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ON THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF CERTAIN ONE-DIMENSIONAL STURM-LIOUVILLE OPERATORS

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1. Introduction.

Consider the formal differential operator

$$a(f) = -f'' - a(x)f$$

acting on functions of one variable. With the domain

$$\mathscr{D}(A) = \{f;\ f \in \mathscr{H},\ f' \ \exists \text{ loc.abs.cont.},\ a(f) \in \mathscr{H}\}\,,$$

where $\mathscr{H} = L^2(-\infty, +\infty)$, a becomes a closed operator A from $\mathscr{H}$ into $\mathscr{H}$. If a is real, locally integrable and bounded from above for large $|x|$, A is known to be self-adjoint (see e.g. [4]) and its spectrum to be discrete [5] below

$$d = \underline{\lim}\,(-a(x)), \qquad |x| \to +\infty\,.$$

Thus, if $d = +\infty$, the entire spectrum is discrete, which is known already from [12]; for a Hilbert space proof, see [6]. For locally regular a with $d = +\infty$, the eigenvalues are distributed [10, Ch. VII] in a fashion described by the asymptotic formula

$$N(\lambda) \sim \pi^{-1}\int\big(a(x)+\lambda\big)^{\frac{1}{2}}, \qquad \lambda \nearrow d\,, \tag{1}$$

where $N(\lambda)$ denotes the number of eigenvalues $< \lambda$ and where the integral is taken over the part of the axis where the integrand is real. In [11] it is shown that if a tends to a finite limit $-d$, the integral in (1) approximates $N(\lambda)$ for $\lambda < d$ with an error majorized by a certain remainder. On applying this result to operators where a is of the form

$$a(x) \sim |x|^{-2\alpha}, \qquad |x| \to +\infty\,, \tag{2}$$

it is observed that the remainder given in [11] is too crude to establish (1) if $\alpha \geqq \frac{1}{2}$.

The object of the present note is to show that (1) holds for a class of

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operators which is somewhat larger than that for which the question is settled by [11]. In particular, our class will include the case where the coefficient a satisfies (2) with $0<\alpha<1$.

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2. Application of Weyl-Courant's principle.

Consider in a Hilbert space $\mathscr{H}$ with norm $\| \;\|$ a symmetric bilinear form $A[\;,\;]$ with domain $\mathscr{D}[A]$, and let $A[\;]$ be the corresponding quadratic form. We suppose that $\mathscr{D}[A]$ is dense in $\mathscr{H}$ and that

$$A[f] \geqq -(\tau-1)\|f\|^2 \tag{3}$$

for some $\tau<+\infty$. Further, using the notations

$$(f,g)_A = A[f,g]+\tau(f,g), \qquad \|f\|_A{}^2 = (f,f)_A ,$$

for elements in $\mathscr{D}[A]$, we suppose that

$$\mathscr{D}[A] \text{ is complete in the norm } \| \;\|_A . \tag{4}$$

Then there exists (at least) one self-adjoint operator A with domain $\mathscr{D}(A)\subseteq\mathscr{D}[A]$ and with

$$(Af,g) = A[f,g], \qquad f\in\mathscr{D}(A),\ g\in\mathscr{D}[A] .$$

In particular, we can choose

$$\mathscr{D}(A) = \{f;\ f\in\mathscr{D}[A],\ |A[f,g]|\leqq C_f\|g\|,\ \forall g\in\mathscr{D}[A]\} ,$$

where C_f is a constant depending only on f. If we use the Riesz representation theorem for bounded linear functionals, we get an operator A which we shall call the Friedrichs operator generated by the form $A[\;]$; cf. [8, p. 35]. The fact that this operator is symmetric is evident from the definition. To see that it is self-adjoint, we use Schwarz' inequality and (3) to get

$$|(f,g)| \leqq \|f\|\,\|g\|_A, \qquad f\in\mathscr{H},\ g\in\mathscr{D}[A] ,$$

and thus

$$(f,g) = (Tf,g)_A$$

defines a bounded operator T with domain $\mathscr{H}$ and with range included in $\mathscr{D}[A]$. In fact, the range is even included in $\mathscr{D}(A)$. It is easy to verify that T is symmetric, and hence self-adjoint. Furthermore, this operator is connected with the Friedrichs operator A by

$$T^{-1} = A+\tau E ,$$

where E is the identity transformation, and thus the self-adjointness of A is established.

