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ON TWO PROPERTIES OF FREE ALGEBRAS

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1. Introduction.[1]

In this note we shall discuss the following two properties of a class $\boldsymbol{K}$ of algebras concerning finitely generated free algebras in this class:

1. *An algebra in* $\boldsymbol{K}$ *which is freely generated by a set with finitely many elements is never generated (freely or not) by a set with fewer elements.*

2. *If an algebra* $\mathfrak{A} \in \boldsymbol{K}$ *is freely generated by a set with finitely many elements, then any other set with the same number of elements which generates* $\mathfrak{A}$ *generates* $\mathfrak{A}$ *freely.*

Many familiar classes $\boldsymbol{K}$, such as the class of groups, the class of rings, and the class of lattices, have both of the properties 1 and 2.

We shall establish here some sufficient conditions for a class $\boldsymbol{K}$ of algebras to have the properties discussed. In fact, we shall show that if $\boldsymbol{K}$ contains a finite algebra with more than one element, then $\boldsymbol{K}$ has the property 1, and if every equation which is identically satisfied in all finite algebras of $\boldsymbol{K}$ is also identically satisfied in all algebras of $\boldsymbol{K}$, then $\boldsymbol{K}$ has the property 2. In addition, we shall give examples of classes $\boldsymbol{K}$ which have one of the properties 1 and 2 without having the other, as well as classes which have neither of these properties.

2. Notation.

If X, Y are sets, X^Y denotes the set of all functions on Y to X. We identify an ordinal with the set of all smaller ordinals. Thus if α is any ordinal and X is any set, X^α is the set of all functions on α to X, that

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[1] The principal results of this paper were stated without proof in [3] (see the bibliography at the end of the paper). The paper was prepared for publication when the authors were working on research projects on the foundations of mathematics supported by the U. S. National Science Foundation. The authors greatly appreciate the help given by H. Jerome Keisler in preparing the paper for publication; this help resulted in particular in some improvement of the results originally formulated in [3]. In addition, Keisler's article [4] contains some supplements to this paper. Also some remarks kindly communicated to the authors by Professor Jerzy Łoś have been embodied in the paper. For some related results see [5].

is the set of all sequences of type α (α-termed sequences) with terms belonging to X. By $\mathfrak{c}(X)$ we denote the cardinality of the set X.

Let α be an ordinal, and $\mu=\langle\nu_\xi\rangle_{\xi<\alpha}$ an α-termed sequence of (finite or infinite) ordinals ν_ξ. An *algebra of type* μ is a sequence $\mathfrak{A}=\langle A,O_\xi\rangle_{\xi<\alpha}$, where A is a non-empty set and, for each $\xi<\alpha, O_\xi$ is an operation (function) on A^{ν_ξ} to A. Throughout the note, $\boldsymbol{K}$ will denote an arbitrary class of algebras of type μ, while $\mathfrak{A}=\langle A,O_\xi\rangle_{\xi<\alpha}$, $\mathfrak{B}=\langle B,P_\xi\rangle_{\xi<\alpha}$ will be two arbitrary algebras of type μ. An algebra $\mathfrak{A}$ is called *finite* if the set A is finite. By $\boldsymbol{F}(\boldsymbol{K})$ we shall denote the class of all finite algebras in $\boldsymbol{K}$.

A function f is a *homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$ if $f\in B^A$ and for every $\xi<\alpha$ and every sequence $\langle a_\gamma\rangle_{\gamma<\nu_\xi}\in A^{\nu_\xi}$, we have

$$f\big(O_\xi(\langle a_\gamma\rangle_{\gamma<\nu_\xi})\big) = P_\xi\big(\langle f(a_\gamma)\rangle_{\gamma<\nu_\xi}\big)\,.$$

A subset $Y\subseteq A$ is said to be *closed* under the operation O_ξ if whenever $\{a_\gamma \mid \gamma<\nu_\xi\}\subseteq Y$, we have $O_\xi(\langle a_\gamma\rangle_{\gamma<\nu_\xi})\in Y$. A set X is said to *generate* $\mathfrak{A}$ if $X\subseteq A$ and A has no proper subset which includes X and is closed under every operation O_ξ, $\xi<\alpha$.

