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ON SOME RESULTS OF JÓNSSON AND TARSKI CONCERNING FREE ALGEBRAS

H. JEROME KEISLER

1. Introduction.

In the paper [3] which precedes this note, Jónsson and Tarski give sufficient conditions for a class $\boldsymbol{K}$ of algebras to have certain properties, which they have named Property 1 and Property 2 (see [3, pp. 95 and 96). However, in many cases the results of [3] are not adequate to settle the question of whether $\boldsymbol{K}$ has Properties 1 and 2. In this article we shall prove two theorems which extend the applicability of the results of [3].

The main results of this paper, Theorems 1 and 2, involve the notion of a *reduct* (cf. Tarski [4]) of an algebra $\mathfrak{A}$, which is obtained from $\mathfrak{A}$ by doing away with some of the operations of $\mathfrak{A}$. Roughly speaking, the main results state that if for every sufficiently small set of operations, the class of reducts of members of $\boldsymbol{K}$ to this set of operations has Property 1 or 2, then $\boldsymbol{K}$ itself also has the same property.

As an example we shall see that the class of vector spaces over the field of rational numbers can be shown, by combining our theorems with the results of [3], to have both Properties 1 and 2; however, this class does not satisfy the sufficient conditions for Properties 1 and 2 given in [3].

The questions answered by Theorems 1 and 2 were proposed by Professor Jerzy Łoś. The author is grateful to both Łoś and Tarski for their interest in this work.

2. Notation.

We shall here assume all the terminology which was introduced in [3]. Throughout this note, we shall let α be an arbitrary ordinal, $\mu = \langle \nu_\xi \rangle_{\xi<\alpha}$ be an α-termed sequence of ordinals, $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary algebras of type μ, and $\boldsymbol{K}$ be an arbitrary class of algebras of type μ.

Let s be any subset of α. We shall call the algebra $\langle A, O_\xi \rangle_{\xi \in s}$ of type $\langle \nu_\xi \rangle_{\xi \in s}$ the *s-reduct* (cf. Tarski [2]) $\mathfrak{A} \rceil s$ of the algebra $\mathfrak{A}$. Similarly, we

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shall use the notation $\boldsymbol{K} \rceil s$ for the class $\{\mathfrak{A} \rceil s \mid \mathfrak{A} \in \boldsymbol{K}\}$ of algebras of type $\langle \nu_\xi \rangle_{\xi \in s}$.

We shall call a cardinal $\mathfrak{m}$ *regular* if any family of fewer than $\mathfrak{m}$ cardinals, each of which is less than $\mathfrak{m}$, has a sum less than $\mathfrak{m}$. For example, the cardinals $\aleph_0, \aleph_1, \aleph_2, \ldots$ are all regular, but the cardinal $\aleph_\omega$ is not regular. Throughout our discussion we shall assume that $\mathfrak{m}$ is the smallest infinite regular cardinal such that $\mathfrak{c}(\nu_\xi) < \mathfrak{m}$ for all $\xi < \alpha$.

$\mathfrak{B}$ is said to be a *subalgebra* of $\mathfrak{A}$ if $B \subseteq A$, and for each $\xi < \alpha$, B is closed under the operation O_ξ, and P_ξ is the restriction of O_ξ to B. Every non-empty set $X \subseteq A$ generates a unique subalgebra of $\mathfrak{A}$. If $\nu_\xi = 0$ for some $\xi < \alpha$, then the empty set also generates a unique subalgebra of $\mathfrak{A}$. Otherwise, the empty set never generates any subalgebra of $\mathfrak{A}$. Whenever we speak of the subalgebra generated by X, we shall implicitly assume that X does generate a subalgebra.

