MATH. SCAND. 9 (1961), 129–146

SOME REMARKS ON THE EXPONENTIAL MAPPING FOR AN AFFINE CONNECTION

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1. Introduction and notation.

Let M be a manifold with an affine connection. The normal coordinates at a point p in M give rise to a topological mapping of a neighborhood of p in M onto a neighborhood of 0 in the tangent space M_p to M at p. Much is gained by replacing this mapping by its inverse, because this last mapping can be extended to the entire M_p (at least if M is complete). This mapping of $M_p \to M$ was called by Ambrose [1] the Exponential mapping, because it reduces to the ordinary exponential mapping for a Lie group G when applied to a suitable left invariant affine connection on G. For this affine connection, the Levi–Civita parallelism amounts to left translation on G; in particular, the parallelism arises here from a point transformation. For a general manifold M, this is of course no longer true; nevertheless, the analogy between parallelism and left translation is useful.

In this article we prove a formula for the differential of the Exponential mapping for an affine connection. This formula is motivated by the known formula for the differential of the exponential mapping of a Lie group, through the analogy stressed above. In Section 4 some applications are made; we give there a direct derivation of the classical formula for the sectional curvature of a Riemannian manifold in terms of the metric and curvature tensor. Although this derivation involves a certain amount of computation, it is conceptually very simple and is independent of the differential geometry of curves and surfaces. In another application we establish some geometric properties of the Exponential mapping for a pseudo-riemannian manifold. In Section 5 we determine the pseudo-riemannian manifolds of constant sectional curvature up to local isometry; these manifolds are represented by quadrics in Euclidean space or equivalently, by coset spaces of orthogonal groups.

Received August 11, 1960.
Work supported in part by the National Science Foundation.

The following notation will be used. If M is a differentiable manifold of class C^∞, we denote by $C^\infty(M)$ the set of differentiable functions on M. The set $C^\infty(M)$ is an algebra over the real numbers $\mathbf{R}$, the addition and multiplication in $C^\infty(M)$ being pointwise addition and multiplication of functions. The vector fields on M can be identified with the derivations of the algebra $C^\infty(M)$; they form a Lie algebra $\mathfrak{D}^1(M)$ under the bracket operation $[X, Y] = XY - YX$. Let $X \in \mathfrak{D}^1(M)$. The mapping $Y \to [X, Y]$ of $\mathfrak{D}^1(M)$ into itself is called Lie derivation with respect to X, and will be denoted by $\theta(X)$. The vector fields on M are the tensor fields on M, which are contravariant of degree 1, covariant of degree 0. More generally, let $\mathfrak{D}^r{}_s(M)$ denote the set of all tensor fields on M, of type (r, s), (that is, contravariant of degree r, covariant of degree s). The set $\mathfrak{D}^r{}_s(M)$ is a module over the ring $C^\infty(M)$. The direct sum $\mathfrak{D}(M) = \sum_{r,s=0}^{\infty} \mathfrak{D}^r{}_s(M)$ is the mixed tensor algebra over M; the addition and multiplication $\otimes$ in $\mathfrak{D}(M)$ is given by pointwise addition and pointwise tensor product of tensors.

If p is a point in M, the tangent space to M at p will be denoted M_p. If $X \in \mathfrak{D}^1(M)$, X_p denotes the value of X at the point $p \in M$. Then $X_p \in M_p$. More generally, the value of a tensor field $T \in \mathfrak{D}^r{}_s(M)$ at a point $p \in M$ will be denoted T_p. In particular, the value of a function $f \in C^\infty(M)$ will often be denoted f_p instead of $f(p)$.

2. Preliminaries.

In order to fix the terminology, we recall briefly some elementary, well-known facts concerning affine connections. Let M be a manifold of class C^∞. An *affine connection* on M is a rule ∇ which to each $X \in \mathfrak{D}^1(M)$ assigns a linear mapping ∇_X (the covariant differentiation with respect to X) of the vector space $\mathfrak{D}^1(M)$ into itself, satisfying the two following conditions:

$$\nabla_1) \qquad \nabla_{fX+gY} = f\nabla_X + g\nabla_Y$$
$$\nabla_2) \qquad \nabla_X(fY) = f\nabla_X Y + (Xf)Y$$