We say that an algebra $\mathfrak{A}$ is $\boldsymbol{K}$-freely generated by the set X if X generates $\mathfrak{A}$ and if for every $\mathfrak{B}\in\boldsymbol{K}$, every function $f\in B^X$ can be extended to a homomorphism f' of $\mathfrak{A}$ into $\mathfrak{B}$.

$\boldsymbol{K}$ is said to *have Property* 1 if an algebra which is $\boldsymbol{K}$-freely generated by a finite set with n elements is never generated (freely or not) by a set with fewer than n elements. $\boldsymbol{K}$ is said to *have Property* 2 if, whenever an algebra $\mathfrak{A}$ is $\boldsymbol{K}$-freely generated by a finite set with n elements, any other n-element set which generates $\mathfrak{A}$ generates $\mathfrak{A}$ $\boldsymbol{K}$-freely. Notice that in this formulation of Properties 1 and 2 (as opposed to that given in the introduction), the algebra $\mathfrak{A}$ is not assumed to belong to $\boldsymbol{K}$.

3. General theorems.

The main results of this section, Theorems 1 and 2, provide sufficient conditions for a class $\boldsymbol{K}$ to have Properties 1 and 2.

We shall make use of the following simple and well-known lemma.

LEMMA 1. *If a set X generates the algebra $\mathfrak{A}$, then any function $f\in B^X$ can be extended to at most one homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.*

PROOF. Let f', f'' be any two homomorphisms of $\mathfrak{A}$ into $\mathfrak{B}$ to which f is extended. The set $A_0=\{a \mid a\in A, f'(a)=f''(a)\}$ includes X and is closed under all operations O_ξ, $\xi<\alpha$. Since X generates $\mathfrak{A}$, we have $A_0=A$, and thus $f'=f''$.

LEMMA 2. *Suppose* $\mathfrak{A}$ *is* $\boldsymbol{K}$*-freely generated by the set* X, $\mathfrak{A}$ *is generated by the set* Y, *and* $\mathfrak{B}$ *is an algebra in* $\boldsymbol{K}$. *Then we have* $\mathfrak{c}(B^X) \leqq \mathfrak{c}(B^Y)$.

PROOF. Each $f \in B^X$ can be extended to a homomorphism f' of $\mathfrak{A}$ into $\mathfrak{B}$, and this extension is unique in view of Lemma 1. Let $\hat{f}$ be the restriction of f' to Y. If $f,g \in B^X$ and $\hat{f}=\hat{g}$, then f' and g' agree on the generating set Y of $\mathfrak{A}$. Hence, by Lemma 1, $f'=g'$, and thus $f=g$. Therefore by correlating $\hat{f}$ with f we obtain a one-one mapping on B^X to B^Y, and consequently $\mathfrak{c}(B^X) \leqq \mathfrak{c}(B^Y)$.

THEOREM 1. *If* $\boldsymbol{K}$ *contains a finite algebra* $\mathfrak{B}$ *with more than one element, then* $\boldsymbol{K}$ *has Property* 1.

PROOF. This is an obvious consequence of Lemma 2.

It is easily seen that two algebras in $\boldsymbol{K}$ which are $\boldsymbol{K}$-freely generated by sets of the same cardinality are necessarily isomorphic. Therefore if $\boldsymbol{K}$ has Property 1 and if X and Y are two finite sets, we can draw the following conclusion: two algebras $\mathfrak{A}$, $\mathfrak{B}$ in $\boldsymbol{K}$ which are $\boldsymbol{K}$-freely generated by X and Y, respectively, are isomorphic if and only if X and Y have the same cardinality. On the other hand, Fujiwara in [2] has shown that if at least one of the sets X, Y is infinite, and if each of the ordinals ν_ξ is finite, $\xi < \alpha$, then the above conclusion is always valid, whether or not $\boldsymbol{K}$ has Property 1. Theorem 1 is an improvement of another theorem proved in [2].

