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SOME ESTIMATES FOR EIGENFUNCTION EXPANSIONS AND SPECTRAL FUNCTIONS CORRESPONDING TO ELLIPTIC DIFFERENTIAL OPERATORS

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1. Introduction.

Let S be an open connected subset of R^n ($n \geqq 2$) and let $C = C(S)$ be the space of all infinitely differentiable functions in S and $C_0 = C_0(S)$ the subspace of C whose elements have compact supports. In C we use the topology of uniform convergence of all derivatives on compact subsets of S. Further let $\mathscr{L}_k^p = \mathscr{L}_k^p(S)$ be the space of all functions in S having derivatives of order k which are locally in L^p ($p \geqq 1$) equipped with its natural topology.

Put $H = L^2(S) \oplus H_1$, where H_1 is an arbitrary Hilbert space, for example $L^2(S_1 - S)$, S_1 being an open set containing S, so that $H = L^2(S_1)$. If T is a topological space of functions in S, we shall say that e.g. a sequence $\{f_i\}$ in H converges in T, if its projection $\{Pf_i\}$ on $L^2(S)$ belongs to T and converges in T.

Let a be a partial differential operator in the variales $x_1, \ldots ,x_n$ (coordinates ind R^n) and with coefficients in C. Acting on C_0 it becomes a linear operator from H to H. We assume that it has at least one self-adjoint extension

$$A = \int \lambda \, dE(\lambda) .$$

Then, if f is in H, $\big(E(\lambda) - E(-\mu)\big)f$ converges to f in H as $\lambda, \mu \to +\infty$. In particular, we have convergence in $L^2(S)$, but with suitable conditions one can say more. Let us assume that a is elliptic and that f is in the domain of A^t with t a positive integer. Then we shall show that if t is larger than a certain number, depending on k and p, then we have convergence in $\mathscr{L}_k^p$, and we shall give an estimate for the convergence (theorem 1). The proof uses a well-known interior estimate for elliptic differential operators and an inequality of the Soboleff–Ehrling type. If we take S bounded and with a regular boundary, let $H = L^2(S)$ and consider a self-adjoint extension A which is regular in a certain sense, we also get the result globally (theorem 2).

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Let us now choose the sign of a so that its characteristic form becomes positive definite in S. Then the convergence in μ is faster than the convergence in λ. In fact, if $f \in H$, then $E(-\mu)f$ tends exponentially to zero in C as $\mu \to +\infty$ (theorem 3). In order to show this, we first prove an estimate for a local fundamental solution of $(a-\lambda)$ when λ is large and negative. Then we obtain theorem 3 using a well-known integral representation for the "almost eigenelements" of A, involving the constructed fundamental solution of $(a-\lambda)$. We also show that

$$E(\lambda)f(x) = \int_S \overline{e_\lambda(x,y)}\, f(y)\, dy \qquad (\text{for } f \in L^2(S))$$

is given by a spectral function e_λ in $C(S \times S)$, converging exponentially to zero in $C(S \times S)$ as $\lambda \to -\infty$ (theorem 4).

At last we prove an asymptotic estimate for the spectral function when $\lambda \to +\infty$ (theorem 5). The proof uses the corresponding estimate for a semi-bounded operator, which has been obtained by Bergendal [1]. It may be noted that Levitan, e.g. in [5], has earlier considered spectral functions of not necessarily semi-bounded operators corresponding to elliptic operators of the second order. He has also proved asymptotic relations for them.

2. Notations. The self-adjoint operator A.

Let R^n be the n-dimensional Cartesian space with elements $x = (x_1 \ldots, x_n)$. We put $|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. The closure of a subset N of R^n will be denoted $\bar{N}$. If $\mu = (\mu_1, \ldots, \mu_n)$, where the μ_i are non-negative integers, D^μ will denote the derivation symbol $D_1^{\mu_1} \ldots D_n^{\mu_n}$ with $D_k = (2\pi i)^{-1}\partial/\partial x_k$ $(k = 1, \ldots, n)$. We put $|\mu| = \mu_1 + \ldots + \mu_n$. For a measurable subset N of R^n we shall write

$$|g, N|_{r,p} = \left(\int_N \sum_{|\mu|=r} |D^\mu g(x)|^p \, dx \right)^{1/p} .$$

Here r is an integer $\geqq 0$, $1 \leqq p < +\infty$, and g an arbitrary function having weak derivatives of order r such that the right hand side is finite. We also define

$$|g, N|_{r,+\infty} = (\text{ess}) \sup_{x \in N} \sum_{|\mu|=r} |D^\mu g(x)| .$$

The Banach space of all (equivalence classes of) complex-valued functions g with $|g, N|_{0,p} < +\infty$ is as usual denoted $L^p(N)$, and for $p = 2$ it is a Hilbert space with the scalar product

$$(f, g) = \int_N f(x)\, \overline{g(x)}\, dx .$$

$\mathscr{L}^p(N)$ will be the space of all functions which are in $L^p(M)$ for every compact subset M of N. In $\mathscr{L}^p(N)$ we shall use the topology generated by the semi-norms $|g,M|_{0,p}$. By $C(N)$ we denote the set of those functions which have all their derivatives of any order continuous on N. $C_0(N)$ will be the set of all functions in $C(N)$ whose supports are compact subsets of N.