for $f, g \in C^\infty(M)$ and $X, Y \in \mathfrak{D}^1(M)$. This definition is due to J. Koszul and is used in Nomizu's paper [6]. It follows directly from ∇_1) and ∇_2) that if X or Y vanishes on an open set U then $\nabla_X(Y)$ vanishes on U. A stronger statement holds for X, namely: If X vanishes at a point $p \in M$, then the same holds for $\nabla_X(Y)$. To see this, let $\{x_1, \ldots, x_m\}$ be a local coordinate system valid on an open neighborhood U of p. On the set U we can write $X = \sum_i f_i \partial/\partial x_i$ where $f_i \in C^\infty(U)$ and $f_i(p) = 0$, $1 \leqq i \leqq m$. As remarked above, $\nabla_X(Y)$ is given on U by the values of X on U; hence we have, writing ∇_i instead of $\nabla_{\partial/\partial x_i}$,

$$\big(\nabla_X(Y)\big)_p = \sum_i f_i(p)\big((\nabla_i(Y)\big)_p = 0 .$$

Since an affine connection ∇ gives rise to an affine connection on any open submanifold, ∇ gives rise to a set of functions $\Gamma_{ij}{}^k$ on the coordinate neighborhood U; these functions are given by the formulas $\nabla_i(\partial/\partial x_j) = \sum_k \Gamma_{ij}{}^k \partial/\partial x_k$ and they in turn determine the affine connection on U.

A *curve* in a C^∞ manifold M is a regular mapping of an open interval $I \subset \boldsymbol{R}$ into M. The restriction of a curve to a closed subinterval is called a curve segment. Let γ: $t \to \gamma(t)$, $t \in I$, be a curve in M and let $X(t)$ denote the family of tangent vectors to the curve, that is $X(t) = d\gamma(d/dt)_t$, $t \in I$. Suppose that $Y(t)$ is a family of tangent vectors along γ, that is $Y(t) \in M_{\gamma(t)}$ for each $t \in I$. Assuming $Y(t)$ to vary differentiably with t, we shall now define what it means for the family $Y(t)$ to be parallel with respect to γ. Let J be a compact subinterval of I such that the finite curve segment γ_J: $t \to \gamma(t)$, $t \in J$, has no double points and such that $\gamma(J)$ is contained in a coordinate neighborhood U. Let $\{x_1, \ldots, x_m\}$ be a coordinate system on U. It can be shown that there exist vector fields $X, Y \in \mathfrak{D}^1(M)$ such that

$$X_{\gamma(t)} = X(t), \qquad Y_{\gamma(t)} = Y(t)$$

for $t \in J$. The family $Y(t)$, $t \in J$, is said to be *parallel* with respect to γ_J (or parallel along γ_J) if

$$\nabla_X(Y)_{\gamma(t)} = 0 \qquad \text{for all} \quad t \in J . \tag{1}$$

In order to show that this definition is independent of the choice of X and Y, we express (1) in the coordinates $\{x_1, \ldots, x_m\}$. There exist functions X^i, Y^j on U such that

$$X = \sum_i X^i \partial/\partial x_i, \qquad Y = \sum_i Y^i \partial/\partial x_i$$

on U. We write also for simplicity

$$x_i(t) = x_i\big(\gamma(t)\big), \qquad X^i(t) = X^i\big(\gamma(t)\big), \qquad Y^i(t) = Y^i\big(\gamma(t)\big)$$

for $t \in J$ and $1 \leqq i \leqq m$. Then $X^i(t) = dx_i/dt$ and since

$$\nabla_X(Y) = \sum_k \left(\sum_i X^i \frac{\partial Y^k}{\partial x_i} + \sum_{i,j} X^i Y^j \Gamma_{ij}{}^k\right) \frac{\partial}{\partial x_k} \qquad \text{on} \quad U , \tag{2}$$

we see that (1) is equivalent to

$$\frac{dY^k}{dt} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{dt} Y^j = 0, \qquad t \in J , \tag{3}$$

In particular, the parallelism definition (1) is independent of the choice of X and Y. It is now obvious how to define parallelism with respect to any curve segment γ_J and finally with respect to the entire curve. A curve $\gamma: t \to \gamma(t)$, $t \in I$, is called a geodesic if the family of tangent vectors $X(t) = d\gamma(d/dt)_t$ is parallel with respect to γ. It follows immediately from (3) that the system

$$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \tag{4}$$

is a necessary and sufficient condition for the curve $t \to \big(x_1(t), \ldots, x_m(t)\big)$ to be a geodesic. From the existence and uniqueness theorems for the system (4) it follows that given $p \in M$ and $X \in M_p$ there exists a unique maximal geodesic $t \to \gamma(t)$, $t \in I$, in M such that $\gamma(0) = p$, $d\gamma(d/dt)_0 = X$. This geodesic will be denoted γ_X. The mapping $X \to \gamma_X(1)$ is defined on a subset of M_p. It is called the Exponential mapping at p and is denoted Exp_p (or simply Exp when no misunderstanding can arise). An open neighborhood N_0 of the origin in M_p is said to be *normal* if: 1) the mapping Exp_p is a diffeomorphism of N_0 onto an open neighborhood N_p of p in M; 2) N_0 is star-shaped, that is if $X \in N_0$ and $0 \leqq t \leqq 1$, then $tX \in N_0$. If N_0 is normal, the diffeomorphic neighborhood N_p is also called normal. The manifold M (with the affine connection ∇) is said to be *complete* if for each $p \in M$, the mapping Exp_p is defined on the entire tangent space M_p.

