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THE q-SERIES GENERALIZATION OF A FORMULA OF SPARRE ANDERSEN

H. W. GOULD

1. Introduction.

In a recent paper [2] the writer gave a proof of two formulas of Sparre Andersen using the device of a general binomial coefficient series transformation. The two formulas of Sparre Andersen are

$$(1.1)\qquad \sum_{k=0}^{a} \binom{x}{k}\binom{-x}{n-k} = -\frac{x-a}{n}\binom{x}{a}\binom{-x-1}{n-a-1}, \qquad n \geqq 1,\ 0 \leqq a \leqq n\ ,$$

and

$$(1.2)\qquad \sum_{k=0}^{a} \binom{x}{k}\binom{1-x}{n-k} = \frac{(n-1)(1-x)-a}{n(n-1)}\binom{x-1}{a}\binom{-x}{n-a-1}, \qquad n \geqq 2,\ 0 < a \leqq n-1\ ,$$

which are valid for all real x.

It may be of interest to exhibit and prove a q-series analogue of the first of these formulas. Not all results known to be true for binomial coefficients pass readily over to the q-series. However we shall show that all the formulas of the type in [2] carry over in this case very easily. Our method is to first prove the q-analogue of the series transformation used in [2] to prove (1.1) and (1.2).

Our main results are formulas (3.2) and (4.1) below. We remark that Carlitz [1] (and other numerous papers) has made detailed use of q-analogues of Bernoulli and Euler numbers.

2. Preliminaries.

By a q-number we mean $[x]=(q^x-1)/(q-1)$, where x is a real number. It follows that

$$[-x] = -q^{-x}[x]\ .$$

The q-generalizations of the ordinary factorials and binomial coefficients are then defined by the following notation:

$$(2.1)\qquad \begin{bmatrix} x \\ n \end{bmatrix} = [x]_n/[n]_n, \qquad \begin{bmatrix} x \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} n \\ k \end{bmatrix} = 0,\ \ n < k\ ,$$

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where
$$[x]_n = [x][x-1][x-2] \ldots [x-n+1], \qquad [0]_0 = 1 ,$$

n and k being positive integers.

From which relations it follows easily that the q-numbers and q-binomial coefficients satisfy the following elementary relations:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}, \qquad 0 \leqq k \leqq n , \tag{2.2}$$

$$\begin{bmatrix} x \\ n \end{bmatrix} = \begin{bmatrix} x-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} x-1 \\ n \end{bmatrix}, \tag{2.3}$$

$$[x+n] = q^n[x] + [n] , \tag{2.4}$$

$$\begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} x \\ j \end{bmatrix} \begin{bmatrix} x-j \\ k-j \end{bmatrix} \tag{2.5}$$

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x]}{[n]} \begin{bmatrix} x-1 \\ n-1 \end{bmatrix}, \tag{2.6}$$

$$\begin{bmatrix} -x \\ n \end{bmatrix} = (-1)^n q^{-nx} \begin{bmatrix} x+n-1 \\ n \end{bmatrix} q^{-n(n-1)/2} \tag{2.7}$$

from which we have for later use also

$$\begin{bmatrix} -1 \\ k \end{bmatrix} = (-1)^k q^{-k(k+1)/2} . \tag{2.8}$$

We also need to know certain facts about q-differences. We define

$$\Delta f(x) = f(x+1) - f(x), \qquad \Delta^{n+1} f(x) = \Delta^n f(x+1) - q^n \Delta^n f(x) , \tag{2.9}$$

from which it follows by induction that

$$\Delta^n f(x) = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} f(x+n-j) . \tag{2.10}$$

Then if $f(x)$ is a polynomial of degree $\leqq n$ in q^x it is easily proved that we have the finite Taylor expansion

$$f(x+y) = \sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \Delta^k f(y) . \tag{2.11}$$

One result of (2.11) which we need is the q-Vandermonde formula. It is readily seen that we may choose $f(x) = \begin{bmatrix} x \\ n \end{bmatrix}$. We note that

$$\Delta^k \begin{bmatrix} x \\ n \end{bmatrix} = \begin{bmatrix} x \\ n-k \end{bmatrix} q^{k(x-n+k)} , \tag{2.12}$$

and thus the q-Vandermonde convolution reads

$$\sum_{k=0}^{n} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{k(y-n+k)} = \begin{bmatrix} x+y \\ n \end{bmatrix}. \tag{2.13}$$

The first lemma we need is the formula:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} x \\ k \end{bmatrix} q^{k(k-1)/2} = (-1)^n \begin{bmatrix} x-1 \\ n \end{bmatrix} q^{n(n+1)/2}. \tag{2.14}$$

This is easily derived from the preceding by setting $y = -1$ and making use of (2.8). Thus it is possible to sum a truncated series of alternating q-binomial coefficients just as in the case of the binomial coefficients. This is what suggests trying the same for the relations of Sparre Andersen.