We shall deal with a differential operator a of order m,

$$a = a(x,D) = \sum_{|\mu|\leqq m} a_\mu(x) D^\mu ,$$

defined in an open connected subset S of R^n $(n \geqq 2)$. The coefficients $a_\mu(x)$ are assumed to be in $C(S)$. By $p_a(x,D)$ we denote the principal part of a:

$$p_a(x,D) = \sum_{|\mu|=m} a_\mu(x) D^\mu .$$

The characteristic form of a is

$$p_a(x,\xi) = \sum_{|\mu|=m} a_\mu(x) \xi^\mu ,$$

where $\xi^\mu = \xi_1^{\mu_1} \ldots \xi_n^{\mu_n}$. We then say that $a(x,D)$ is elliptic in S, if $p_a(x,\xi) \neq 0$ for all real $\xi \neq 0$ and all $x \in S$. Throughout the paper we shall assume that this is the case. Further let us suppose that $a(x,D)$ is formally self-adjoint in S, that is

$$\sum_{|\mu|\leqq m} a_\mu(x) D^\mu = \sum_{|\mu|\leqq m} D^\mu \overline{a_\mu(x)} .$$

It is then easily seen that $p_a(x,D)$ must be real and its order m even and that $p_a(x,\xi)$ must be either positive or negative definite for all $x \in S$.

As the Hilbert space in which we shall work we take the orthogonal sum H of $L^2(S)$ and H_1, where H_1 is an arbitrary Hilbert space. The scalar product in H is denoted (f,g) and the norm $\|f\| = (f,f)^{\frac{1}{2}}$. Speaking of the properties in S of elements in H we shall always refer to their orthogonal projections on $L^2(S)$. Thus, $f \in C(S)$ will mean that the projection of f on $L^2(S)$ is in $C(S)$. However, saying that $f \in C_0(S)$, we shall also require that f has zero projection on H_1.

Now, defining a on $C_0(S)$, we get a linear operator a_0 from H to H. Since a was supposed to be formally self-adjoint, it follows from Green's formula that a_0 is symmetric, that is, $(a_0\varphi,\psi) = (\varphi, a_0\psi)$ for all φ and ψ in $C_0(S)$. We are going to consider the case where a_0 has a self-adjoint extension $A = A^*$, the star denoting the adjoint in H. If f is in the domain $D(A)$ of A, we have in particular $(f, a_0\varphi) = (Af, \varphi)$ for all φ in $C_0(S)$. This implies that Af agrees with af in S, where a is taken in the weak (distributional) sense. Similarly it follows that $A^t f$ agrees with

$a^t f$ in S, if t is a positive integer and $f \in D(A^t)$. Since A is self-adjoint, it has a spectral resolution $E(\lambda)$ which is a projection-valued function defined for $-\infty \leqq \lambda \leqq +\infty$, satisfying $A = \int_{-\infty}^{+\infty} \lambda dE(\lambda)$ (Nagy [6, p. 50]).

3. The convergence of the eigenfunction expansion.

LEMMA 1. (see e.g. Nirenberg [8, p. 519]) *Let Ω be an open subset of S, M a compact subset of Ω and t an arbitrary positive integer. Then*

$$|u, M|_{mt,2} \leqq K(|a^t u, \Omega|_{0,2} + |u, \Omega|_{0,2})$$

for all functions u for which the right hand side is defined with a^t taken in the weak sense. K is a number independent of u but may depend on Ω, M, t and the coefficients of a.

LEMMA 2. *Let F be a sphere in R^n, let $2 \leqq p \leqq +\infty$, let s and r be non-negative integers and put $l = (np + 2ps - 2n)/2p$ (in particular, $l = (n+2s)/2$ when $p = +\infty$). Then for all functions v for which the right hand side below is finite we have the inequality*

$$|v, F|_{s,p} \leqq C(h^{l-r}|v, F|_{r,2} + h^l |v, F|_{0,2})$$

provided $r \geqq l$ or $r > l$ according as $p < +\infty$ or $p = +\infty$. The factor h is an arbitrary positive number, and C is independent of v and h when h is large but may depend on F, r, s, and p.

PROOF. By inequalities of Soboleff [11] and Ehrling [2, p. 270–273] there is a constant K such that for all v for which the right hand side is defined we have

$$|v, F|_{s,p} \leqq K(|v, F|_{r,2} + |v, F|_{0,2}) , \tag{1}$$

where r, s and p satisfy the condition of the lemma. Let us for $h \geqq 1$ consider the spheres hF with radius hR, where R is the radius of F. We cover hF with domains $F_1^{(h)}, \ldots, F_{n_h}^{(h)}$, for which the inequality (1) (with F replaced by the domain in question) holds with the same constant, independent of h. This can be done in such a way that

$$\bigcup_{i=1}^{n_h} F_i^{(h)} = hF$$

and that no point of hF is covered by more than L of the domains $F_i^{(h)}$, where L is an integer, independent of h. Then

$$|v, hF|_{s,p}^2 \leqq \sum_{i=1}^{n_h} |v, F_i^{(h)}|_{s,p}^2 .$$

This follows from the elementary inequality

$$\left(\sum a_i^p\right)^{1/p} \leqq \left(\sum a_i^2\right)^{1/2} \quad (p \geqq 2,\ a_i \geqq 0)\,.$$

Thus by (1) (for the domains $F_i^{(h)}$)

$$\begin{aligned}(2)\qquad |v,hF|_{s,p} &\leqq 2K^2 \sum \left(|v,F_i^{(h)}|_{r,2}{}^2 + |v,F_i^{(h)}|_{0,2}{}^2\right.\\ &\leqq 2K^2 L\left(|v,hF|_{r,2}{}^2 + |v,hF|_{0,2}{}^2\right)\,,\end{aligned}$$

where $2K^2L$ is evidently independent of h. If we suppose that the origin is the centre of all our spheres and take a function v defined in F, then the function $v_h(x) = v(x/h)$ is defined in hF, and

$$\begin{aligned}|v_h, hF|_{s,p} &= h^{n/p-s}|v,F|_{s,p}\,,\\ |v_h, hF|_{r,2} &= h^{n/2-r}|v,F|_{r,2}\,,\\ |v_h, hF|_{0,2} &= h^{n/2}|v,F|_{0,2}\,.\end{aligned}$$

(For $p = +\infty$ we put 0 for n/p.) Introducing this in (2), we find

$$|v,F|_{s,p} \leqq K'\left(h^{l-r}|v,F|_{r,2} + h^l|v,F|_{0,2}\right)\,,$$

where K' is independent of v and h, which proves the lemma.