If we assume that the bottom part of the spectrum of A is discrete with eigenvalues

$$\lambda_1 \leqq \lambda_2 \leqq \dots \leqq \lambda_n \leqq \dots ,$$

the spectral representation of A shows that the Weyl–Courant characterization of the eigenvalues takes the form

$$(5) \quad \lambda_n = \sup_{\mathscr{L}} \inf_{f} A[f]\,\|f\|^{-2}, \qquad f \in \mathscr{L} \cap \mathscr{D}[A], \quad \dim(\mathscr{H} \ominus \mathscr{L}) < n .$$

Thus, if we have two quadratic forms $A_1[\;]$ and $A_2[\;]$ both of which satisfy the above conditions on $A[\;]$, then

$$\mathscr{D}[A_1] \subseteq \mathscr{D}[A_2], \qquad A_1[f] \geqq A_2[f], \quad f \in \mathscr{D}[A_1],$$

imply the inequality

$$\lambda_n(A_1) \geqq \lambda_n(A_2)$$

for the eigenvalues of the corresponding Friedrichs operators. In other words, $\lambda_n(A)$ increases when the corresponding form increases and its domain is fixed, and $\lambda_n(A)$ increases when the form is restricted to a smaller domain.

Now we leave the general theory and return to the operator A described in the introduction. Concerning the function a we shall until further notice only assume that it is real, bounded for large $|x|$ and locally integrable. Let z be a division of the axis,

$$-\infty = z_0 < z_1 < \dots < z_t < z_{t+1} = +\infty ,$$

and let I_k be the interval (z_k, z_{k+1}). We introduce the following subspaces of $\mathscr{H}$:

$$\mathscr{D}_z[\underline{A}] = \{f;\ \|f\| + \|f'\|_{I_k} < +\infty (0 \leqq k \leqq t),\ f(z_k) = 0 (0 < k \leqq t)\},$$
$$\mathscr{D}[A] = \{f;\ \|f\| + \|f'\| < +\infty\},$$
$$\mathscr{D}_z[\bar{A}] = \{f;\ \|f\| + \|f'\|_{I_k} < +\infty (0 \leqq k \leqq t)\},$$

where $\|f\|_{I_k}$ denotes the norm of f considered as an element of $L^2(I_k)$. Note that the functions in $\mathscr{D}_z[\bar{A}]$ need not be continuous at the points of z. Further, let $\underline{a}$ and $\bar{a}$ be locally integrable functions, bounded for large $|x|$ and with the property

$$\underline{a} \leqq a \leqq \bar{a} .$$

Let $A[f]$ be the form

$$A[f] = \int_{-\infty}^{+\infty} (f'^2 - a(x) f^2)$$

with domain $\mathscr{D}[A]$, and let $\underline{A}[f]$ and $\bar{A}[f]$ be the corresponding forms with a replaced by $\underline{a}$ and $\bar{a}$ and with domains $\mathscr{D}_z[\underline{A}]$ and $\mathscr{D}_z[\bar{A}]$, respectively. These three forms are seen to satisfy conditions (3) and (4). On forming the corresponding Friedrichs operators it will be seen that A, the operator generated by the form $A[\ ]$, is identical with the operator A defined in the introduction, while the operators $\underline{A}$ and $\bar{A}$ are

$$\underline{A}f = -f'' - \underline{a}(x)f, \qquad \bar{A}f = -f'' - \bar{a}(x)f$$