LEMMA 3. *Suppose* $\mathfrak{A}$ *is* $\boldsymbol{K}$*-freely generated by a finite set* X, $\mathfrak{A}$ *is generated by* Y, *and* $\mathfrak{c}(Y)=\mathfrak{c}(X)$. *Then* $\mathfrak{A}$ *is* $\boldsymbol{F}(\boldsymbol{K})$*-freely generated by* Y.

PROOF. Consider an arbitrary algebra $\mathfrak{B}$ in $\boldsymbol{F}(\boldsymbol{K})$. Any $f \in B^X$ can be extended to a unique homomorphism f' of $\mathfrak{A}$ into $\mathfrak{B}$ (cf. Lemma 1). Let f be the restriction of f' to Y. As in the proof of Lemma 2, we obtain a one-one mapping on B^X to B^Y by correlating $\hat{f}$ with f. Since B^X and B^Y have the same finite number of elements, this mapping is onto B^Y, in other words for every $h \in B^Y$ there exists $f \in B^X$ such that $\hat{f}=h$. Hence h can be extended to the homomorphism f' of $\mathfrak{A}$ into $\mathfrak{B}$. It follows that $\mathfrak{A}$ is $\boldsymbol{F}(\boldsymbol{K})$-freely generated by Y.

THEOREM 2. *Suppose that whenever an algebra* $\mathfrak{A}$ *is* $\boldsymbol{F}(\boldsymbol{K})$*-freely generated by a finite set* X, $\mathfrak{A}$ *is also* $\boldsymbol{K}$*-freely generated by* X. *Then* $\boldsymbol{K}$ *has Property* 2.

PROOF. Obvious, by Lemma 3.

It can be shown that, in metamathematical terminology, an algebra $\mathfrak{A}$ is $\boldsymbol{K}$-freely generated by a set X if and only if X generates $\mathfrak{A}$ and every equation which is satisfied in $\mathfrak{A}$ by some particular elements of X is identically satisfied in every algebra of $\boldsymbol{K}$. Hence the hypothesis of Theorem 2 can be reformulated as follows: every equation which is identically satisfied in every algebra of $\boldsymbol{F}(\boldsymbol{K})$ is also identically satisfied in every algebra of $\boldsymbol{K}$. Theorem 2 thus reformulated essentially coincides with (and actually slightly improves) a result mentioned in the introduction.

In connection with Theorems 1 and 2 notice that, whenever a class $\boldsymbol{K}$ has been shown to have Property 1 or 2, the result extends automatically to various other classes of algebras. Indeed the following facts can easily be established:

(1) If $\boldsymbol{K}$ has Property 1, then every class of algebras (of the given type μ) which includes $\boldsymbol{K}$ also has this property.

(2) If each of the classes $\boldsymbol{K}_i$ with $i \in I$ has Property 2, then the class $\bigcup_{i \in I} \boldsymbol{K}_i$ also has this property.

(3) Let $\boldsymbol{L}$ be the class of all algebras $\mathfrak{A}$ such that every equation which is identically satisfied in every algebra of $\boldsymbol{K}$ is also identically satisfied in $\mathfrak{A}$. If $\boldsymbol{K}$ has one of the properties 1 and 2, then $\boldsymbol{L}$ also has the same property, and conversely.

4. Special theorems.

Theorems 3 and 4 of this section provide us with examples of algebras which have one of the properties 1 and 2 without having the other; in Theorem 5 we find an example of a class $\boldsymbol{K}$ which has neither of these properties.

THEOREM 3. (i) *The empty class of algebras and any class of one-element algebras* (*of a given type* μ) *have Property* 2 *but do not have Property* 1.

(ii) *Any other class* $\boldsymbol{K}$ *of algebras which has Property* 2 *also has Property* 1.