Now let us return to the self-adjoint extension A of a_0, with the spectral resolution $E(\lambda)$. We have the following theorem.

THEOREM 1. *Suppose that* $2 \leqq p \leqq +\infty$ *and let* t *and* s *be non-negative integers such that* $mt \geqq l \equiv (np+2ps-2n)/2p$, *if* $p < +\infty$, *and* $mt > l \equiv (n+2s)/2$, *if* $p = +\infty$. *Then for any function* f *in* $D(A^t)$, $\big(E(\lambda)-E(-\mu)\big)f$ *is in* $C(S)$ *for* λ *and* μ *finite, and if* $|\alpha| = s$ *and* $\lambda,\mu \to +\infty$, $D^\alpha\big(E(\lambda)-E(-u)\big)f$ *converges to* $D^\alpha f$ *in* $\mathscr{L}^p(S)$. *For the convergence we have the following estimates*

$$(3)\qquad |E(-\mu)f,K|_{s,p} = o(1)\,\mu^{l/m-t}\|A^tf\|\,,$$

$$(4)\qquad |f-E(\lambda)f,K|_{s,p} = o(1)\,\lambda^{l/m-t}\|A^tf\|\,,$$

where K *is an arbitrary compact subset of* S *and where the functions* $o(1)$ *can be majorized by a constant independent of* f *(for* λ *and* μ *large).*

PROOF. If λ and μ are finite, then $\big(E(\lambda)-E(-\mu)\big)f \in D(A^\infty) \equiv \bigcap_{i=1}^\infty D(A^i)$. From the fact that a^i is elliptic and of order mi and has its coefficients in $C(S)$ it follows by the lemmas 1 and 2, taking $p = +\infty$ and s as large as we like, that $\big(E(\lambda)-E(-\mu)\big)f \in C(S)$. Let us consider $E(-\mu)f$. Since $f \in D(A^t)$ we have

$$\|A^tE(-\mu)f\| = o(1)\|A^tf\|$$

as $\mu \to +\infty$. But

$$\|E(-\mu)f\| \leqq \mu^{-t}\|A^tE(-\mu)f\| \qquad (\mu > 0)\,.$$

Thus by lemma 1

$$|E(-\mu)f,F|_{0,2} = o(1)\mu^{-t}\|A^tf\|\,,$$
$$|E(-\mu)f,F|_{mt,2} = o(1)\|A^tf\|\,,$$

if F is a sphere whose closure is contained in S. Thus by lemma 2

$$|E(-\mu)f,F|_{s,p} \leqq g(\mu)\,(h^{l-mt}+h^l\mu^{-t})\,\|A^tf\|\,,$$

where $g(\mu) \to 0$ as $\mu \to +\infty$. Choosing $h=\mu^{1/m}$ we find

$$|E(-\mu)f,F|_{s,p} = o(1)\,\mu^{l/m-t}\|A^tf\|\,.$$

By the Heine–Borel theorem this estimate follows also for an arbitrary compact subset of S. All the functions $o(1)$ entering in the proof can evidently be majorized by a constant independent of f, and so this holds true also for the $o(1)$ in (3) and (4). For $\big(f-E(\lambda)f\big)$ the proof is exactly the same, and since

$$f - \big(E(\lambda)-E(-\mu)\big)f = \big(f-E(\lambda)f\big) + E(-\mu)f\,,$$

it follows that $D^\alpha\big(E(\lambda)-E(-\mu)\big)f$ converges to $D^\alpha f$ in $\mathscr{L}^p$ for $|\alpha|=s$. This completes the proof of the theorem.

Now let us further assume that S is bounded and has an infinitely differentiable boundary and that the coefficients of $a(x,D)$ are in $C(\bar{S})$. We take $H_1=\{0\}$, so that $H=L^2(S)$. Then let us call the selfadjoint extension A of a_0 regular, if there are numbers C_t independent of f such that

$$|f,S|_{mt,2} \leqq C_t(\|A^tf\|+\|f\|) \qquad \big(f \in D(A^t),\ t=1,2,\ldots\big)\,.$$

For instance, if we restrict the maximal operator $a_0{}^*$ to those functions which have zero Dirichlet data at the boundary of S, we get such a regular self-adjoint extension of a_0. More generally, the self-adjoint extension A of a_0 is regular, if we get it by restriction of the maximal operator to those functions u, for which $b_iu=0$ $(i=1,\ldots,\frac{1}{2}m)$, where $\{b_i\}$ is a "normal" set of differential operators with infinitely differentiable coefficients on the boundary of S such that $\{b_i\}$ "covers" $a(x,D)$ (Schechter [9, p. 564]). We have

THEOREM 2. *Let A be a regular self-adjoint extension of a_0. Then for A theorem* 1 *holds globally, i.e. we have convergence in $L^p(S)$ and* (3) *and* (4) *hold with K replaced by S.*

PROOF. Evidently S can be covered—so that no point of its complement is covered—with a finite number of regions for which the inequality of lemma 2 holds. For instance, we can take domains which can be mapped one to one on the sphere F by sufficiently differentiable mappings. But then lemma 2 holds also for S and we can make the proof as for theorem 1.

4. An estimate for a certain fundamental solution.

Now we are going to prove an estimate for a local fundamental solution of $(a(x,D)-\lambda)$ where λ is large and negative and where the sign of $a(x,D)$ is chosen so that its characteristic form is positive definite. The estimate is to be used in the next section. We start by quoting the following lemma.