with domains

$$\mathscr{D}(\underline{A}) = \{f;\ \|f\| + \|f'\|_{I_k} + \|f'' + \underline{a}f\|_{I_k} < +\infty,\ f(z_k) = 0\},$$
$$\mathscr{D}(\bar{A}) = \{f;\ \|f\| + \|f'\|_{I_k} + \|f'' + \bar{a}f\|_{I_k} < +\infty,\ f'(z_k \pm 0) = 0\},$$

where I_k ranges over all the intervals of the division z, i.e., $0 \leqq k \leqq t$, and where z_k ranges over all finite points in z, i.e. $0 < k \leqq t$. Since, obviously,

$$\mathscr{D}_z[\underline{A}] \subseteq \mathscr{D}[A] \subseteq \mathscr{D}_z[\bar{A}]$$

and since

$$\underline{A}[f] \geqq A[f], \qquad f \in \mathscr{D}_z[\underline{A}],$$
$$A[f] \geqq \bar{A}[f], \qquad f \in \mathscr{D}[A],$$

the maximin-principle (5) gives the inequality

$$\lambda_n(\underline{A}) \geqq \lambda_n(A) \geqq \lambda_n(\bar{A}) \tag{6}$$

for the eigenvalues of $\underline{A}$, A and $\bar{A}$, provided the spectra of these operators are discrete below the eigenvalue in question. If $N(\lambda, \cdot)$ denotes the number of eigenvalues $< \lambda$ of the operator indicated, (6) implies

$$N(\lambda, \underline{A}) \leqq N(\lambda, A) \leqq N(\lambda, \bar{A}).$$

Since the operators $\underline{A}$ and $\bar{A}$ are direct sums

$$\underline{A} = \sum \oplus \underline{A}_k, \quad \bar{A} = \sum \oplus \bar{A}_k, \qquad 0 \leqq k \leqq t,$$

of certain operators $\underline{A}_k$ and $\bar{A}_k$, self-adjoint when regarded as operators in $L^2(I_k)$, we get

$$\sum N(\lambda, \underline{A}_k) \leqq N(\lambda, A) \leqq \sum N(\lambda, \bar{A}_k), \qquad 0 \leqq k \leqq t, \tag{7}$$

still assuming discreteness below λ of the spectra of all the operators involved.

Until now we have only assumed that

(8) $\quad a$ is real, bounded for $|x| > X_0$ and locally integrable.

The nature of the problem makes the additional assumption

$$a(x) = o(1), \qquad |x| \to +\infty, \tag{9}$$

a natural one. Further, our methods require a to satisfy

(10) a is non-decreasing for $x < -X_0$, non-increasing for $x > X_0$.

Now suppose $\varepsilon > 0$ is given sufficiently small,

$$0 < \varepsilon < \min\bigl(a(-X_0),\ a(X_0)\bigr),$$

and let w_- and w_+ be defined by

$$a(w_\pm) = \varepsilon, \quad w_- < -X_0, \quad X_0 < w_+ . \tag{11}$$

In the above-mentioned division z of the axis, put

$$z_1 = w_-, \qquad z_t = w_+$$

and suppose that for a certain j we have

$$z_j = -X_0, \qquad z_{j+1} = X_0 .$$

Then an admissible choice of $\bar{a}$ in I_0 and I_t is

$$\bar{a}(x) = \varepsilon, \qquad x \in I_0 \cup I_t$$

and the spectra of the associated operators $\bar{A}_0$ and $\bar{A}_t$ will be empty below $-\varepsilon$, i.e.

$$N(-\varepsilon,\bar{A}_0) = N(-\varepsilon,\bar{A}_t) = 0 .$$

In I_j, the interval where a may be unbounded, we take

$$\bar{a}(x) = a(x), \qquad x \in I_j .$$

The operator $\bar{A}_j$ will be regular and thus its spectrum will be entirely discrete ([1, p. 310]), i.e.

$$N(-\varepsilon,\bar{A}_j) = O(1), \qquad \varepsilon \searrow 0 .$$

Consequently, with these choices the inequality (7) reduces to

$$\sum N(-\varepsilon,\underline{A}_k) \leqq N(-\varepsilon,A) \leqq \sum N(-\varepsilon,\bar{A}_k) + O(1), \tag{12}$$
$$0<k<t, \quad k \neq j ,$$

for $\varepsilon \searrow 0$. Here we have omitted three non-negative terms on the smallest side.