PROOF. Part (i) is obvious. To prove part (ii), suppose $\boldsymbol{K}$ contains an algebra $\mathfrak{B}$ with more than one element, and suppose $\boldsymbol{K}$ does not have Property 1. Then there is an algebra $\mathfrak{A}$ which is $\boldsymbol{K}$-freely generated by a finite set X, and is generated by a set Y which has fewer elements than X. Let Y' be any set such that $Y \subset Y' \subseteq A$, and which has the same number of elements as X. Clearly Y' generates $\mathfrak{A}$. Let $f \in B^Y$. We can find two different extensions $f', f'' \in B^{Y'}$ of f to Y', because B has more than one element. By Lemma 1, f can be extended to at most one homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Therefore at most one of the functions f', f'' can be extended to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. It follows that $\mathfrak{A}$ is not $\boldsymbol{K}$-freely generated by Y', and thus $\boldsymbol{K}$ does not have Property 2.

In proving the remaining two theorems, it will be convenient to use a well-known result concerning the existence of free algebras. We now state this result as a lemma; the proof can be found, for example, in [1, p. viii].

Lemma 4. *Let E be any set of equations, and define $\boldsymbol{K}$ as the class of all algebras in which every equation in E is identically satisfied. Suppose $\boldsymbol{K}$ contains at least one algebra which has more than one element. Then for any non-empty set X there is an algebra $\mathfrak{A}$ in $\boldsymbol{K}$ which is $\boldsymbol{K}$-freely generated by X.*

If some $\nu_\xi = 0$, $\xi < \alpha$, the conclusion remains valid even if X is empty. However, if every ν_ξ is positive, $\xi < \alpha$, then no algebra of type μ is generated by the empty set.

Theorem 4. *Let $\boldsymbol{K}_1$ be the class of algebras $\mathfrak{A} = \langle A, {}^+, {}^\times \rangle$ of type $\langle 1,1 \rangle$ such that for all $a \in A$, we have $(a^+)^\times = a$. Then $\boldsymbol{K}_1$ has Property 1 but does not have Property 2. In fact, there is an algebra $\mathfrak{A}$ in $\boldsymbol{K}_1$ such that a one-element set $\{x\}$ $\boldsymbol{K}_1$-freely generates $\mathfrak{A}$, and the one-element set $\{x^+\}$ also generates $\mathfrak{A}$ but not $\boldsymbol{K}_1$-freely.*

Proof. Any algebra $\mathfrak{A}$ for which both $^+$ and $^\times$ are the identity function clearly belongs to $\boldsymbol{K}_1$. Such an algebra can obviously be found which is finite but has more than one element. Therefore, by Theorem 1, $\boldsymbol{K}_1$ has Property 1.

We shall now prove the last assertion of the theorem. Since $\boldsymbol{K}_1$ satisfies the hypotheses of Lemma 4, there exists an algebra $\mathfrak{A}$ in $\boldsymbol{K}_1$ which is $\boldsymbol{K}_1$-freely generated by the one-element set $\{x\}$. The one-element set $\{x^+\}$ also generates $\mathfrak{A}$, because $(x^+)^\times = x$ and $\{x\}$ generates $\mathfrak{A}$. It remains to show that $\{x^+\}$ does not $\boldsymbol{K}_1$-freely generate $\mathfrak{A}$. We observe that the algebra $\langle \omega, {}^\oplus, {}^\otimes \rangle$, where ω is the set of finite ordinals, $n^\oplus = n+1$ for any $n \in \omega$, $n^\otimes = n-1$ for any positive $n \in \omega$, and $0^\otimes = 0$, is a member of $\boldsymbol{K}_1$. Let f be the mapping of $\{x^+\}$ to ω defined by $f(x^+) = 0$. If f could be extended to a homomorphism f' of $\mathfrak{A}$ into $\langle \omega, {}^\oplus, {}^\otimes \rangle$, then we would have

$$0 = f'(x^+) = f'\big(((x^+)^\times)^+\big) = \big((f'(x^+))^\otimes\big)^\oplus = (0^\otimes)^\oplus = 1\,,$$

which is a contradiction. Therefore $\mathfrak{A}$ is not $\boldsymbol{K}_1$-freely generated by $\{x^+\}$.

It follows that $\boldsymbol{K}_1$ does not have Property 2.