Lemma 3 (Gårding [3, p. 241]). *Let $p(\xi)=p(\xi_1,\ldots,\xi_n)$ be a polynomial of degree m whose coefficients are majorized by a number c_1, and suppose that for some positive number c_2*

$$|p(\xi)| \geqq c_2(1+|\xi|^m) .$$

Then the inverse Fourier transform of $1/p(\xi)$ (in the sense of Schwartz [10, Chapter VII]) is an infinitely differentiable function $P(x)$ in the region $|x| \neq 0$ satisfying

$$|D^\alpha P(x)| \leqq C e_{|\alpha|}(x)(1+|x|^N)^{-1} ,$$

where $e_{|\alpha|}(x)=1$, if $m-|\alpha|-n>0$, and $e_{|\alpha|}(x)=|x|^{m-|\alpha|-n-\varepsilon}$, if $m-|\alpha|-n\leqq 0$. Here $N\geqq 0$ and $0<\varepsilon<1$ are arbitrary, and the number C depends on c_1, c_2, $|\alpha|$, N and ε but is otherwise independent of the polynomial p.

If $b(D)$ is a differential operator with constant coefficients such that $b(\xi)\neq 0$ for ξ real, then, according to Schwartz [10, II, p. 142], it has a unique temperate fundamental solution with pole zero, namely the inverse Fourier transform of $1/b(\xi)$. If $b(D)$ is elliptic, we see from lemma 3 that this fundamental solution is infinitely differentiable outside the pole. The following lemma gives an estimate for a fundamental solution in the case of constant coefficients.

Lemma 4. *Let $a(D)$ be an elliptic operator of order m with constant coefficients such that its characteristic form is positive definite. Let $g_\lambda(x)$ be the temperate fundamental solution (with pole zero) of $(a(D)-\lambda)$ for λ large and negative. Then there are three positive numbers b, C and $-\lambda_0$ such that*

$$|D^\alpha g_\lambda(x)| \leqq C|\lambda|^{-\varepsilon/m} e_{|\alpha|}(x)\exp(-b|\lambda|^{1/m}|x|) \qquad (\lambda \leqq \lambda_0) .$$

Here ε is the number entering in $e_{|\alpha|}(x)$, if $e_{|\alpha|}(x)\neq 1$; otherwise ε is an arbitrary positive number $\leqq 1$. For α, m and ε fixed, C, b and λ_0 can be chosen as continuous functions of the coefficients of $a(D)$ $(a(D)$ all the time being an operator permitted in the lemma$)$.

PROOF. Let $h_\lambda(x)$ be the temperate fundamental solution with pole zero of

$$H_\lambda(D) = \big(|\lambda|^{-1}a(|\lambda|^{1/m}D)+1\big);$$

hence $h_\lambda(x)$ is the inverse Fourier transform of $1/H_\lambda(\xi)$. Then

$$g_\lambda(x) = |\lambda|^{(n-m)/m}\, h_\lambda(|\lambda|^{1/m}x) \tag{5}$$

which is easily proved by a homothety transformation. Since the characteristic form of $a(D)$ was supposed to be positive definite, the polynomial $H_\lambda(\xi_1+z,\xi_2,\ldots,\xi_n)$ will for λ sufficiently large have no zero for any real ξ or complex z with $|\mathrm{Im}(z)|<c$, hwere c is some positive number independent of ξ and λ. For $|\mathrm{Im}(z)|<c$, considering $1/H_\lambda(\xi_1+z,\xi_2,\ldots,\xi_n)$ as a function of z, whose range consists of temperate distributions on R^n, we easily find that it is an analytic function of z (this means that for every φ in the Schwartz class of rapidly decreasing, infinitely differentiable functions the value of the distribution at φ is an ordinary analytic function of z). Hence for $|\mathrm{Im}(z)|<c$ the inverse Fourier transform of $1/H_\lambda(\xi_1+z,\xi_2,\ldots,\xi_n)$ is an analytic, distribution-valued function of z. If z is real, it is equal to $\exp(-2\pi i z x_1)\, h_\lambda(x)$. Considering $\exp(-2\pi i z x_1)\, h_\lambda(x)$ for general complex z as a (not necessarily temperate) distribution-valued function of z, we find that it is an entire analytic function. And since it agrees with the inverse Fourier transform of $1/H_\lambda(\xi_1+z,\xi_2,\ldots,\xi_n)$ for z real, the two functions agree also for all z with $|\mathrm{Im}(z)|<c$, by the uniqueness theorem for analytic functions. If b' is some fixed number, $0<b'<c$, and λ is large and negative, then $H_\lambda(\xi_1+ib',\xi_2,\ldots,\xi_n)$ satisfies the requirements on the polynomial of lemma 3 so that the constants c_1 and c_2 there can be taken independent of λ. Thus by lemma 3

$$\big|D^\alpha\big(\exp(2\pi b' x_1)\, h_\lambda(x)\big)\big| \leqq C' e_{|\alpha|}(x)\ ,$$

where C' is independent of x and λ and α an arbitrary derivation index. From this it follows that

$$|D^\alpha h_\lambda(x)| \leqq K \exp(-2\pi b' x_1) \max\big(e_{|\alpha|}(x), 1\big)\ ,$$

where K is independent of x and λ. Such an estimate also holds with $-c<b'<0$ and with x_1 replaced by any of the other variables. Hence we can conclude that

$$|D^\alpha h_\lambda(x)| \leqq C \exp(-b|x|)\, e_{|\alpha|}(x) \qquad (\lambda \leqq \text{some } \lambda_0)\ ,$$

where C and b are positive and independent of x and λ. Then it only remains to use (5), since it follows from the proof that C, b and λ_0 can be taken as continuous functions of the coefficients of the operator $a(D)$.

Now, with the help of lemma 4, we shall prove an estimate for a certain local fundamental solution of $(a(x,D)-\lambda)$, where $a(x,D)$ is our original differential operator. Its sign is chosen so that $p_a(x,\xi)$ is positive definite.