In each of the remaining intervals we choose $\underline{a}$ and $\bar{a}$ constant,

$$\underline{a}(x) = \underline{a}_k, \quad \bar{a}(x) = \bar{a}_k, \quad x \in I_k, \qquad 0<k<t, \ k \neq j .$$

The same remark as above shows that the spectra of the corresponding operators are discrete, and, remembering the statement in [5] on the discreteness below 0 of the spectrum of A, we conclude that (7) is applicable for $\lambda = -\varepsilon$ and thus that (12) holds. Since the operators $\underline{A}_k$ and $\bar{A}_k$ involved in (12) have constant coefficients, their eigenvalues may be explicitly computed,

$$\lambda_n(\underline{A}_k) = n^2\pi^2|I_k|^{-2} - \underline{a}_k ,$$
$$\lambda_n(\bar{A}_k) = (n-1)^2\pi^2|I_k|^{-2} - \bar{a}_k ,$$

and hence we have

$$N(-\varepsilon,\underline{A}_k) \geqq \pi^{-1}|I_k|(\underline{a}_k-\varepsilon)^{\frac{1}{2}} - 1 ,$$
$$N(-\varepsilon,\bar{A}_k) \leqq \pi^{-1}|I_k|(\bar{a}_k-\varepsilon)^{\frac{1}{2}} + 1 ,$$

where $|I_k|$ denotes the length of the interval I_k and where we have supposed $\underline{a}_k \geqq \varepsilon$.

Combining the information obtained we get the following

Lemma 1. *Let the function a satisfy* (8)–(10). *Then* $N(-\varepsilon)$, *the number of eigenvalues* $< -\varepsilon$ *of the operator A defined in the introduction, satisfies*

$$-S(\underline{a})+\pi^{-1}\int_{\Omega}(\underline{a}-\varepsilon)^{\frac{1}{2}} \leqq N(-\varepsilon) \leqq S(\bar{a})+\pi^{-1}\int_{\Omega}(\bar{a}-\varepsilon)^{\frac{1}{2}}+O(1)$$

for $\varepsilon \searrow 0$. *Here* Ω *denotes* $(w_-, -X_0)\cup(X_0,w_+)$ *with* $w_\pm$ *defined by* (11), *while* $\underline{a}$ *and* $\bar{a}$ *are piecewise constant functions with* $\varepsilon \leqq \underline{a} \leqq a \leqq \bar{a}$ *and* $S(\cdot)$ *denotes the number of jumps of the function indicated.*

3. Proof of the asymptotic relation.

The lemma of the preceeding section reduces the problem of estimating $N(-\varepsilon)$ to the quite elementary problem of approximating the integral of a certain function over a bounded set by Riemann sums formed with piecewise constant minorants and majorants, the only complication being that the number of discontinuities of the approximating function is subtracted from the Riemann sum or added to it, respectively.

To show the validity of the expected asymptotic relation

$$N(-\varepsilon) \sim J(-\varepsilon), \qquad \varepsilon \searrow 0 ,$$

where

$$J(-\varepsilon) = \pi^{-1}\int_{-\infty}^{+\infty}\big(\max\big(0,a(x)-\varepsilon\big)\big)^{\frac{1}{2}} , \tag{13}$$

we first prove a lemma.

