MATH. SCAND. 9 (1961), 122–128

# DETERMINANTS OF A CERTAIN CLASS OF NON-HERMITIAN TOEPLITZ MATRICES

GLEN BAXTER and PALLE SCHMIDT

## 1. Introduction.

In this paper we will establish an elementary identity between two determinants and illustrate its use as at a tool for investigating a certain class of Toeplitz determinants. The feature of our identity which makes it useful in such investigations is that it equates a determinant of "large" order $n$ to one of fixed "small" order $k$. Thus, using only elementary techniques, precise information can be obtained on the asymptotic behavior of certain $n \times n$ Toeplitz determinants as $n$ becomes infinite.

We first introduce some notation. With any formal Laurent series

$$f = \sum_{-\infty}^{\infty} A_m z^m$$

we associate a sequence of $n \times n$ Toeplitz determinants defined by $D_0(f) = 1$ and

$$D_n(f) = \begin{vmatrix} A_0 & A_{-1} & \ldots & A_{-n+1} \\ A_1 & A_0 & \ldots & A_{-n+2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n-1} & A_{n-2} & \ldots & A_0 \end{vmatrix}, \qquad n = 1,2,\ldots .$$

If the Laurent series $f$ actually converges for some $z$, the function thus defined in a certain set in the complex plane will also be denoted by $f$.

The case that will be of interest to us in this paper is that in which $f$ is a "one sided" Laurent series in the sense that $f$ has only a finite number $k$ of negative powers of $z$. In that case, only the first $k$ diagonals above the main diagonal in the determinants $D_n(f)$ can consist of non-zero elements.

---

Received November 29, 1960.

This work was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF AFOSR-61-4, and through its European Office, under contract No. AF 61-052-42.

We now state the identity.

THEOREM 1. *Let g and h be formal Laurent series, which are formal power series,*

$$g = \sum_{0}^{\infty} a_m z^m, \qquad a_0 = 1,\ a_m = 0 \text{ for } m < 0\ ,$$

$$h = \sum_{0}^{\infty} b_m z^m, \qquad b_0 = 1,\ b_m = 0 \text{ for } m < 0\ ,$$

*and let g and h be formally inverse of each other, i.e.*

$$gh = \left(\sum_{0}^{\infty} a_m z^m\right)\left(\sum_{0}^{\infty} b_m z^m\right) = 1\ .$$

*Then for all* $n, k \geqq 0$,

(1.1) $$D_n(z^{-k}g) = (-1)^{nk} D_k(z^{-n}h)\ ,$$

*that is*

$$\begin{vmatrix} a_k & a_{k-1} & \cdots & a_{k-n+1} \\ a_{k+1} & a_k & \cdots & a_{k-n+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k+n-1} & a_{k+n-2} & \cdots & a_k \end{vmatrix} = (-1)^{nk} \begin{vmatrix} b_n & b_{n-1} & \cdots & b_{n-k+1} \\ b_{n+1} & b_n & \cdots & b_{n-k+2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n+k-1} & b_{n+k-2} & \cdots & b_n \end{vmatrix}\ .$$

Although Theorem 1 shows a complete duality between the roles of $n$ and $k$, we will be primarily interested in the case in which $k$ remains fixed and $n$ becomes infinite. We use Theorem 1 to investigate the asymptotic behavior of the Toeplitz determinants associated with a "one sided" Laurent series of the form

(1.2) $$f = z^{-k} g = z^{-k} \sum_{0}^{\infty} a_m z^m, \qquad a_0 = 1,\ a_m = 0 \text{ for } m < 0\ .$$

In that special non-Hermitian case we obtain the following analogue of G. Szegö's theorem on the asymptotic behavior of Toeplitz determinants (cf. [2] or [1, p. 76]).

THEOREM 2. *Let the Laurent series* (1.2) *satisfy the conditions*

(i) $$\sum_{0}^{\infty} |a_m| < \infty\ ,$$

(ii) $$\log f(e^{i\theta}) = \sum_{-\infty}^{\infty} h_m e^{im\theta} \quad \textit{with} \quad \sum_{-\infty}^{\infty} |h_m| < \infty\ .$$

*Then*

(1.3) $$\lim_{n\to\infty} \frac{D_n(f)}{\exp[nh_0]} = \exp\left[\sum_{1}^{\infty} m h_m h_{-m}\right]\ .$$

Condition (ii) states that $\log f(e^{i\theta})$ has an absolutely convergent Fourier series. Provided (i) holds, this is equivalent, by the Wiener–Levy theorem [4, Chapter VI, (5.2)], to the condition that the curve traversed in the complex plane by $f(z)$ as $z=e^{i\theta}$ traverses the unit circle has winding number zero around the origin.

