hj},Pf_{hc} involving hm, hj and hc respectively.

Now for $t = 1$ consider the following embedding of the projection of the hyperplane $f = m$ from the coordinate point with only nonzero coordinate $h = 1$:

$$\begin{aligned}(x, a_{1,1}, a_{1,2}, a_{1,3}, a_{2,2}, a_{2,3}, a_{3,3}, b_{1,1}, b_{1,2}, b_{1,3}, b_{2,2}, b_{2,3}, b_{3,3}, y)\\ = (n, f, c, -g - k, e, a, c + i, g, -d, -f, \ell, -b, -d, j)\end{aligned}$$

and check directly that ideal of the image of G_2 under the projection coincides with the ideal of the pre-image of $LG(3, 6)$ via this embedding. In other terms the equations of the projection are restrictions of equations of $LG(3, 6)$ to

$$(x, A, B, z) = \left(n, \begin{pmatrix} -f & c & -g-k \\ c & e & a \\ -g-k & a & c+i \end{pmatrix}, \begin{pmatrix} g & -d & -m \\ -d & \ell & -b \\ -m & -b & -d \end{pmatrix}, j \right).$$

Note that the ideal of the projection contains the quadric $a^2 + ng - e(c+i) = b = c = d = f = \ell = j = g + k = 0$, which is a 3-dimensional linear section of the quadric Q_1.

In the example corresponding to $t = 0$ i.e. $j = c + f$, which is the general case, the equations of the projected variety define the same ideal as the equations of $LG(3, 6)$ restricted to

$$(x, A, B, z) = \left(a-d, \begin{pmatrix} i+e+b & d & -g \\ d & -b & c \\ -g & c & -e-m \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -d & -m & -j \\ -m & -a+n+d & g+k \\ -j & g+k & d-\ell \end{pmatrix}, m+b \right),$$

i.e. the projection is isomorphic to a codimension 2 linear section of the Lagrangian Grassmannian containing the 3-dimensional quadric defined by $m = b = c = d = f = g + k = \ell = a(a-n) - g^2 - e(i+e) = 0$ which is also a linear section of the quadric Q_0.

The equality of the above ideals can be easily checked by hand, writing each equation of one variety as a linear combination of equations defining the other. To save space we wont write down all the equations here, instead we provide a simple Macaulay 2 script, Listing 2, performing the computations. Note that the previous script needs to be compiled for this script to work.

For the other direction we observed that all maximal dimensional quadrics in $LG(3, 6)$ are equivalent by the action of the symplectic group. We can also observe that two general codimension 2 sections containing a fixed quadric are linearly isomorphic. Indeed, we have a 5-dimensional family of 3-dimensional quadrics and each of them spans a $\mathbb{P}^4 \subset \mathbb{P}^{13}$ hence is contained in a $G(2, 9)$ of codimension 2 spaces. This means that the family of codimension two sections containing a quadric is of dimension 19. Take the representation of $Sp(6, \mathbb{C})$ acting on the space $\mathbb{P}\left(\bigwedge^2\left(\bigwedge^{\langle 3\rangle} V_6^*\right)\right)$ containing the Grassmannian of 2-spaces orthogonal to codimension 2 sections of $LG(3, V_6)$. Now, consider the orbit of the line orthogonal to the codimension 2 section of $LG(3, 6)$. To

LISTING 2. Macaulay 2 script checking equality of ideals.

```
R=QQ[a,b,c,d,e,f,g,h,i,j,k,l,m,n,t];
M=matrix{{0,-f,e,g,h,i,a},{f,0,-d,j,k,l,b},
    {-e,d,0,m,n,-g-k,c},{-g,-j,-m,0,c,-b,d},
    {-h,-k,-n,-c,0,a,e},{-i,-l,g+k,b,-a,0,f},
    {-a,-b,-c,-d,-e,-f,0}};
G2=pfaffians(4,M);
H=G2+ideal(t*m+(1-t)*(j-c)-f);
pr=saturate eliminate(h,H);
ProjectionG2=substitute(pr,f=>t*m+(1-t)*(j-c));
x1=n;
A1=matrix{{-m,c,-g-k},{c,e,a},{-g-k,a,c+i}};
B1=matrix{{g,-d,-m},{-d,l,-b},{-m,-b,-d}};
z1=j;
LG1=LG(x1,A1,B1,z1);
LG1==sub(ProjectionG2,t=>1)
x2=a-d;
A2=matrix{{i+e+b,d,-g},{d,-b,c},{-g,c,-e-m}};
B2=matrix{{-d,-m,-j},{-m,-a+n+d,g+k},{-j,g+k,d-l}};
z2=m+b;
LG0=LG(x2,A2,B2,z2);
LG0==sub(ProjectionG2,t=>0)
```