LEMMA 5. *For an arbitrary point $x_0 \in S$ and for large negative λ there is a neighbourhood ω of x_0 and a fundamental solution $g_\lambda(x,y)$ of $\big(a(x,D)-\lambda\big)$ (we suppose that it is a fundamental solution with respect to y, with pole x) defined and infinitely differentiable in $\omega \times \omega$ for $x \neq y$ and such that*

$$(6) \qquad |g_\lambda(x,y)| \leqq K_1 \, |x-y|^{\delta-n} ,$$

$$(7) \qquad |D_y{}^\alpha g_\lambda(x,y)| \leqq K_2 \exp(-b|\lambda|^{1/m}) \qquad (|x-y| \geqq \tau) .$$

Here K_1, δ, K_2 and b are positive and independent of x, y and λ, and α is an arbitrary derivation index, while K_2 and b may depend on τ which is an arbitrary positive number.

PROOF. We shall follow Gårding [3, p. 244–245]. As in [3] we find by the parametrix method the fundamental solution, infinitely differentiable in $\omega \times \omega$ for $x \neq y$, in the form

$$(8) \qquad g_\lambda(x,y) = g_\lambda'(x,y) + \int_\omega u_\lambda(x,z) \, g_\lambda'(z,y) \, dz ,$$

where $g_\lambda'(x,y)$ is the temperate fundamental solution with pole x of the operator $\big(a(x,D_y)-\lambda\big)$ and ω is, for example, a sphere with centre x_0, $\bar{\omega} \subset S$. Further $u_\lambda(x,z)$ is a solution of the integral equation

$$(9) \qquad u_\lambda(x,z) - \int_\omega u_\lambda(x,y) \, \beta_\lambda(y,z) \, dy = \beta_\lambda(x,z) ,$$

where $\beta_\lambda(x,z) = \big(a(x,D_z) - a(z,D_z)\big) g_\lambda'(x,z)$. By lemma 4 we get for $\beta_\lambda(x,z)$ the estimate

$$|\beta_\lambda(x,z)| \leqq C \, |\lambda|^{-\varepsilon/m} |x-z|^{1-\varepsilon-n} \exp(-b|\lambda|^{1/m}|x-z|) \qquad (\lambda \leqq \lambda_0) ,$$

where $b>0$ and where b, C and λ_0 can be chosen independent of x, z and λ for x and z in ω. Then we can solve the integral equation (9) by means of its Neumann series

$$\beta_\lambda(x,z) + \int_\omega \beta_\lambda(x,y) \, \beta_\lambda(y,z) \, dy + \dots$$

since this can be majorized by

$$\Big(C \, |\lambda|^{-\varepsilon/m} |x-z|^{1-\varepsilon-n} + (C \, |\lambda|^{-\varepsilon/m})^2 \int_\omega |x-y|^{1-\varepsilon-n} |y-z|^{1-\varepsilon-n} \, dy + \dots \Big) \cdot$$
$$\cdot \exp(-b|\lambda|^{1/m}|x-z|) ,$$

where the trivial inequality

$$\exp(-b|\lambda|^{1/m}|x-z|) \geqq \exp(-b|\lambda|^{1/m}|x-y|)\exp(-b|\lambda|^{1/m}|y-z|)$$

has been used. Thus for λ sufficiently large the Neumann series will converge to a solution $u_\lambda(x,z)$ of the integral equation (9), and we have the estimate

$$(10) \qquad |u_\lambda(x,z)| \leqq C'|\lambda|^{-\varepsilon/m}|x-z|^{1-\varepsilon-n}\exp(-b|\lambda|^{1/m}|x-z|) \qquad (\lambda \leqq \lambda_1),$$

where C' is independent of x, z and λ. From (8), (10) and lemma 4 we now get

$$(11) \qquad |g_\lambda(x,y)| \leqq K|x-y|^{\delta-n}\exp(-b|\lambda|^{1/m}|x-y|),$$

where K and δ are independent of x, y and λ and where $\delta>0$. Thus we have proved (6). For $y \neq x$ we have

$$\big(a(y,D_y)\big)^k g_\lambda(x,y) = \lambda^k g_\lambda(x,y) \qquad (k = 1,2,\ldots),$$

and so by (11) and the lemmas 1 and 2 we also get (7) (in a sphere strictly contained in ω) which completes the proof.

5. The convergence of the negative part of the expansion.

Let us choose the sign of our operator $a(x,D)$ so that $p_a(x,\xi)$ becomes positive definite. As before we consider an arbitrary self-adjoint extension A of a_0 in H, again H_1 being arbitrary. Then it can be shown that the eigenelements or "almost eigenelements" corresponding to large negative λ are small in the interior of S. As a consequence of that we get for the negative part of the expansion a faster convergence than that asserted by theorem 1, where we did not use this fact. We have the following theorem.

Theorem 3. *For an arbitrary f in H and $\lambda \neq +\infty$ we have $E(\lambda)f \in C(S)$, and corresponding to any derivative D^α and every compact subset M of S there are positive numbers b and C_α independent of f and λ such that*

$$\sup_{x\in M}|D^\alpha E(\lambda)f(x)| \leqq C_\alpha \exp(-b|\lambda|^{1/m})\,\|f\| \qquad (\lambda \leqq 0).$$