Lemma 2. *Suppose a satisfies the assumptions in lemma* 1 *and that*

$$a(x) \leqq |x|^{-2\alpha}, \qquad |x| > X_0 , \tag{14}$$

for some α *with* $0<\alpha<1$. *Then we have*

$$N(-\varepsilon) = J(-\varepsilon)+O(\varepsilon^{-\gamma}), \qquad \varepsilon \searrow 0 ,$$

where

$$2\alpha\gamma(2-\alpha) = 1-\alpha .$$

PROOF. Let $w_\pm$ be defined by (11) and consider the equidistant division

$$X_0 = x_0 < x_1 < \ldots < x_p = w_+$$

of the interval (X_0, w_+). The integer p is to be determined later. Choose in best possible way a piecewise constant minorant $\underline{a}$ of a in each of the intervals of the above division. By means of a rectangle estimate we get

$$\int_{x_0}^{w_+} (\underline{a}-\varepsilon)^{\frac12} \geqq \int_{x_1}^{w_+} (a-\varepsilon)^{\frac12} .$$

A similar treatment of the interval $(w_-, -X_0)$, combined with lemma 1 gives

$$N(-\varepsilon) \geqq -2p + \pi^{-1}\int_{\Omega} (a-\varepsilon)^{\frac12} - \pi^{-1}\int_{\Omega_0} (a-\varepsilon)^{\frac12} ,$$

where

$$\Omega = (w_-, -X_0) \cup (X_0, w_+), \qquad \Omega_0 = (x_{-1}, -X_0) \cup (X_0, x_1) .$$

Estimating the last integral by (14) and using the obvious fact that

$$J(-\varepsilon) = \pi^{-1}\int_{\Omega} (a-\varepsilon)^{\frac12} + O(1), \qquad \varepsilon \searrow 0 , \tag{15}$$

we obtain

$$N(-\varepsilon) \geqq -2p+J(-\varepsilon)-C_\alpha(x_{-1}^{1-\alpha}+x_1^{1-\alpha})+O(1), \qquad \varepsilon \searrow 0 ,$$

where C_α depends only on α. However, (14) implies

$$|w_\pm| \leqq \varepsilon^{-\frac12/\alpha} \tag{16}$$

and thus

$$N(-\varepsilon) \geqq -2p+J(-\varepsilon)-C_\alpha p^{\alpha-1}\varepsilon^{\frac12(1-1/\alpha)}+O(1), \qquad \varepsilon \searrow 0 .$$

Choosing p near $\varepsilon^{-\gamma}$, we get $\geqq$ in the desired equality.

To prove a reverse inequality, we start with the same divisions of $(w_-, -X_0)$ and (X_0, w_+) as before. Choose a best possible constant majorant in each of the intervals except in $(x_{-1}, -X_0)$ and (X_0, x_1), which require a special care. Again referring to the fact that a is monotone in each of the intervals, we get

$$\int_{x_1}^{w_+} (\bar{a}-\varepsilon)^{\frac12} \leqq \int_{x_0}^{w_+} (a-\varepsilon)^{\frac12}$$

and similarly for the integral over (w_-,x_{-1}). Using lemma 1 and (15) we find

$$N(-\varepsilon) \leqq S(\bar{a})+J(-\varepsilon)+\pi^{-1}\int_{\Omega_0}(\bar{a}-\varepsilon)^{\frac{1}{2}}+O(1), \qquad \varepsilon \searrow 0\,, \tag{17}$$

where the function $\bar{a}$ is yet to be determined in Ω_0. Assuming, as we obviously may, that $X_0>0$, we divide the interval (X_0,x_1) by the sequence

$$y_n = X_0 n^{1/(1-\alpha)}, \qquad 1 \leqq n \leqq p_+\,,$$

where p_+ is the greatest integer q with $y_q<x_1$. In each of the subintervals of (X_0,x_1) thus obtained, choose $\bar{a}$ constant and best possible. Then we get

$$\begin{aligned}\int_{x_0}^{x_1}(\bar{a}-\varepsilon)^{\frac{1}{2}} &\leqq \int_{x_0}^{x_1}(\bar{a})^{\frac{1}{2}} \\ &\leqq \sum X_0\big((k+1)^{1/(1-\alpha)}-k^{1/(1-\alpha)}\big)\big(a(y_k)\big)^{\frac{1}{2}} \\ &\leqq \sum C_{\alpha a}(k+1)^{1/(1-\alpha)-1}k^{-\alpha/(1-\alpha)} \\ &\leqq \sum C_{\alpha a} = C_{\alpha a}p_+\end{aligned} \tag{18}$$

where the summations are extended over $1\leqq k\leqq p_+$ and where $C_{\alpha a}$ depends only on α and X_0. Of course we can estimate the integral of a suitable $\bar{a}$ over $(x_{-1},-X_0)$ in the same way. From (17) and (18) we conclude

$$N(-\varepsilon) \leqq (2p+p_-+p_+)+J(-\varepsilon)+C_{\alpha a}(p_-+p_+)+O(1), \qquad \varepsilon \searrow 0\,.$$