3. General theorems.

LEMMA 1. *Suppose the element a is contained in the subalgebra of $\mathfrak{A}$ generated by the subset $X \subseteq A$. Then there is a subset $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$ and a is contained in the subalgebra of $\mathfrak{A} \rceil s$ generated by X.*

PROOF. Assume the hypothesis is satisfied. Without loss of generality, we may assume that X generates $\mathfrak{A}$. Consider the set $A_0 = \{b \mid b \in A$, and for some $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$, b is contained in the subalgebra of $\mathfrak{A} \rceil s$ generated by $X\}$. Obviously $X \subseteq A_0$. Suppose $\xi < \alpha$ and $\langle a_\varkappa \rangle_{\varkappa < \nu_\xi}$ is a sequence of elements of A_0. For each $\varkappa < \nu_\xi$, there exists $s_\varkappa \subseteq \alpha$ such that $\mathfrak{c}(s_\varkappa) < \mathfrak{m}$ and $a_\varkappa$ is contained in the subalgebra of $\mathfrak{A} \rceil s_\varkappa$ generated by X. Let

$$s = \{\xi\} \cup \bigcup_{\varkappa < \nu_\xi} s_\varkappa .$$

Since $\mathfrak{m}$ is regular and infinite, $\mathfrak{c}(s) < \mathfrak{m}$. But $O_\xi(\langle a_\varkappa \rangle_{\varkappa < \nu_\xi})$ is clearly in the subalgebra of $\mathfrak{A} \rceil s$ generated by X, and is therefore contained in A_0. Hence A_0 is closed under all operations $O_\xi, \xi < \alpha$. Since X generates $\mathfrak{A}$, we have $A_0 = A$, and therefore $a \in A_0$.

LEMMA 2. *Suppose the subsets $X \subseteq A$ and $Y \subseteq A$ each generate the algebra $\mathfrak{A}$. Moreover, suppose $\mathfrak{c}(X) < \mathfrak{m}$ and $\mathfrak{c}(Y) < \mathfrak{m}$. Then there is a subset $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$ and X, Y each generate the same subalgebra of $\mathfrak{A} \rceil s$.*

PROOF. Assume the hypothesis is satisfied. By Lemma 1, for each $x \in X$ there exists a subset $t_x \subseteq \alpha$ such that $\mathfrak{c}(t_x) < \mathfrak{m}$ and x is in the subalgebra of $\mathfrak{A} \rceil t_x$ generated by Y. Let $t = \bigcup_{x \in X} t_x$. Since $\mathfrak{m}$ is regular,

$\mathfrak{c}(t) < \mathfrak{m}$. Moreover, X is included in the subalgebra of $\mathfrak{A} \rceil t$ generated by Y. By the same argument, there exists a subset $u \subseteq \alpha$ such that $\mathfrak{c}(u) < \mathfrak{m}$ and Y is included in the subalgebra of $\mathfrak{A} \rceil u$ generated by X. Now let $s = t \cup u$. Since $\mathfrak{m}$ is infinite, $\mathfrak{c}(s) < \mathfrak{m}$. Also, each of X, Y is included in the subalgebra of $\mathfrak{A} \rceil s$ which is generated by the other. Therefore X and Y generate the same subalgebra of $\mathfrak{A} \rceil s$.

THEOREM 1. *Suppose that for each subset $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$, $\boldsymbol{K} \rceil s$ has Property* 1. *Then* $\boldsymbol{K}$ *has Property* 1.

PROOF. Assume the hypothesis is satisfied. Suppose the algebra $\mathfrak{A}$ is $\boldsymbol{K}$-freely generated by the finite set X. Let Y be a subset of A such that $\mathfrak{c}(Y) < \mathfrak{c}(X)$, and suppose Y generates $\mathfrak{A}$. We shall arrive at a contradiction.

Since $\mathfrak{m}$ is infinite, we have $\mathfrak{c}(X) < \mathfrak{m}$ and $\mathfrak{c}(Y) < \mathfrak{m}$. By Lemma 2, there is a subset $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$ and X, Y generate the same subalgebra $\mathfrak{A}_0$ of $\mathfrak{A} \rceil s$. However, it is easily seen that X $\boldsymbol{K} \rceil s$-freely generates $\mathfrak{A}_0$, because X $\boldsymbol{K}$-freely generates $\mathfrak{A}$. Therefore $\boldsymbol{K} \rceil s$ does not have Property 1, contrary to hypothesis.

THEOREM 2. *Suppose that for each subset $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$, $\boldsymbol{K} \rceil s$ has Property* 2. *Then* $\boldsymbol{K}$ *has Property* 2.