Proof. Let us introduce the spaces $H(\lambda_1,\lambda_2)$ for $\lambda_1<\lambda_2$ defined by

$$H(\lambda_1,\lambda_2) = \big(E(\lambda_2)-E(\lambda_1)\big)H.$$

Hence, if the intervals (λ_1,λ_2) and (λ_1',λ_2') have at most one point in common, then $H(\lambda_1,\lambda_2)$ is orthogonal to $H(\lambda_1',\lambda_2')$. From lemma 1 and lemma 2 it follows that all the elements in a space $H(\lambda_1,\lambda_2)$ with λ_1 and λ_2 finite are in $C(S)$ and that

$$\sup_{x \in M} |D^\alpha f(x)| \leqq c\|f\| \qquad \left(f \in H(\lambda_1,\lambda_2)\right), \tag{12}$$

where α is an arbitrary derivation index and c is independent of f but may depend on α, λ_1, λ_2, M and the coefficients of $a(x,D)$. Taking an arbitrary point x_0 in S there is a neighbourhood $\omega \subset S$ of x_0 with a fundamental solution $g_\lambda(x,y)$ of $\left(a(x,D)-\lambda\right)$ satisfying the estimates of lemma 5. Let ψ be in $C_0(\omega)$ and vanish outside a sphere $F_1 \subset \omega$ with centre x_0 and be equal to 1 in another sphere F_2, concentric with F_1. Then, if u is e.g. m times continuously differentiable, we have the representation formula (Gårding [4]; Nilsson [7, p. 4])

$$u(x) = \int \overline{B_\lambda(x,y)}\, u(y)\, dy + \int \overline{\psi(y)\, g_\lambda(x,y)}\, \left(a(y,D_y)-\lambda\right) u(y)\, dy \qquad (x \in F_2), \tag{13}$$

where $B_\lambda(x,y) = \left(a(y,D_y)-\lambda\right)\left(\left(1-\psi(y)\right)g_\lambda(x,y)\right)$ vanishes when y is outside $(F_1 - F_2)$. According to the estimate (7) of lemma 5 we have

$$|B_\lambda(x,y)| \leqq K \exp(-b'|\lambda|^{1/m}) \qquad (x \in F_3,\ \lambda \leqq \lambda_0), \tag{14}$$

where K and $b' > 0$ are independent of x, y and λ and where F_3 is a sphere with centre x_0, strictly contained in F_2. By (13), (14) and the estimate (6) of lemma 5 we get for a sufficiently regular function u

$$|u,F_3|_{0,2} \leqq K' \left(\exp(-b'|\lambda|^{1/m})\, |u,F_1-F_2|_{0,2} + |(a-\lambda)u,F_1|_{0,2}\right), \tag{15}$$

where K' is a constant. For an f in $H(\lambda-\varepsilon,\lambda)$ $(\varepsilon > 0)$ we have

$$|(a-\lambda)f,F_1|_{0,2} \leqq \|(A-\lambda)f\| \leqq \varepsilon\|f\|.$$

Hence by (15) we get for all f in $H(\lambda-\varepsilon,\lambda)$

$$|f,F_3|_{0,2} \leqq K' \left(\exp(-b'|\lambda|^{1/m}) + \varepsilon\right)\|f\|. \tag{16}$$

For an arbitrary f in $H(\lambda-k\varepsilon,\lambda)$, where k is a positive integer, we have $f = f_1 + \ldots + f_k$ where $f_i \in H\left(\lambda - i\varepsilon,\ \lambda-(i-1)\varepsilon\right)$ for $i=1,\ldots,k$. For such an element f we get by (16) and the Cauchy–Schwarz inequality

$$\begin{aligned} |f,F_3|_{0,2} \leqq \sum_{i=1}^{k} |f_i,F_3|_{0,2} &\leqq K'\left(\exp(-b'|\lambda|^{1/m})+\varepsilon\right)\sum_{i=1}^{k}\|f_i\| \\ &\leqq K'\left(\exp(-b'|\lambda|^{1/m})+\varepsilon\right)k^{\frac{1}{2}}\|f\|. \end{aligned}$$

Let us choose $\varepsilon = \exp(-b'|\lambda|^{1/m})$ and k as the largest integer less than $2\exp(b'|\lambda|^{1/m})$. Then $k\varepsilon$ will tend to 2 when λ tends to $-\infty$. Hence for $f \in H(\lambda-1,\lambda)$ and λ sufficiently large we have

$$|f,F_3|_{0,2} \leqq K'' \exp(-\tfrac{1}{2}b'|\lambda|^{1/m})\, \|f\|, \tag{17}$$

where K'' is independent of f and λ. But with f also $Af, A^2f, \ldots$ are in

$H(\lambda-1,\lambda)$, and for such an f we have $\|A^k f\| \leqq (|\lambda|+1)^k \|f\|$ $(k=1,2,\ldots)$. Thus by (17) there exist positive numbers C_k, b_1 and $-\lambda_0'$ independent of f and λ such that for f in $H(\lambda-1,\lambda)$

$$|A^k f, F_3)|_{0,2} \leqq C_k \exp(-b_1|\lambda|^{1/m}) \|f\| \qquad (\lambda \leqq \lambda_0') .$$

From this we get, using lemma 1 and lemma 2, that for f in $H(\lambda-1,\lambda)$

$$\sup_{x \in F_4} |D^\alpha f(x)| \leqq C_\alpha' \exp(-b_1|\lambda|^{1/m}) \|f\| , \tag{18}$$

where F_4 is a sphere with centre x_0, strictly contained in F_3, α an arbitrary derivation index, and C_α' is independent of f and λ. For a general f in H we may write $E(\lambda)f = f_1 + f_2 + \ldots$, where $\lambda \neq +\infty$ and

$$f_i \in H(\lambda-i, \lambda-i+1) \qquad (i = 1,2,\ldots) ,$$

and from (18) it follows that $E(\lambda)f$ is infinitely differentiable in F_4 and that

$$\sup_{x \in F_4} |D^\alpha E(\lambda) f(x)| \leqq C_\alpha' \left(\sum_{i=1}^{\infty} \exp(-b_1|\lambda-i+1|^{1/m}) \right) \|f\| .$$

Thus we can easily conclude that

$$\sup_{x \in F_4} |D^\alpha E(\lambda) f(x)| \leqq K_\alpha \exp(-b_2|\lambda|^{1/m}) \|f\| \qquad (f \in H) ,$$

where K and b_2 are positive constants. By the Heine–Borel theorem and (12) we then get the theorem.