Using the definition of p_+ and (16) we have

$$p_+ \leqq p^{\alpha-1}\varepsilon^{\frac{1}{2}(1-1/\alpha)}$$

and similarly for p_-. On choosing the integer p in a way to minimize the maximum order of the remainder terms, we get the desired inequality, and the proof of the lemma is complete.

Now we are in a position to prove our main theorem.

THEOREM 1. *If a is real, locally integrable, non-decreasing for $x<-X_0$, non-increasing for $x>X_0$ and satisfies*

$$a(x) \leqq |x|^{-2\alpha} \quad \text{for} \quad |x| > X_0\,,$$

$$a(x) \geqq |x|^{-2\beta} \quad \text{for} \quad x < -X_0 \quad \text{or for} \quad x > X_0$$

where

$$0 < \alpha \leqq \beta < 1\,, \tag{19}$$

$$\beta < \alpha(2-\alpha)(1+\alpha-\alpha^2)^{-1}\,, \tag{20}$$

then $N(-\varepsilon)$, *the number of eigenvalues* $< -\varepsilon$ *of the operator* A *defined in the introduction, satisfies*

$$N(-\varepsilon) \sim J(-\varepsilon), \qquad \varepsilon \searrow 0\,,$$

where J *is defined by* (13).

PROOF. The assumed lower estimate for a implies that

$$J(-\varepsilon) \geqq \big(C_\beta + o(1)\big)\varepsilon^{\frac{1}{2}(1-1/\beta)}, \qquad \varepsilon \searrow 0\,,$$

with some $C_\beta > 0$. We prefer to write this inequality in the form

$$\varepsilon^{\frac{1}{2}(1-1/\beta)} = O\big(J(-\varepsilon)\big), \qquad \varepsilon \searrow 0\,.$$

Thus (20) tells us that

$$\varepsilon^{\frac{1}{2}(1-1/\alpha)/(2-\alpha)} = o\big(J(-\varepsilon)\big), \qquad \varepsilon \searrow 0\,.$$

Now we observe that the left side of the last equality is exactly what stands inside the ordo term in lemma 2, and since our assumptions on a imply that the lemma is applicable, we have

$$N(-\varepsilon) = J(-\varepsilon) + o\big(J(-\varepsilon)\big), \qquad \varepsilon \searrow 0\,,$$

which proves the assertion.

It should be observed that if $0 < \alpha < 1$, there always exists a β satisfying (19)–(20), and similarly if $0 < \beta < 1$. In particular, if $\alpha = \beta$, (20) is valid for $0 < \alpha < 1$, and thus our theorem is applicable to certain operators to which [11] does not apply.

Concerning the necessity of the assumption (19) it may be noted that if $\alpha > 1$ we have ([7] or [2, p. 114])

$$N(-\varepsilon) = O(1), \qquad \varepsilon \searrow 0\,,$$

and it is clear that the same holds for $J(-\varepsilon)$. On the other hand, if $\alpha \leqq 1 \leqq \beta$, the asymptotic formula breaks down entirely, as is shown by the example

$$a(x) \sim k|x|^{-2}, \qquad |x| \to +\infty\,,$$

where $J(-\varepsilon) \to +\infty$ but where $N(-\varepsilon)$ stays bounded if k is sufficiently small but tends to infinity if k is sufficiently large; cf. [7], [2, p. 121]. However, the reverse calamity cannot occur, since known results ([7], [2, p. 114]) show that if a is monotone in the sense of theorem 1 and if $J(0) < +\infty$, then there is only a finite number of negative eigenvalues.