3. The q-binomial series transformation.

For an arbitrary function f we define its transform F by

$$\begin{aligned} F(n) &= \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} f(j) \\ &= (-1)^n \Delta^n f(x) \Big|_{x=0} \quad \text{in terms of (2.10)}. \end{aligned} \tag{3.1}$$

Our first main result then is the formula

$$\begin{aligned} &\sum_{k=0}^{a} (-1)^k \begin{bmatrix} x \\ k \end{bmatrix} F(k) \\ &= (-1)^a \begin{bmatrix} x-1 \\ a \end{bmatrix} \sum_{j=0}^{a} (-1)^j \begin{bmatrix} a \\ j \end{bmatrix} \frac{[x]}{[x-j]} q^{(a-j)(a-j+1)/2} f(j). \end{aligned} \tag{3.2}$$

This is the q-analogue of the corresponding formula in [2]. The proof is as follows. From (3.1) and (2.5) we find

$$\begin{aligned} \sum_{k=0}^{a} (-1)^k \begin{bmatrix} x \\ k \end{bmatrix} F(k) &= \sum_{j=0}^{a} (-1)^j \begin{bmatrix} x \\ j \end{bmatrix} f(j) \sum_{k=j}^{a} (-1)^k \begin{bmatrix} x-j \\ k-j \end{bmatrix} q^{(k-j)(k-j-1)/2} \\ &= \sum_{j=0}^{a} \begin{bmatrix} x \\ j \end{bmatrix} f(j) \sum_{k=0}^{a-j} (-1)^k \begin{bmatrix} x-j \\ k \end{bmatrix} q^{k(k-1)/2} \\ &= \sum_{j=0}^{a} \begin{bmatrix} x \\ j \end{bmatrix} f(j) (-1)^{a-j} \begin{bmatrix} x-j-1 \\ a-j \end{bmatrix} q^{(a-j)(a-j+1)/2}, \quad \text{by (2.14)}, \end{aligned}$$

and again using (2.5) and (2.6) this reduces to the proposed value.

We next need to note that (2.12) may be expressed in the form

$$(3.3)\qquad \sum_{j=0}^{k}(-1)^j\begin{bmatrix}k\\ j\end{bmatrix}\begin{bmatrix}x+j\\ n\end{bmatrix}q^{(k-j)(k-j-1)/2} = (-1)^k\begin{bmatrix}x\\ n-k\end{bmatrix}q^{k(x-n+k)}\,.$$

With these preliminary remarks we may now proceed to the q-analogue of the formula of Sparre Andersen.

4. The q-analogue.

Proceeding along the same lines as in [2] we now choose

$$f(j) = \begin{bmatrix}-x+j\\ n\end{bmatrix}$$

and apply (3.2). We find

$$\sum_{j=0}^{a}(-1)^j\begin{bmatrix}a\\ j\end{bmatrix}\frac{[x]}{[x-j]}\begin{bmatrix}-x+j\\ n\end{bmatrix}q^{(a-j)(a-j+1)/2}$$

$$= -[x]\sum_{j=0}^{a}(-1)^j\begin{bmatrix}a\\ j\end{bmatrix}\begin{bmatrix}-x+j\\ n\end{bmatrix}\frac{q^{-x+j}}{[-x+j]}q^{(a-j)(a-j+1)/2}$$

$$= -\frac{[x]}{[n]}q^{-x}\sum_{j=0}^{a}(-1)^j\begin{bmatrix}a\\ j\end{bmatrix}\begin{bmatrix}-x+j-1\\ n-1\end{bmatrix}q^j q^{(a-j)(a-j+1)/2}$$

$$= -\frac{[x]}{[n]}q^{a-x}\sum_{j=0}^{a}(-1)^j\begin{bmatrix}a\\ j\end{bmatrix}\begin{bmatrix}-x+j-1\\ n-1\end{bmatrix}q^{(a-j)(a-j-1)/2}$$

$$= -\frac{[x]}{[n]}(-1)^a\begin{bmatrix}-x-1\\ n-1-a\end{bmatrix}q^{(a-x)(a+1)}q^{-an},\qquad \text{by (3.3)}\,.$$