Remark 1. It follows from our proof that the constants C_α and b can be taken independent of the particular choice of self-adjoint extension A and of the space H_1.

Remark 2. If $S = R^n$ and p_a has, e.g., constant coefficients, then we can take b as large as we want. In fact, it can be seen from the proofs of the lemmas 4 and 5 that then we can take the neighbourhood ω of lemma 5 and the number b in (7) as large as we like (for τ sufficiently large). Hence, taking the radius of F_2 in the proof of theorem 3 sufficiently large, we can make b' in (14) as large as we want, and the statement easily follows.

6. The spectral function.

Consider the differential operator $a(x,D)$ of theorem 3 with the self-adjoint extension A of a_0 and the corresponding resolution of the identity $E(\lambda)$. For an arbitrary point $x \in S$ let us define the mapping

$T_\lambda^{(\alpha)}(x)$ from H to the complex numbers by $T_\lambda^{(\alpha)}(x)f = D^\alpha E(\lambda)f(x)$. By theorem 3 we have

$$|T_\lambda^{(\alpha)}(x)f| \leqq C_\alpha(x)\exp\left(-b(x)|\lambda|^{1/m}\right)\|f\| \qquad (f \in H,\ \lambda \leqq 0)\,,$$

where $C_\alpha(x)$ and $b(x)$ denote the positive numbers which by theorem 3 correspond to the compact subset $\{x\}$ of S. Thus there is an element $e_\lambda^{(\alpha)}(x,\cdot)$ in H such that

$$T_\lambda^{(\alpha)}(x)f = \left(f, e_\lambda^{(\alpha)}(x,\cdot)\right) \qquad (x \in S,\ f \in H)\,, \tag{19}$$

where

$$\|e_\lambda^{(\alpha)}(x,\cdot)\| \leqq C_\alpha(x)\exp\left(-b(x)|\lambda|^{1/m}\right). \tag{20}$$

For all φ in $H(\lambda, +\infty)$ we have $\left(\varphi, e_\lambda^{(\alpha)}(x,\cdot)\right) = 0$, and so

$$e_\lambda^{(\alpha)}(x,\cdot) \in H(-\infty,\lambda)\,.$$

Hence by theorem 3

$$e_\lambda^{(\alpha)}(x,\cdot) \in C(S)$$

and

$$\begin{aligned} |D_y^\beta e_\lambda^{(\alpha)}(x,y)| &\leqq C_\beta(y)\exp\left(-b(y)|\lambda|^{1/m}\right)\|e_\lambda^{(\alpha)}(x,\cdot)\| \\ &\leqq C_\alpha(x)C_\beta(y)\exp\left(-\left(b(x)+b(y)\right)|\lambda|^{1/m}\right), \end{aligned} \tag{21}$$

where (20) has also been used. Let h be a function in $C_0(R^n)$ with $\int h(x)\,dx = 1$. Then

$$e_\lambda^{(\alpha)}(x,y) = \lim_{\delta\to+0} \delta^{-n}\int e_\lambda^{(\alpha)}(x,z)\,h\left((z-y)/\delta\right)dz \qquad (x,y \in S)\,.$$

Here all the functions under the limit sign are easily seen to be continuous in $S\times S$, and from (21) it follows that the convergence is uniform on compact subsets of $S\times S$. Hence $e_\lambda^{(\alpha)}(x,y)$ is continuous in $S\times S$. For u and v in H we always have $\left(E(\lambda)u,v\right) = \left(u,E(\lambda)v\right)$. Here we let u and v be in $C_0(S)$. Using (19) (with $|\alpha| = 0$) we get

$$\int u(y)\overline{e_\lambda(x,y)\,v(x)}\,dx\,dy = \int u(y)\,e_\lambda(y,x)\,\overline{v(x)}\,dx\,dy\,,$$

where e_λ is $e_\lambda^{(\alpha)}$ for $|\alpha| = 0$. Since u and v are arbitrary in $C_0(S)$, we get $e_\lambda(x,y) = \overline{e_\lambda(y,x)}$ for $x,y \in S$, and it follows that $e_\lambda(x,y) \in C(S\times S)$. We also find $e_\lambda^{(\alpha)}(x,y) = (-1)^{|\alpha|}D_x^\alpha e_\lambda(x,y)$, and so (21) gives an estimate for $D_x^\alpha D_y^\beta e_\lambda(x,y)$. Since it follows from (12) that we have no difficulty in defining $e_\lambda(x,\cdot)$ also for $\lambda > 0$ we have the following theorem.

THEOREM 4. *For every $x \in S$ and every real λ there is an element $e_\lambda(x,\cdot)$ of H such that*

$$E(\lambda)f(x) = \left(f, e_\lambda(x,\cdot)\right) \qquad (f \in H)\,.$$

Here $e_\lambda(x,y)$ *is infinitely differentiable in* $S\times S$, $e_\lambda(x,y)=\overline{e_\lambda(y,x)}$ *and*

$$D_x^{\alpha}D_y^{\beta}e_\lambda(x,y) \leqq C_\alpha(x)C_\beta(y)\exp\bigl(-(b(x)+b(y))|\lambda|^{1/m}\bigr) \qquad (\lambda \leqq 0)\,,$$

where $C_\gamma(z)$ *and* $b(z)$ *are the positive numbers which by theorem* 3 *correspond to the compact subset* $\{z\}$ *of* S *and the derivation index* γ.