For any formal Laurent series $f$ and any complex number $r$ we denote by $f(rz)$ the formal Laurent series obtained by replacing $z$ in $f$ by $rz$. It is easy to see that for $r \neq 0$

$$D_n(f) = D_n(f(rz)), \qquad n = 0,1,2,\ldots .$$

Hence, if for some $r>0$ the Laurent series $f(rz)$ satisfies conditions (i) and (ii), the asymptotic behavior of $D_n(f)$ can be found from Theorem 2.

Finally it is obvious that the requirement $a_0=1$ in (1.2) can be replaced by $a_0 \neq 0$.

As another application of Theorem 1 we give an expression for the Toeplitz determinants of an arbitrary Laurent polynomial

$$(1.4) \qquad f = z^{-k}g = z^{-k}\sum_0^q a_m z^m, \qquad q \geqq k,\ a_0=1,\ a_q \neq 0\ ,$$

in terms of the zeros of that polynomial, or equivalently, in terms of the zeros $\varrho_1,\varrho_2,\ldots,\varrho_q$ of the polynomial

$$z^q g(z^{-1}) = z^q + a_1 z^{q-1} + \ldots + a_q\ .$$

We will prove

THEOREM 3. *If the Laurent polynomial* (1.4) *has no multiple zeros, then for* $n=0,1,2,\ldots$

$$(1.5) \qquad D_n(f) = (-1)^{nk}\sum_I \left(\prod_{i\in I}\varrho_i^{\,n+q-k}\right)\left(\prod_{\substack{i\in I\\ j\in \bar I}}(\varrho_i-\varrho_j)^{-1}\right),$$

*where* $I$ *runs through the set of all subsets of cardinality* $k$ *of the set* $\{1,2,\ldots,q\}$ *and where* $\bar I=\{1,2,\ldots,q\}-I$.

Theorem 3 is a sharpening of a result due to Harold Widom, who proved in [3] that if a Laurent polynomial $f$ has no multiple zeros, then $D_n(f)=0$ if and only if the right hand side of (1.5) vanishes.

Theorems 1 to 3 are proved in Sections 2 to 4, respectively.

## 2. The proof of the identity (1.1).

We use the notation introduced in Theorem 1. The proof is based on the simple trick of multiplying by the determinants $D_k(g)=D_n(h)=1$. We consider, for $n \geqq k$,

$$D_n(z^{-k}g)\,D_n(h) = \det\left(\sum_{\nu=1}^{n} a_{k+i-\nu}\, b_{\nu-j}\right), \qquad i,j = 1,2,\ldots,n\,.$$

Using the reciprocal property

$$\sum_{\nu=0}^{m} a_\nu b_{m-\nu} = \delta_{m0}, \qquad m = 0,1,2,\ldots\,,$$

of the Laurent series $g$ and $h$, we find that

$$\text{(2.1)} \qquad \sum_{\nu=1}^{n} a_{k+i-\nu}\, b_{\nu-j} = \begin{cases} \delta_{k+i,\,j} & \text{for } i \leqq n-k \\ -\displaystyle\sum_{\nu=n+1}^{n+k} a_{k+i-\nu}\, b_{\nu-j} & \text{for } i > n-k\,. \end{cases}$$

Thus,

$$D_n(z^{-k}g)\,D_n(h) = \left|\begin{array}{c|c} 0 & I \\ \hline N & M \end{array}\right| \begin{array}{l} \}n-k \\ \}k \end{array},$$

with the column blocks of widths $k$ and $n-k$,

where $I$ is the identity matrix of order $n-k$ and where $N$ and $M$ are suitably chosen matrices. In simpler terms

$$D_n(z^{-k}g)\,D_n(h) = (-1)^{k(n-k)} \det(N)\,.$$

The elements of $N$ can be found from (2.1) by the simple change of subscripts $i' = i-(n-k)$. Dropping primes, we have

$$\begin{aligned} D_n(z^{-k}g)\,D_n(h) &= (-1)^{k(n-k)} \det\left(-\sum_{\nu=n+1}^{n+k} a_{n+i-\nu}\, b_{\nu-j}\right) \\ &= (-1)^{nk} \det\left(\sum_{\mu=1}^{k} a_{i-\mu}\, b_{n+\mu-j}\right), \qquad i,j = 1,2,\ldots,k\,. \end{aligned}$$

Hence for $n \geqq k$

$$D_n(z^{-k}g)\,D_n(h) = (-1)^{nk}\, D_k(g)\, D_k(z^{-n}h)\,,$$

which proves Theorem 1.