Let p and q be two points in M and γ a curve segment from p to q. It follows from the existence and uniqueness theorems for the system (3) that the parallelism along γ induces an isomorphism of M_p onto M_q. Let N_p be a normal neighborhood of p in M. Let $X \in M_p$ and for each $q \in N_p$ put $(X^*)_q = \tau_{pq} \cdot X$, where τ_{pq} is the parallel translation along the unique geodesic segment in N_p from p to q. The assignment $q \to (X^*)_q$ is a vector field on N_p, which is said to be *adapted* to the tangent vector X.

In order to extend the covariant differentiation to the mixed tensor algebra $\mathfrak{D}(M)$ it is convenient to describe ∇_X by means of the parallelism.

LEMMA 1. *Let M be a C^∞ manifold with an affine connection. Let $p \in M$ and $X, Y \in \mathfrak{D}^1(M)$ such that $X_p \neq 0$. Let $\gamma: t \to \gamma(t)$ be an integral curve of X through $p = \gamma(0)$, and let τ_t denote the parallel translation from p to $\gamma(t)$ along γ. Then*

$$\big(\nabla_X(Y)\big)_p = \lim_{s \to 0} \frac{1}{s}\left(\tau_s^{-1} \cdot Y_{\gamma(s)} - Y_p\right).$$

This lemma is well known, but for completeness we sketch a direct proof. Consider a fixed number $s \geqq 0$ and the vector field $Z_{\gamma(t)}$, $0 \leqq t \leqq s$, which is parallel with respect to the curve γ such that $Z_{\gamma(0)} = \tau_s^{-1} \cdot Y_{\gamma(s)}$. Let $\{x_1, \ldots, x_m\}$ be a system of coordinates in an open neighborhood U of p such that $x_1(p) = \ldots = x_m(p) = 0$. Let $X^j(t)$, $Y^j(t)$, $Z^j(t)$ denote the coordinates of the vectors $X_{\gamma(t)}$, $Y_{\gamma(t)}$, $Z_{\gamma(t)}$ with respect to the basis $\partial/\partial x_i$, $1 \leqq i \leqq m$. Then

$$\frac{dZ^k}{dt} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{dt} Z^j = 0, \qquad 0 \leqq t \leqq s,$$

and $Z^k(s) = Y^k(s)$, $1 \leqq k \leqq m$. By the mean value theorem,

$$Z^k(s) = Z^k(0) + s\left(\frac{dZ^k}{dt}\right)(t^*),$$

where t^* is a suitable number between 0 and s. Hence the k^{th} component of $(1/s)(\tau_s^{-1} \cdot Y_{\gamma(s)} - Y_p)$ is

$$\begin{aligned}\frac{1}{s}\big(Z^k(0) - Y^k(0)\big) &= \frac{1}{s}\left\{Z^k(s) - s\left(\frac{dZ^k}{dt}\right)(t^*) - Y^k(0)\right\} \\ &= \sum_{i,j} Z^j(t^*)\left(\Gamma_{ij}{}^k \frac{dx_i}{dt}\right)(t^*) + \frac{1}{s}\big(Y^k(s) - Y^k(0)\big).\end{aligned}$$

As $s \to 0$, this expression has the limit

$$\frac{dY^k}{ds} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{ds} Y^j.$$

Since $X^i(t) = dx_i/ds$, this expression is the k^{th} component of $\big(\nabla_X(Y)\big)_p$, and the lemma follows.

The covariant derivative ∇_X can now be extended to the mixed tensor algebra $\mathfrak{D}(M)$. For a function $f \in C^\infty(M)$ one puts

$$(\nabla_X f)_p = \lim_{s \to 0} \frac{1}{s}\{f(\gamma(s)) - f(p)\}$$

and for a tensor field $T \in \mathfrak{D}(M)$

$$(\nabla_X T)_p = \lim_{s \to 0} \frac{1}{s}(\tau_s^{-1} \cdot T_{\gamma(s)} - T_p).$$

Then $\nabla_X f = Xf$ and ∇_X is a derivation of the tensor algebra $\mathfrak{D}(M)$, commuting with contractions.