Theorem 5. *Let $\boldsymbol{K}_2$ be the class of algebras $\mathfrak{A} = \langle A, \cdot, {}^+, {}^\times \rangle$ of type $\langle 2,1,1 \rangle$ such that for all $a, b \in A$ we have $(a \cdot b)^+ = a$, $(a \cdot b)^\times = b$, and*

$a^+ \cdot a^\times = a$. Then $\boldsymbol{K}_2$ has neither Property 1 nor Property 2. In fact, any two algebras in $\boldsymbol{K}_2$ which are $\boldsymbol{K}_2$-freely generated by finite sets are isomorphic.

PROOF. We shall first prove the last assertion of the theorem. No algebra in $\boldsymbol{K}_2$ is ever generated by the empty set. Thus, in view of the the remark following Theorem 3, it suffices to prove that if m, n are positive integers with $m < n$, and if the algebra $\mathfrak{A}$ in $\boldsymbol{K}_2$ is $\boldsymbol{K}_2$-freely generated by an n-element set, then $\mathfrak{A}$ is $\boldsymbol{K}_2$-freely generated by some m-element set. It follows by induction that it is sufficient to assume $n = m+1$. We shall actually prove the slightly stronger assertion that if an algebra $\mathfrak{A}$ in $\boldsymbol{K}_2$ is $\boldsymbol{K}_2$-freely generated by the set $X \cup \{x_0, x_1\}$, and neither x_0 nor x_1 is in X, then A is also $\boldsymbol{K}_2$-freely generated by $X \cup \{x_0 \cdot x_1\}$. First, $X \cup \{x_0 \cdot x_1\}$ generates $\mathfrak{A}$, for $x_0 = (x_0 \cdot x_1)^+$, and $x_1 = (x_0 \cdot x_1)^\times$. Now let $\mathfrak{B} = \langle B, \odot, \oplus, \otimes \rangle$ be any algebra in $\boldsymbol{K}_2$ and let $f \in B^{X \cup \{x_0 \cdot x_1\}}$. Define $f' \in B^{X \cup \{x_1 \cdot x_0\}}$ as follows:

$$f'(x) = f(x) \text{ for } x \in X,$$
$$f'(x_0) = \big(f(x_0 \cdot x_1)\big)^{\oplus}, \qquad f'(x_1) = \big(f(x_0 \cdot x_1)\big)^{\otimes}.$$

By hypothesis, f' can be extended to a homomorphism f'' of $\mathfrak{A}$ into $\mathfrak{B}$. For each $x \in X$ we have $f''(x) = f'(x) = f(x)$. Also

$$\begin{aligned} f''(x_0 \cdot x_1) &= f''(x_0) \odot f''(x_1) \\ &= f'(x_0) \odot f'(x_1) \\ &= \big(f(x_0 \cdot x_1)\big)^{\oplus} \odot \big(f(x_0 \cdot x_1)\big)^{\otimes} = f(x_0 \cdot x_1). \end{aligned}$$

Thus f'' is an extension of f. Consequently $\mathfrak{A}$ is $\boldsymbol{K}_2$-freely generated by $X \cup \{x_0 \cdot x_1\}$, as we were to show.

The class $\boldsymbol{K}_2$ contains the algebra $\langle \omega, \cdot, {}^+, {}^\times \rangle$, where ω is the set of finite ordinal numbers, $\cdot$ is any one-one function of ω^2 onto ω, and ${}^+, {}^\times$ are defined by the conditions $(a \cdot b)^+ = a, (a \cdot b)^\times = b$, for all $a, b \in \omega$. Therefore $\boldsymbol{K}_2$ satisfies the hypotheses of Lemma 4. It follows from Lemma 4 that there is an algebra $\mathfrak{A}$ in $\boldsymbol{K}_2$ which is $\boldsymbol{K}_2$-freely generated by the two-element set $\{x_0, x_1\}$. As we have shown, $\mathfrak{A}$ is also $\boldsymbol{K}_2$-freely generated by the one-element set $\{x_0 \cdot x_1\}$. Therefore $\boldsymbol{K}_2$ does not have Property 1

Finally, since $\boldsymbol{K}_2$ contains an algebra with more than one element, it follows by Theorem 3 that $\boldsymbol{K}_2$ does not have Property 2.

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