## 3. The asymptotic behavior of the Toeplitz determinants of a "one-sided" Laurent series.

We assume as given a Laurent series $f = z^{-k}g$ of type (1.2) satisfying (i) and (ii) of Theorem 2. By the principle of the argument, condition (ii) implies that $g$ has exactly $k$ zeros inside and no zeros on the unit circle, and that we can write

$$\text{(3.1)} \qquad g^{-1} = Q(z) \prod_{\nu=1}^{k} (1+\sigma_\nu z)^{-1}\,,$$

where $1 < |\sigma_1| \leqq |\sigma_2| \leqq \ldots \leqq |\sigma_k|$, and where

$$Q(z) = \sum_0^\infty q_m z^m \quad \text{with} \quad \sum_0^\infty |q_m| < \infty .$$

Furthermore, $Q(z) \neq 0$ for $|z| \leqq 1$.

The proof of Theorem 2 will be essentially contained in the proof of the following asymptotic formula:

$$(3.2) \qquad D_n(f) \sim (\sigma_1 \ldots \sigma_k)^n \prod_{\nu=1}^{k} Q(-\sigma_\nu^{-1}), \qquad n \to \infty .$$

To prove (3.2) we introduce the quantities $D_m(i,j)$, $i=1,2,\ldots,k$, $j=1,2,\ldots,k$, $m=0,1,2,\ldots$, defined by

$$(3.3) \qquad \sum_{m=0}^{\infty} D_m(i,j) z^m = Q(z) \prod_{\nu=i}^{k} (1+\sigma_\nu z)^{-1} \prod_{\nu=j+1}^{k} (1+\sigma_\nu z) .$$

From (3.1) we see that the coefficient of $z^m$ in $h=g^{-1}$ is $D_m(1,k)$. Thus, by Theorem 1 we have

$$D_n(f) = (-1)^{nk} \det\bigl(D_{n+i-j}(1,k)\bigr), \qquad i,j = 1,2,\ldots,k .$$

Let us now consider the two $k \times k$ matrices $(m_{ij})$ and $(n_{ij})$ whose entries are defined by the relations

$$\sum_{j=1}^{k} m_{ij} z^{i-j} = \prod_{\nu=1}^{i-1} (1+\sigma_\nu z), \qquad i = 1,2,\ldots,k ,$$

and

$$\sum_{i=1}^{k} n_{ij} z^{i-j} = \prod_{\nu=j+1}^{k} (1+\sigma_\nu z), \qquad j = 1,2,\ldots,k .$$

Note that both $(m_{ij})$ and $(n_{ij})$ are subdiagonal with $m_{ii}=n_{ii}=1$, $i=1,2,\ldots,k$, so that

$$\det(m_{ij}) = \det(n_{ij}) = 1 .$$

Note also that the $ij$'th element of the matrix product

$$(m_{ij})\bigl(D_{n+i-j}(1,k)\bigr)(n_{ij})$$

is exactly the coefficient of $z^{n+i-j}$ in

$$\prod_{\nu=1}^{i-1} (1+\sigma_\nu z) \prod_{\nu=j+1}^{k} (1+\sigma_\nu z) \sum_{m=0}^{\infty} D_m(1,k)\, z^m = \sum_{m=0}^{\infty} D_m(i,j)\, z^m .$$

In other words

$$\det\bigl(D_{n+i-j}(1,k)\bigr) = \det\bigl(D_{n+i-j}(i,j)\bigr), \qquad i,j = 1,2,\ldots,k ,$$

and we have reduced the problem to finding the asymptotic behavior of the determinant on the right in the above equation.

By tedious but elementary calculations we find from (3.3)

$$\lim_{m\to\infty} D_m(i,j) = 0 \qquad \text{for } i > j ,$$
$$|D_m(i,j)| \leqq \text{const.}\, n^k |\sigma_j|^n \qquad \text{for } i < j ,$$
$$D_n(j,j) = (-\sigma_j)^n Q_n(-\sigma_j^{-1}) ,$$

where $Q_n(z)$ is the $n$'th partial sum of $Q(z)$. Thus, dividing the $j$'th column of $\det\big(D_{n+i-j}(i,j)\big)$ by $D_n(j,j)$ we see that

$$D_n(f) = (-1)^{nk} \det\big(D_{n+i-j}(i,j)\big) \sim (\sigma_1 \ldots \sigma_k)^n \prod_{\nu=1}^{k} Q(-\sigma_\nu^{-1}), \qquad n \to \infty ,$$

and (3.2) is proved.