PROOF. Assume the hypothesis is satisfied. Suppose $\mathfrak{A}$ is $\boldsymbol{K}$-freely generated by the finite set X. Let Y be another subset of A such that $\mathfrak{c}(Y) = \mathfrak{c}(X)$ and Y generates $\mathfrak{A}$. We wish to show that Y generates $\mathfrak{A}$ $\boldsymbol{K}$-freely.

Let $\mathfrak{B}$ be any algebra in $\boldsymbol{K}$, and consider an arbitrary mapping $f \in B^Y$. We must show that f can be extended to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Since $\mathfrak{c}(X) = \mathfrak{c}(Y) < \mathfrak{m}$, we may apply Lemma 2 to show the existence of a subset $s \subseteq \alpha$ such that $\mathfrak{c}(s) < \mathfrak{m}$ and X, Y each generate the same subalgebra $\mathfrak{A}_0$ of $\mathfrak{A} \rceil s$. As in the proof of Theorem 1, X generates $\mathfrak{A}_0$ $\boldsymbol{K} \rceil s$-freely. Since $\boldsymbol{K} \rceil s$ has Property 2, Y also generates $\mathfrak{A}_0$ $\boldsymbol{K} \rceil s$-freely. Because the algebra $\mathfrak{B} \rceil s$ is a member of $\boldsymbol{K} \rceil s$, f can be extended to a homomorphism f' of $\mathfrak{A}_0$ into $\mathfrak{B} \rceil s$. Let $\hat{f}$ be the restriction of f' to X. Since $\hat{f} \in B^X$ and X $\boldsymbol{K}$-freely generates $\mathfrak{A}$, $\hat{f}$ can be extended to a homomorphism f'' of $\mathfrak{A}$ into $\mathfrak{B}$. The restriction of f'' to the set A_0 is obviously a homomorphism of $\mathfrak{A}_0$ into $\mathfrak{B} \rceil s$. However, f' is the only extension of $\hat{f}$ to a homomorphism of $\mathfrak{A}_0$ into $\mathfrak{B} \rceil s$, because X generates $\mathfrak{A}_0$ (cf. [3], Lemma 1). Therefore f' itself is the restriction of f'' to A_0. Since Y is included in A_0, f is the restriction of f'' to Y. Thus f can be extended to the homomorphism f'' of $\mathfrak{A}$ into $\mathfrak{B}$, as we were to show.

4. An example.

We conclude with an example to show that Theorems 1 and 2 of this paper actually do enable one to apply the results in [3] to special cases not covered by the sufficient conditions given in [3] for classes $\boldsymbol{K}$ to have Properties 1 and 2.

To facilitate our discussion, let H_1 denote the hypothesis of Theorem 1 of [3], namely that $\boldsymbol{K}$ contains a finite algebra with more than one element. Let H_2 denote the hypothesis of Theorem 2 of [3], namely that every equation which is identically satisfied in every algebra of $\boldsymbol{F}(\boldsymbol{K})$ is also identically satisfied in $\boldsymbol{K}$. Thus H_1 and H_2 are sufficient conditions for Property 1 and Property 2, respectively, according to the results of [3].

Let $\boldsymbol{E}(\boldsymbol{K})$ denote the class of all algebras which identically satisfy every equation which is identically satisfied by every member of $\boldsymbol{K}$. It is easily seen that $\boldsymbol{K} \subseteq \boldsymbol{E}(\boldsymbol{K})$, and that if $\boldsymbol{K}$ satisfies H_1 or H_2, then $\boldsymbol{E}(\boldsymbol{K})$ satisfies the same condition. Moreover, $\boldsymbol{K}$ has Property 1 or Property 2 if and only if $\boldsymbol{E}(\boldsymbol{K})$ has the same property. Therefore to demonstrate the value of the results of this paper, we must find, for $i = 1, 2$, a class $\boldsymbol{K}$ such that $\boldsymbol{E}(\boldsymbol{K})$ does not satisfy H_i, but nevertheless $\boldsymbol{K}$ can be shown using our methods to have Property i.

We see at once that Theorems 1 and 2 can give new information only when $\mathfrak{m} \leqq \mathfrak{c}(\alpha)$. The simplest case where this occurs is when $\mathfrak{m} = \aleph_0$ and $\alpha = \omega$.