compute the dimension of the orbit it is enough to compute the dimension of its stabilizer. For simplicity of calculations, we can perform the computation on the Lie algebra representation. We use the representation of $\mathfrak{sp}(6,\mathbb{C}) + \mathfrak{gl}(1)$ where the $\mathfrak{gl}(1)$ represents the $\mathbb{C}^*$ action corresponding to the projectivization. The representation ϕ on V_6^* induces a representation on $\bigwedge^3 V_6^*$ and further on $\bigwedge^2(\bigwedge^3 V_6^*)$. The space $\bigwedge^2(\bigwedge^{\langle 3\rangle} V_6^*)$ is a subset of the latter hence for stabilizer computation we can perform the computation in the bigger space. For that, we choose $V_1, V_2 \in \bigwedge^{\langle 3\rangle} V_6^*$ corresponding to two linear equations cutting from $LG(3,6)$ the result of the projection above. We write down the coefficients of the action of the Lie algebra on the 2-vector $V_1 \wedge V_2 \in \bigwedge^2(\bigwedge^{\langle 3\rangle} V_6^*)$. The tangent to the stabilizer of the action of $Sp(6)\times\mathbb{C}^*$ on $V_1 \wedge V_2 \in \bigwedge^2(\bigwedge^{\langle 3\rangle} V_6^*)$ is given by the vanishing of all those coefficients. We conclude by the fact that the dimension of the latter stabilizer is equal to the dimension of the stabilizer of $[V_1 \wedge V_2]$ under the $Sp(6)$ action on $\mathbb{P}(\bigwedge^2(\bigwedge^{\langle 3\rangle} V_6^*))$. The computation is performed by the script Listing 3.

We deduce that the dimension of the stabilizer of this line is 2 i.e. the orbit is of dimension $\dim(Sp(6,\mathbb{C})) - 2 = 21 - 2 = 19$. It follows that the orbit of the codimension 2 section described in the general case is open and

LISTING 3. Macauly 2 script computing dimension of stabiliser.

```
R=QQ[a_1..a_9,b_1..b_6,c_1..c_6,d]
M=matrix{{a_1+d,a_2,a_3,b_1,b_2,b_3},
  {a_4,a_5+d,a_6,b_2,b_4,b_5},
  {a_7,a_8,a_9+d,b_3,b_5,b_6},
  {c_1,c_2,c_3,-a_1+d,-a_4,-a_7},
  {c_2,c_4,c_5,-a_2,-a_5+d,-a_8},
  {c_3,c_5,c_6,-a_3,-a_6,-a_9+d}}
S=R[x_1..x_6, SkewCommutative => true]
phi=map(S,S,transpose(M*transpose(vars(S))))
phi(x_1)*x_4+x_1*phi(x_4)+phi(x_2)*x_5+x_2*phi(x_5)+
  phi(x_3)*x_6+x_3*phi(x_6)
V1=-x_2*x_5*x_6+x_1*x_4*x_6+2*x_2*x_3*x_4
V2=2*x_2*x_4*x_6+x_2*x_3*x_5-x_1*x_3*x_4-2*x_1*x_2*x_3
PHIV1=
  -phi(x_2)*x_5*x_6-x_2*phi(x_5)*x_6-x_2*x_5*phi(x_6)+
  phi(x_1)*x_4*x_6+x_1*phi(x_4)*x_6+x_1*x_4*phi(x_6)+
  2*phi(x_2)*x_3*x_4+2*x_2*phi(x_3)*x_4+2*x_2*x_3*phi(x_4)
PHIV2=
  2*phi(x_2)*x_4*x_6+phi(x_2)*x_3*x_5-phi(x_1)*x_3*x_4-
  2*phi(x_1)*x_2*x_3+2*x_2*phi(x_4)*x_6+x_2*phi(x_3)*x_5-
  x_1*phi(x_3)*x_4-2*x_1*phi(x_2)*x_3+2*x_2*x_4*phi(x_6)+
  x_2*x_3*phi(x_5)-x_1*x_3*phi(x_4)-2*x_1*x_2*phi(x_3)
VAR3=mingensideal(vars(S)**vars(S)**vars(S))
phiv1=(coefficients(PHIV1,Monomials=> VAR3))_1
v1=(coefficients(V1,Monomials=>VAR3))_1
phiv2=(coefficients(PHIV2,Monomials=> VAR3))_1
v2=(coefficients(V2,Monomials=> VAR3))_1
LL=exteriorPower(2,v1|phiv2)+exteriorPower(2,phiv1|v2)
STAB= (map(R,S))(ideal(LL))
dim STAB
```

dense in the variety of all codimension 2 sections of $LG(3,6)$ containing a three-dimensional quadric.