4. Equations of higher order.

There is a straightforward generalization to the operator $A^{(m)}$ defined by

$$a^{(m)}(f) = (-1)^m f^{(2m)} - a(x)f$$

on the domain

$$\{f; f \in \mathscr{H}, f^{(2m-1)} \exists \text{ loc. abs. cont.}, a^{(m)}(f) \in \mathscr{H}\} .$$

Again referring to Weyl–Courant's principle we have

$$\sum N(\lambda, \underline{A}_k^{(m)}) \leqq N(\lambda, A^{(m)}) \leqq \sum N(\lambda, \bar{A}_k^{(m)}) .$$

Here

(21) $$\underline{A}_k^{(m)} f = (-1)^m f^{(2m)} - \underline{a}_k(x) f$$

with domain the subset of $L^2(I_k)$, where

(22) $$\|f\|_{I_k} + \|(-1)^m f^{(2m)} - \underline{a}_k(x) f\|_{I_k} < +\infty$$

and where

$$f^{(j)}(z_k) = f^{(j)}(z_{k+1}) = 0, \qquad 0 \leqq j < m ,$$

while $\bar{A}_k^{(m)}$ is the obvious analogue of (21) on the subset of $L^2(I_k)$, where (22) holds with $\underline{a}_k$ replaced by $\bar{a}_k$ and where the natural boundary conditions

$$f^{(j)}(z_k + 0) = f^{(j)}(z_{k+1} - 0) = 0, \qquad m \leqq j < 2m ,$$

are fulfilled. Even if these operators have constant coefficients, their eigenvalues cannot be explicitly computed if $m > 1$. However, it is well known ([3] and [9, p. 224]) that the eigenvalues μ_n of any self-adjoint operator in $L^2(0,1)$ generated by

$$(-1)^m f^{(2m)}$$

satisfy

$$\mu_n = \big(n\pi + O(1)\big)^{2m}, \qquad n \to +\infty ,$$

and thus the eigenvalues of the above operators satisfy

$$\lambda_n(\underline{A}_k^{(m)}) \leqq -\underline{a}_k + |I_k|^{-2m}(n\pi + C_m)^{2m} ,$$
$$\lambda_n(\bar{A}_k^{(m)}) \geqq -\bar{a}_k + |I_k|^{-2m}(n\pi - C_m)^{2m} ,$$

where C_m depends only on m (and not, e.g., on $|I_k|$). Hence, with the same notations and assumptions as before, we get

$$-C_m S(\underline{a}) + \pi^{-1}\int_\Omega (\underline{a} - \varepsilon)^{\frac{1}{2}/m} \leqq N(-\varepsilon) \leqq C_m S(\bar{a}) + \pi^{-1}\int_\Omega (\bar{a} - \varepsilon)^{\frac{1}{2}/m} + O(1) ,$$

and in the same way as before we can prove the following theorem.

THEOREM 2. *If a is real, locally integrable, non-decreasing for $x < -X_0$, non-increasing for $x > X_0$ and satisfies*

$$a(x) \leqq |x|^{-2m\alpha} \quad \textit{for} \quad |x| > X_0 ,$$
$$a(x) \geqq |x|^{-2m\beta} \quad \textit{for} \quad x < -X_0 \quad \textit{or for} \quad x > X_0 ,$$

where

$$0 < \alpha \leqq \beta < 1 ,$$
$$\beta < \alpha(2-\alpha)(1+\alpha-\alpha^2)^{-1} ,$$

then $N(-\varepsilon)$, *the number of eigenvalues* $< -\varepsilon$ *of the operator* $A^{(m)}$ *defined above, satisfies*

$$N(-\varepsilon) \sim \pi^{-1}\int (a-\varepsilon)^{\frac{1}{2}/m}, \qquad \varepsilon \searrow 0 ,$$

where the integral is taken over the part of the axis where the integrand is real.

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