Also, we have then after a slight simplification

$$\sum_{k=0}^{a}(-1)^k\begin{bmatrix}x\\ k\end{bmatrix}F(k) = -\frac{[x-a]}{[n]}\begin{bmatrix}x\\ a\end{bmatrix}\begin{bmatrix}-x-1\\ n-a-1\end{bmatrix}q^{(a-x)(a+1)}q^{-an}\,.$$

On the other hand we find

$$F(k) = (-1)^k\begin{bmatrix}-x\\ n-k\end{bmatrix}q^{k(-x-n+k)}$$

directly by (3.3). Therefore the q-analogue of Sparre Andersens's formula (1.1) is

$$(4.1)\quad \sum_{k=0}^{a}\begin{bmatrix}x\\ k\end{bmatrix}\begin{bmatrix}-x\\ n-k\end{bmatrix}q^{k(-x-n+k)} = -\frac{[x-a]}{[n]}\begin{bmatrix}x\\ a\end{bmatrix}\begin{bmatrix}-x-1\\ n-a-1\end{bmatrix}q^{(a-x)(a+1)-an}\,.$$

Presumably in a fashion similar to what was done in [2] formula number (1.2) possesses a q-analogue. Since (4.1) is clearly a polynomial identity in q^x it is true for all real x.

5. Further generalization.

The writer is indebted to Professor L. Carlitz who has indicated the following generalization which gives not only a formula containing (1.1) and (1.2) as special cases but in the more general q-setting.

Consider the sum

$$\sum_{k=0}^{a} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} r-x \\ n-k \end{bmatrix} q^{k(-x-n+k)} = \sum_{k=0}^{a} \begin{bmatrix} x \\ k \end{bmatrix} q^{k(-x-n+k)} \sum_{s=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix} \begin{bmatrix} -x \\ n-k-s \end{bmatrix} q^{s(-x-n+k+s)}$$

by (2.13). If we next apply (4.1) this becomes

$$\sum_{s=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix} q^{s(-x-n+s)} \sum_{k=0}^{a} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} -x \\ n-k-s \end{bmatrix} q^{k(-x-n+k+s)}$$

$$= -\sum_{s=0}^{r} \begin{bmatrix} r \\ s \end{bmatrix} q^{s(-x-n+s)} \frac{[x-a]}{[n-s]} \begin{bmatrix} x \\ a \end{bmatrix} \begin{bmatrix} -x-1 \\ n-s-a-1 \end{bmatrix} q^{(a-x)(a+1)-a(n-s)} .$$

Therefore we have the generalization using r as a parameter: $(n > r)$

$$(5.1) \qquad \sum_{k=0}^{a} \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} r-k \\ n-k \end{bmatrix} q^{k(-x-n+k)}$$

$$= q^{a(a+1)} \begin{bmatrix} x-1 \\ a \end{bmatrix} \sum_{s=0}^{r} \frac{[n-s-a]}{[n-s]} \begin{bmatrix} r \\ s \end{bmatrix} \begin{bmatrix} -x \\ n-s-a \end{bmatrix} q^{-(a+s)(x+n-s)} .$$

It is readily verified that $r=0$ is the q-analogue of (1.1) and that $r=1$ is the q-analogue of (1.2).

For $q=1$ this reduces to

$$(5.2) \qquad \sum_{j=0}^{a} \binom{x}{j} \binom{r-x}{n-j} = \sum_{s=0}^{r} \binom{r}{s} \frac{n-s-a}{n-s} \binom{x-1}{a} \binom{-x}{n-s-a},$$

where r is an integer subject to $n > r$. This last relation may also be written in the form

$$(5.3) \quad \sum_{j=0}^{a} \binom{x}{j} \binom{r-x}{n-j} = \binom{x-1}{a} \binom{r-x}{n-a} - a \binom{x-1}{a} \sum_{s=0}^{r} \binom{r}{s} \binom{-x}{n-s-a} \frac{1}{n-s}.$$

REFERENCES

1. L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. 15 (1948), 987–1000.
2. H. W. Gould, *Note on a paper of Sparre Andersen*, Math. Scand. 6 (1958), 226–230.

WEST VIRGINIA UNIVERSITY, MORGANTOWN, WEST VIRGINIA, U.S.A.