5. Geometric transitions

In this section, we shall see that Theorems 2.1, 2.2, 1.4 and 2.3 provide a very concrete and geometrically simple way to connect different families of prime Fano threefolds of index 1 and genus ≤ 10. These connections are natural from the point of view of Mirror Symmetry and analogous to the so-called conifold transitions well known in the context of Calabi-Yau threefolds. Let us give more details of the theory here. A geometric transition between two

Calabi-Yau threefolds is a transformation consisting of a contraction morphism followed by a flat deformation. More precisely:

DEFINITION 5.1. A geometric transition from a smooth Calabi-Yau threefold X to a smooth Calabi-Yau threefold Y is a pair consisting of a birational morphism $f\colon X \to Z$ and a flat family over a disc with central fiber Z and some other fiber Y. Here, Z is a singular Calabi-Yau threefold. In this context, the latter deformation is called a smoothing of Z or a degeneration of Y depending on the direction from which we look. A geometric transition is a conifold transition if the singularities of Z are only ordinary double points and $f\colon X \to Z$ is a small resolution.

Conifold transitions are the most natural transformation from the point of view of mirror symmetry, in particular they admit interpretations in terms of physics. It is conjectured that any two Calabi-Yau manifold can be connected by a sequence of geometric transitions. It is moreover conjectured that, if two Calabi-Yau threefolds are connected by a geometric transitions then their mirrors are connected by a dual geometric transition.

Since it is not possible to connect two Calabi-Yau manifolds of Picard number one by a single geometric transition (varieties with Picard number one do not admit any nontrivial birational morphisms) the most natural in their context is to study pairs of geometric transitions, we call them geometric bitransitions:

DEFINITION 5.2. We say that two Calabi-Yau threefolds are connected by a geometric bitransition if there exists a Calabi-Yau threefold T such that there are geometric transitions both from T to X and from T to Y.

Notice now that there is a list of Calabi-Yau manifolds of Picard number one related to Mukai varieties. Namely these are Calabi-Yau threefolds which appear as sections of Mukai 4-folds by quadric hypersurfaces. These Calabi-Yau manifolds were extensively studied by Borcea and are called after him. In our context, Theorems 2.1, 2.2, 1.4 and 2.3 provide a way to connect all Borcea Calabi-Yau threefolds by means of a chain of geometric bitransitions that we suggestively call a cascade. More details of this constructions are given in [6, sec 6].

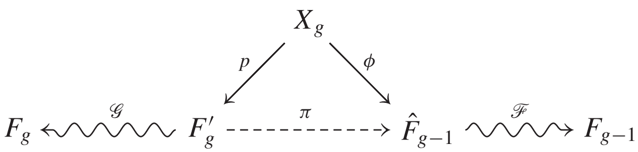
In this section, we concentrate on an analogous construction in the case of prime Fano threefolds of index 1. Knowing that the theory of Landau-Ginzburg models for Fano manifolds is parallel to that of mirror symmetry for Calabi-Yau manifolds, we extend our notions to the context of Fano varieties. We first observe that the definitions of geometric and conifold transitions as well as bitransitions can be literally repeated for Fano manifolds just by replacing the words Calabi-Yau with Fano. Their importance in the study of Landau-Ginzburg models of Fano manifolds is expected to be similar to that of geometric and conifold transitions in classical mirror symmetry. Trying to make a step towards the understanding of mirror symmetry for Fano threefolds we explain how Theorems 2.1, 2.2, 1.4 and 2.3 give rise to geometric bitransitions between prime Fano threefolds of index 1, arranging them into a sequence that we shall again call a cascade.

Let F_g be a general Fano threefold of genus g in its anti-canonical embedding. By Mukai's linear section theorem, F_g is a transverse linear section of M_g. It is hence clear that F_g admits a flat deformation to a nodal Fano threefold F'_g being a transverse linear section of a nodal linear section L of M_g studied in this paper.