Let us now consider the class $\boldsymbol{K}_0$ of all vector spaces $\mathfrak{A}$ over the field of rational numbers (cf. [2]). Each algebra $\mathfrak{A}$ of $\boldsymbol{K}_0$ thus has a binary operation $+$, and every rational number is identified with a unary operation on $\mathfrak{A}$; we may suppose the rational numbers have been ordered in a sequence of type ω, so that $\mathfrak{A}$ is of type $\langle 2, 1, 1, 1, \ldots \rangle$.

In view of our earlier discussion, the following theorem shows that Theorems 1 and 2 really do extend the applicability of the results of [3], and in particular our theorems apply to the class $\boldsymbol{K}_0$.

THEOREM 3. (i) $\boldsymbol{E}(\boldsymbol{K}_0)$ *satisfies neither* H_1 *nor* H_2.
(ii) *For each finite subset* $s \subseteq \omega$, $\boldsymbol{E}(\boldsymbol{K}_0 \rceil s)$ *satisfies both* H_1 *and* H_2.

Notice that by the results of [3], (ii) implies that $\boldsymbol{E}(\boldsymbol{K}_0 \rceil s)$ has both Properties 1 and 2 for each finite $s \subseteq \omega$. Hence by Theorems 1 and 2 with $\mathfrak{m} = \aleph_0$, the class $\boldsymbol{K}_0$ has both Properties 1 and 2. [One can, of course, easily give a direct proof that $\boldsymbol{K}_0$ has Properties 1 and 2 using the special fact that every vector space has a unique dimension. A less well-known example of a class of algebras which satisfies the conditions in Theorem 3 is the class of proper cylindric algebras, of a given infinite dimension, such that $c_0(-d_{01})=1$ (for notation see [1]).]

Proof of Theorem 3. We shall first prove (i). $\boldsymbol{K}_0$ can be defined as the class of all algebras $\mathfrak{A}$ which identically satisfy the infinite set of equations which state that $\langle A, +\rangle$ is an abelian group, that the ring of rational numbers is a ring of operators for $\langle A, +\rangle$, and that 1 is the identity operator on A. Therefore $\boldsymbol{K}_0=\boldsymbol{E}(\boldsymbol{K}_0)$, and it is sufficient to show that $\boldsymbol{K}_0$ satisfies neither H_1 nor H_2. Whenever $\mathfrak{A}$ is in $\boldsymbol{K}_0$, the group $\langle A, +\rangle$ must be completely divisible, and therefore the set A is either infinite or consists of a single element. It follows that $\boldsymbol{K}_0$ does not satisfy H_1. Since $\boldsymbol{F}(\boldsymbol{K}_0)$ consists only of one-element algebras, but $\boldsymbol{K}_0$ contains infinite algebras, $\boldsymbol{K}_0$ does not satisfy H_2.

We shall only outline the proof of (ii). Let s be a finite subset of ω. We may suppose the ordinal 0, which corresponds to the operation $+$, is in s, for if $\boldsymbol{K}_0 \rceil (s \cup \{0\})$ satisfies H_1 and H_2, so does $\boldsymbol{K}_0 \rceil s$. Let $r_1, \ldots, r_n$ be the rational numbers which correspond to the members of s other than 0.

Let p be a prime number which does not divide the denominator of any of $r_1, \ldots, r_n$. Then the additive group $\mathfrak{Z}_p$ of integers modulo p, with $r_1, \ldots, r_n$ defined as operators in the natural way, can be shown to be a member of $\boldsymbol{E}(\boldsymbol{K}_0 \rceil s)$. Thus $\boldsymbol{E}(\boldsymbol{K}_0 \rceil s)$ contains a finite algebra with more than one element, and hence satisfies H_1.

Given any equation which is not identically satisfied by some member of $\boldsymbol{E}(\boldsymbol{K}_0 \rceil s)$, one can show that for any sufficiently large prime q, the group $\mathfrak{Z}_q$ with the operators $r_1, \ldots, r_n$ defined in the natural way also does not identically satisfy this equation. It follows that $\boldsymbol{E}(\boldsymbol{K}_0 \rceil s)$ satisfies H_2.

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UNIVERSITY OF CALIFORNIA AT BERKELEY, U.S.A.