The curvature tensor R and the torsion tensor T of the affine connection ∇ are given by

$$\begin{aligned}T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \\ R(X,Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\end{aligned}$$

for $X, Y \in \mathfrak{D}^1(M)$. Then T is a tensor field of type $(1,2)$ and R is a tensor field of type (1,3).

Let M be a C^∞ manifold. A *pseudo-riemannian structure* on M is a tensor field g of type $(0,2)$ which satisfies the two conditions: 1) g is symmetric, that is, $g(X,Y)=g(Y,X)$ for $X,Y \in \mathfrak{D}^1(M)$; 2) for each $p \in M$, the value g_p of g at p is a non-degenerate bilinear form on M_p.

Let g be a pseudo-riemannian structure on M. There exists a unique torsion-free affine connection on M (the pseudo-riemannian connection) with the property that the parallel translation preserves the inner product on the tangent spaces. This property is equivalent to $\nabla_Z g=0$ for all $Z \in \mathfrak{D}^1(M)$. Let $X, Y \in \mathfrak{D}^1(M)$ and apply ∇_Z to the tensor field $X \otimes Y \otimes g$. Using the fact that ∇_Z is a derivation and commutes with contractions it follows that the relation $\nabla_Z g=0$ is equivalent to

$$(5) \qquad Zg(X,Y) = g(\nabla_Z X, Y)+g(X,\nabla_Z Y), \qquad X,Y \in \mathfrak{D}^1(M)\,.$$

From this relation and the formula

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

(expressing $T=0$), it can be deduced without difficulty, that

$$(6) \quad 2g(X,\nabla_Z Y) = Zg(X,Y)+g(Z,[X,Y])+ Yg(X,Z)+g(Y,[X,Z])- \\ -Xg(Y,Z)-g(X,[Y,Z])$$

for all $X,Y,Z \in \mathfrak{D}^1(M)$. The uniqueness of the pseudo-riemannian connection is obvious from (6); it can also be shown without difficulty that the operator ∇_Z, as defined by (6), satisfies the axioms ∇_1 and ∇_2. Hence the relation (6) defines the pseudo-riemannian connection.

A C^∞ manifold with a pseudo-riemannian structure g is called a pseudo-riemannian manifold. In the case, when g_p is positive definite for each $p \in M$, we drop the prefix „pseudo" and speak of a Riemannian structure, Riemannian connection and Riemannian manifold.

3. The differential of the Exponential mapping.

An affine connection ∇ on an analytic manifold M is called *analytic* if for each point $p \in M$, the vector field $\nabla_X(Y)$ is analytic at p whenever the vector fields X and Y are analytic at p. In terms of local coordinates, analyticity of ∇ is equivalent to the analyticity of the functions $\Gamma_{ij}{}^k$.

Theorem 1. *Let M be an analytic manifold with an analytic affine connection ∇. Let $p \in M$ and $X \in M_p$. Then there exists an $\varepsilon > 0$ such that the differential of* Exp *is given by*

$$(d\,\mathrm{Exp})_{tX}(Y) = \left\{\frac{1-e^{-\theta(tX^*)}}{\theta(tX^*)}(Y^*)\right\}_{\mathrm{Exp}\,tX}, \qquad Y \in M_p\,,$$

for $|t| < \varepsilon$. (Here $(1-e^{-A})/A$ stands for the series $\sum_0^\infty (-1)^m/(m+1)!\,A^m$ and the manifold M_p is identified with its tangent space at each point.)

In the proof below, we sometimes write F_p for the value of a function F at $p \in M$. The mapping Exp is given by the solutions of the system of differential equations (4); since the functions $\Gamma_{ij}{}^k$ are analytic, the mapping Exp is analytic at the origin in M_p. Let f be an analytic function at $p \in M$. Then there exists a star-shaped neighborhood U_0 of 0 in M_p such that

$$f(\mathrm{Exp}\,tZ) = P(tz_1,\ldots,tz_m), \qquad Z \in U_0, \quad 0 \leqq t \leqq 1\,,$$

where P is an absolutely convergent power series and $z_1,\ldots,z_m$ denote the coordinates of $Z \in M_p$ with respect to some basis of M_p. If t is sufficiently small,

$$(Z^*f)_{\mathrm{Exp}\,tZ} = \left\{\frac{d}{du}f\big(\mathrm{Exp}\,(t+u)Z\big)\right\}_{u=0} = \frac{d}{dt}f(\mathrm{Exp}\,tZ)\,,$$

and by induction

$$\big((Z^*)^n f\big)_{\mathrm{Exp}\,tZ} = \frac{d^n}{dt^n}f(\mathrm{Exp}\,tZ)\,.$$