LEMMA 5.3. *Let π be a linear projection of a one nodal proper linear section of M_g from its node, as in Theorems 2.1, 2.2, 1.4 and 2.3, then π is a birational map onto its image.*

PROOF. From the fact that Mukai varieties are generated by quadrics and are not cones we know that their projection from a point lying on them is a birational morphism contracting the tangent cone to its base. To conclude we need only to observe that the considered projections are restrictions of these projections to nodal hyperplane sections which are not cones.

By Theorems 2.1, 2.2, 1.4, 2.3 and Lemma 5.3 the projection of F'_g from its node is a possibly singular Fano threefold $\hat{F}_{g-1}$ containing a quadric surface and obtained as a special proper linear section of M_{g-1}. Using Mukai's linear section theorem, by moving the latter section to a general transversal section of M_{g-1}, we obtain a flat family with $\hat{F}_{g-1}$ as special fiber and general Fano threefolds of genus $g-1$ as general fibers. We hence get the following diagram connecting two general Fano 3-folds F_g and F_{g-1} of genus g and $g-1$ respectively.



Here π is the birational projection which factorizes through the blow up p of the node and a contraction morphism ϕ, whereas $\mathscr{F}$ and $\mathscr{G}$ represent deformation families with the arrows going from the special to the general fiber. To complete the picture we describe also the singularities of $\hat{F}_{g-1}$.

PROPOSITION 5.4. *If F_g is a general three-dimensional one-nodal proper linear section of M_g then all singularities of its projection $\hat{F}_{g-1}$ from the node are again nodes. Moreover the number of nodes on $\hat{F}_{g-1}$ is* $5, 4, 4, 3$ *for* $g = 7, 8, 9, 10$ *respectively.*

PROOF. Observe that by Lemma 5.3 the singularities of $\hat{F}_{g-1}$ are exactly the images of the lines contracted by the projection. Note that the projection factors through a blow up of the node with exceptional divisor a smooth quadric and a morphism contracting proper transforms of lines passing through the node. The latter maps the exceptional quadric onto a smooth quadric which passes through the images of all contracted lines. Let us now compute the singular locus of $\hat{F}_{g-1}$ being the union of images of contracted lines. To compute it observe that each line contracted is contained in the intersection of $F_g \cap T_p M_g$ where p is the center of projection and $T_p M_g$ is the projective tangent space to M_g in p. We now observe that $M_g \cap T_p M_g$ is one of the following:

- a cone over a Grassmannian $G(2, 5)$ for $g = 7$,
- a cone over a product $\mathbb{P}^1 \times \mathbb{P}^3$ for $g = 8$,
- a cone over a Veronese surface for $g = 9$,
- a cone over a twisted cubic for $g = 10$ (note that the cone spans only a $\mathbb{P}^4$ in the tangent which is a $\mathbb{P}^5$).

Now, we observe that it is always a variety of codimension 3 in $T_p M_g$, hence $F_g \cap T_p M_g$ is a union of as many lines as the degree of corresponding cone i.e. $5, 4, 4, 3$ for $g = 7, 8, 9, 10$ respectively. Thus $\hat{F}_{g-1}$ has isolated singularities whose number is given in the assertion.

Finally, we claim that all singularities of $\hat{F}_{g-1}$ as well as F'_g are ordinary double points. Since we have only a few cases to consider one can check every singularity of a representative of each orbit of varieties and check their type of singularities on the computer. We shall however use a more general argument. We just observe that the variety $\hat{F}_{g-1}$ is a general proper linear section of M_{g-1} containing a chosen quadric surface. As the quadric surface is a scheme theoretical proper linear section of M_{g-1} one can use the following proposition which is a reformulation of [5, thm 2.1] in a slightly more general context.

PROPOSITION 5.5. *Let X be a smooth projective variety of dimension $s + 2$. Let $S \subset X$ be a smooth codimension s surface being a scheme theoretical base locus of a linear subsystem $\mathscr{L} \subset |\mathcal{O}_X(d)|$, for some $d \geq 1$. Then the intersection of a set of $s - 1$ generic divisors from the system $\mathscr{L}$ is a threefold with only ordinary double points as singularities.*

PROOF. The proof of [5, thm 2.1] can be reproduced without changes.

REMARK 5.6. In the theory of Landau-Ginzburg models, a counterpart to mirror symmetry for Fano threefolds, the above construction provides the simplest possible way to connect two different Fano threefolds such that one can hope to keep track of the Landau-Ginzburg models (mirrors) involved. Not only the bitransition consists of conifolds but it is also related to a single linear projection. We hope that the results presented in this section will contribute to a better understanding of general mirror symmetry for Fano threefolds.

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