Now let us consider the self-adjoint operator A^2. It is an extension of a^2 defined on $C_0(S)$. Moreover, it is bounded from below. So we have by Bergendal [1, theorem 3.2.1, p. 43] (he considers only the case $H=L^2(S)$, but his proof works as well in the general case) the following estimate for the spectral function $E_\lambda(x,y)$ of A^2, when $\lambda \to +\infty$,

$$(22) \qquad D_x^{\alpha}D_y^{\beta}E_\lambda(x,y) = D_x^{\alpha}D_y^{\beta}E_{x,\lambda}(x,y) + O(1)\lambda^{(n+|\alpha|+|\beta|)/2m}(\log\lambda)^{-1}\,,$$

which is uniform on compact subsets of $S\times S$. Here $E_{z,\lambda}(x,y)$ is the spectral function of the unique self-adjoint extension in $L^2(R^n)$ of the operator with constant coefficients $\bigl(p_a(z,D_x)\bigr)^2$, defined on $C_0(R^n)$, and $D_x^{\alpha}D_y^{\beta}E_{x,\lambda}(x,y)$ is short for $[D_x^{\alpha}D_y^{\beta}E_{z,\lambda}(x,y)]_{z=x}$. By a Fourier transformation we can easily calculate $E_{z,\lambda}(x,y)$ and we find

$$E_{z,\lambda}(x,y) = \int\limits_{p_a(z,\xi)^2\leqq\lambda} \exp\bigl(-2\pi i(x-y)\xi\bigr)\,d\xi\,.$$

Evidently

$$E_{z,\lambda}(x,x) = c(z)\lambda^{n/2m}\,,$$

where $c(z)$ is independent of λ, and so by (22), $E_{x,\lambda}(x,x)$ is an asymptotic estimate for $E_\lambda(x,x)$ as $\lambda \to +\infty$. Now, with the spectral functions corresponding to a we have the following relations ($\lambda > 0$)

$$e_\lambda(x,y) = e_{-\lambda}(x,y)+E_{\lambda^2}(x,y)\,,$$
$$e_{z,\lambda}(x,y) = E_{z,\lambda^2}(x,y)$$

provided that $E(\lambda)$ is defined conveniently when λ is an eigenvalue, which has no influence on the asymptotic formulas. Thus

$$e_\lambda(x,y)-e_{z,\lambda}(x,y) = e_{-\lambda}(x,y) + \bigl(E_{\lambda^2}(x,y)-E_{z,\lambda^2}(x,y)\bigr)$$

and so by theorem 4 and (22)

$$D_x^{\alpha}D_y^{\beta}\bigl(e_\lambda(x,y)-e_{x,\lambda}(x,y)\bigr) = O(1)\lambda^{(n+|\alpha|+|\beta|)/m}(\log\lambda)^{-1}\,,$$

where the estimate is uniform on compact subsets of $S\times S$. Hence we have the following theorem.

THEOREM 5. *For the spectral function* $e_\lambda(x,y)$ *defined in theorem* 4 *we have the estimate* ($x,y \in S$ *and* $\lambda \to +\infty$)

$$D_x^{\alpha} D_y^{\beta} e_{\lambda}(x,y) = D_x^{\alpha} D_y^{\beta} e_{x,\lambda}(x,y) + O(1)\lambda^{(n+|\alpha|+|\beta|)/m}(\log \lambda)^{-1},$$

where $e_{z,\lambda}(x,y)$ is the spectral function of the unique self-adjoint extension in $L^2(R^n)$ of the differential operator with constant coefficients $p_a(z,D_x)$, defined on $C_0(R^n)$. The estimate is uniform on compact subsets of $S \times S$.

REMARK. If the operator a has constant coefficients in S, then

$$D_x^{\alpha} D_y^{\beta} e_{\lambda}(x,y) = D_x^{\alpha} D_y^{\beta} e_{o,\lambda}(x,y) + O(1)\lambda^{(n+|\alpha|+|\beta|-1)/m},$$

where $e_{o,\lambda}(x,y)$ is the spectral function of the unique self-adjoint extension in $L^2(R^n)$ of a, defined on $C_0(R^n)$. In fact, by Bergendal [1, theorem 3.1.1, p. 37] this is true in the semi-bounded case, and so we get the general case, arguing as above.

REFERENCES

1. G. Bergendal, *Convergence and summability of eigenfunction expansions connected with elliptic differential operators*, Thesis, Lund, 1959.
2. G. Ehrling, *On a type of eigenvalue problem for certain elliptic differential operators*, Math. Scand. 2 (1954), 267–285.
3. L. Gårding, *On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators*, Math. Scand. 1 (1953), 237–255.
4. L. Gårding, *On a lemma by H. Weyl*, Kungl. Fysiogr. Sällsk. i Lund Förh. 20 (1950), 250–253.
5. B. M. Levitan, *On the eigenfunction expansion at the equation* $\Delta u + (\lambda - q(x_1, \ldots, x_n))u = 0$, Izvestia Akad. Nauk SSSR 20 (1956), 437–468. (In Russian.)
6. B. Sz.-Nagy, *Spektraldarstellung linearer Tranformationen des Hilbertschen Raumes*, Springer-Verlag, Berlin, 1942.
7. N. Nilsson, *Essential selfadjointness and the spectral resolution of Hamiltonian operators*, Kungl. Fysiogr. Sällsk. i Lund Förh. 29 (1959), 1–19.
8. L. Nirenberg, *Estimates and existence of solutions of elliptic equations*, Comm. Pure Appl. Math. 9 (1956), 509–529.
9. M. Schechter, *Remarks on elliptic boundary value problems*, Comm. Pure Appl. Math. 12 (1959), 561–578.
10. L. Schwartz, *Théorie des distributions* I–II, Paris, 1950–1951.
11. W. I. Soboleff, *On a theorem of functional analysis*, Mat. Sbornik 4 (46) (1938), 471–497.