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VERIFICATION OF A CONJECTURE OF TH. SKOLEM

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In [1], Th. Skolem gives a distribution of the numbers $1,\ldots,2n$ into n disjoint pairs (a_r,b_r) such that $b_r=a_r+r$ for $r=1,2,\ldots,n$ and $n\equiv 0$ or 1 (mod 4). In [2], a set of pairs (a_r,b_r) of this kind was related to a system of disjoint triples (r,a_r+n,b_r+n) over a set of $3n$ elements, which, in turn, was used to construct a system of Steiner triples over a set of $6n+1$ elements. The extension of these methods to the cases $n\equiv 2$ or 3 (mod 4) depends on the verification of the following conjecture of Th. Skolem:

Theorem. *For* $n\equiv 2$ *or* 3 (mod 4), *the numbers* $1,2,\ldots,2n-1,2n+1$ *can be distributed into* n *disjoint pairs* (a_r,b_r) *such that* $b_r=a_r+r$ *for* $r=1,\ldots,n$.

Proof. Let $n\equiv 2$ (mod 4). It is sufficient to describe a system of pairs for $n=4m+2$. Such a system consists of the pairs

A: $(r,4m+2-r)$ for $r=1,\ldots,2m$,
B: $(2m+1,6m+2)$,
C: $(4m+2,6m+3)$,
D: $(4m+3,8m+5)$,
E: $(4m+3+r,8m+4-r)$ for $r=1,\ldots,m-1$,
F: $(5m+2+r,7m+3-r)$ for $r=1,\ldots,m-1$,
G: $(7m+3,7m+4)$.

Let X_1 be the set of elements which are first elements in the pair X, and X_2 be the set of second elements. The above pairs are disjoint and cover the set $1,\ldots,8m+3,8m+5$ because

$1,\ldots,2m$ are in A_1,
$2m+1$ is in B_1,
$2m+2,\ldots,4m+1$ are in A_2,
$4m+2$ is in C_1,
$4m+3$ is in D_1,
$4m+4,\ldots,5m+2$ are in E_1,

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$5m+3,\ldots,6m+1$ are in F_1,
$6m+2$ is in B_2,
$6m+3$ is in C_2,
$6m+4,\ldots,7m+2$ are in F_2,
$7m+3$ is in G_1,
$7m+4$ is in G_2,
$7m+5,\ \ldots,\ 8m+3$ are in E_2,
$8m+5$ is in D_2.

The differences for $r=1,\ldots,4m+2$ are obtained as follows:

$2,4,\ldots,4m$ from A,
$4m+2$ from D,
1 from G,
$3,5,\ldots,2m-1$ from F,
$2m+1$ from C,
$2m+3,2m+5,\ldots,4m-1$ from E,
$4m+1$ from B.

This concludes the proof for $n\equiv 2 \pmod 4$.

Let $n\equiv 3 \pmod 4$. It is sufficient to describe a system of pairs for $n=4m-1$. A system consists of the following pairs

A: $(r,4m-1-r)$ for $r=1,\ldots,m-1$,
B: $(m,m+1)$,
C: $(m+1+r,3m-r)$ for $r=1,\ldots,m-2$,
D: $(2m,4m-1)$,
E: $(4m,8m-1)$,
F: $(4m+r,8m-2-r)$ for $r=1,\ldots,2m-2$,
G: $(2m+1,6m-1)$.

These sets are disjoint and cover the integers $1,\ldots,8m-3,8m-1$ because

$1,\ldots,m-1$ are in A_1,
m is in B_1,
$m+1$ is in B_2,
$m+2,\ldots,2m-1$ are in C_1,
$2m$ is in D_1,
$2m+1$ is in G_1,
$2m+2,\ldots,3m-1$ are in C_2,
$3m,\ldots,4m-2$ are in A_2,
$4m-1$ is in D_2,
$4m$ is in E_1,

$4m+1,\ldots,6m-2$ are in F_1,
$6m-1$ is in G_2,
$6m,\ldots,8m-3$ are in F_2,
$8m-1$ is in E_2.

The differences for $r=1,\ldots,4m-1$ are obtained as follows:

1 from B,
$3,5,\ldots,2m-3$ from C,
$2m-1$ from D,
$2m+1,2m+3,\ldots,4m-3$ from A,
$4m-1$ from E,
$2,4,\ldots,4m-4$ from F,
$4m-2$ from G.

This completes the proof of the theorem.

BIBLIOGRAPHY

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