To finish the proof of Theorem 2 we merely note that from

$$\sum_{-\infty}^{\infty} h_m z^m = \sum_{\nu=1}^{k} \log(z^{-1}+\sigma_\nu) - \log Q(z), \qquad z = e^{i\theta} ,$$

follows

$$h_0 = \log \sigma_1 \ldots \sigma_k$$

and

$$\sum_{1}^{\infty} m h_m h_{-m} = -\sum_{\nu=1}^{k} \sum_{m=1}^{\infty} h_m (-\sigma_\nu^{-1})^m = \log \prod_{\nu=1}^{k} Q(-\sigma_\nu^{-1}) .$$

**4. The Toeplitz determinants of an arbitrary Laurent polynomial expressed by the zeros of that polynomial.**

Let $f = z^{-k} g$ be a Laurent polynomial of form (1.4). Let $\varrho_1, \varrho_2, \ldots, \varrho_q$ be the reciprocals of the zeros of $f$, or equivalently, the zeros of the polynomial $z^q g(z^{-1})$. Finally let $\varrho_i \neq \varrho_j$ for $i \neq j$.

By Theorem 1 we have the following expression for the $n$'th Toeplitz determinant associated with $f$

$$D_n(f) = (-1)^{nk} \det(b_{n+i-j}), \qquad i,j = 1,2,\ldots,k , \tag{4.1}$$

where $b_m = 0$ for $m < 0$ and

$$\sum_{m=0}^{\infty} b_m z^m = g^{-1}(z) = \prod_{\nu=1}^{q} (1-\varrho_\nu z)^{-1} .$$

By expanding in partial fractions we obtain

$$b_m = \sum_{\nu=1}^{q} A_\nu \varrho_\nu^{\,m}, \qquad m = 0,1,2,\ldots , \tag{4.2}$$

where

$$A_\nu = \varrho_\nu^{\,q-1} \prod_{\mu \neq \nu} (\varrho_\nu - \varrho_\mu)^{-1}, \qquad \nu = 1,2,\ldots,q .$$

The expression (4.2) for $b_m$ also holds for $m = -1, -2, \ldots, -q+1$, because for all $m$

$$\sum_{\nu=1}^{q} A_\nu \varrho_\nu^{\,m} = \prod_{\mu>\nu} (\varrho_\mu - \varrho_\nu)^{-1} \begin{vmatrix} 1 & \varrho_1 & \dots & \varrho_1^{\,q-2} & \varrho_1^{\,m+q-1} \\ 1 & \varrho_2 & \dots & \varrho_2^{\,q-2} & \varrho_2^{\,m+q-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varrho_q & \dots & \varrho_q^{\,q-2} & \varrho_q^{\,m+q-1} \end{vmatrix} .$$

Hence by (4.1) and (4.2) we have, for $n=0,1,2,\ldots,$

$$D_n(f) = (-1)^{nk} \det\left(\sum_{\nu=1}^{q} A_\nu \varrho_\nu^{\,n+i-j}\right), \qquad i,j = 1,2,\ldots,k .$$

We can write the $k \times k$ determinant on the right as a product of two $q \times q$ Vandermonde determinants as follows

$$(-1)^{nk} D_n(f) =$$

$$\begin{vmatrix} A_1 & A_2 & \dots & A_q \\ A_1\varrho_1^{-1} & A_2\varrho_2^{-1} & \dots & A_q\varrho_q^{-1} \\ \dots & \dots & \dots & \dots \\ A_1\varrho_1^{-q+1} & A_2\varrho_2^{-q+1} & \dots & A_q\varrho_q^{-q+1} \end{vmatrix} \cdot \begin{vmatrix} 1 & \varrho_1 & \dots & \varrho_1^{\,q-k-1} & \varrho_1^{\,n+q-k} & \dots & \varrho_1^{\,n+q-1} \\ 1 & \varrho_2 & \dots & \varrho_2^{\,q-k-1} & \varrho_2^{\,n+q-k} & \dots & \varrho_2^{\,n+q-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \varrho_q & \dots & \varrho_q^{\,q-k-1} & \varrho_q^{\,n+q-k} & \dots & \varrho_q^{\,n+q-1} \end{vmatrix} .$$

From this we easily derive the expression (1.5) by Laplace expansion of the second Vandermonde determinant above after the last $k$ columns.

## REFERENCES


1. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Berkeley, 1958.
2. G. Szegö, *On certain Hermitian forms associated with the Fourier series of a positive function*, Communications du séminaire mathématique de l'université de Lund, tome supplémentaire (1952) dédié à Marcel Riesz, 228–238.
3. H. Widom, *On the eigenvalues of certain Hermitian operators*, Trans. Amer. Math. Soc. 88 (1958), 491–522.
4. A. Zygmund, *Trigonometrical series* 1, Second ed., Cambridge, 1959.


UNIVERSITY OF AARHUS, DENMARK