Using Taylor's formula we find

$$f(\mathrm{Exp}\,Z) = \sum_0^\infty \frac{1}{n!}\big((Z^*)^n f\big)_p\,. \tag{7}$$

Now, suppose Y is an arbitrary vector in M_p. Then

$$d\,\mathrm{Exp}_{tX}(Y)f = Y_{tX}(f\circ \mathrm{Exp}) = \left\{\frac{d}{du}f\big(\mathrm{Exp}\,(tX+uY)\big)\right\}_{u=0}.$$

If t and u are sufficiently small, we obtain from (7)

$$f\big(\mathrm{Exp}\,(tX+uY)\big) = \sum_0^\infty \frac{1}{r!}\big((tX^*+uY^*)^r f\big)_p = \sum_{m,n\geqq 0}\frac{t^n u^m}{(n+m)!}(S_{n,m}f)_p\,, \tag{8}$$

where $S_{n,m}$ is the coefficient to $t^n u^m$ in $(tX^*+uY^*)^{n+m}$. In particular

$$S_{n,1} = (X^*)^n Y^* + (X^*)^{n-1}Y^*X^* + \ldots + Y^*(X^*)^n\,.$$

We differentiate the expansion (8) with respect to u and put $u=0$. Then we obtain

$$d\,\mathrm{Exp}_{tX}(Y)f = \sum_0^\infty \frac{t^n}{(n+1)!}\big(\{(X^*)^n Y^* + \ldots + Y^*(X^*)^n\}f\big)_p\,.$$

Let $D(N_p)$ denote the algebra of operators on the vector space $C^\infty(N_p)$ generated by the vector fields Z^* as Z varies through M_p. Let L_{X^*} and R_{X^*} denote the linear transformations of $D(N_p)$ given by L_{X^*}: $A \to X^*A$ and R_{X^*}: $A \to AX^*$. Since the Lie derivative $\theta(X^*)$ equals $L_{X^*}-R_{X^*}$ on the subspace $\mathfrak{D}^1(N_p)$ of $D(N_p)$, we extend $\theta(X^*)$ to $D(N_p)$ by the formula $\theta(X^*)=L_{X^*}-R_{X^*}$. Then $\theta(X^*)$ and L_{X^*} commute; hence we have

$$(R_{X^*})^m = \big(L_{X^*}-\theta(X^*)\big)^m = \sum_{p=0}^{m}(-1)^p\binom{m}{p}(L_{X^*})^{m-p}\big(\theta(X^*)\big)^p .$$

Using the relation

$$\sum_{p=0}^{n-k}\binom{n-p}{k} = \binom{n+1}{k+1}$$

it follows that

$$S_{n,1} = \sum_{p=0}^{n}(X^*)^p\sum_{k=0}^{n-p}(-1)^k\binom{n-p}{k}(X^*)^{n-p-k}\theta(X^*)^k(Y^*)$$
$$= \sum_{k=0}^{n}\binom{n+1}{k+1}(X^*)^{n-k}\theta(-X^*)^k(Y^*) ,$$

so

$$d\,\mathrm{Exp}_{tX}(Y)f = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{\frac{(tX^*)^{n-k}}{(n-k)!}\frac{\theta(-tX^*)^k}{(k+1)!}(Y^*)\right\}f\right)_p .$$

For sufficiently small t, the right hand side can be rewritten by the formula

$$\text{(9)}\qquad \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{\frac{(tX^*)^{n-k}}{(n-k)!}\frac{\theta(-tX^*)^k}{(k+1)!}(Y^*)\right\}f\right)_p$$
$$= \sum_{r=0}^{\infty}\left(\frac{(tX^*)^r}{r!}\left[\sum_{m=0}^{\infty}\left\{\frac{\theta(-tX^*)^m}{(m+1)!}(Y^*)\right\}f\right)_p .$$

The justification of this interchange of summation is elementary, but not altogether trivial. We outline the various necessary steps. Consider the right hand side of (9). The inner series converges absolutely at each point q in a suitable neighborhood U of p, provided t is sufficiently small. This is easily seen by choosing a coordinate system $\{x_1, \ldots, x_m\}$ in a neighborhood of p on which X^* has the form $\partial/\partial x_1$. (See e.g. Chevalley, Theory of Lie Groups, Vol. I, p. 89.) The analyticity of ∇ implies that the coefficients g_i given by $Y^*=\sum g_i\partial/\partial x_i$ are analytic functions; hence the sequence $(X^*)^n g_i$, $n=0,1,\ldots$, grows no faster than $n!$. In this manner one can also prove that the outer series on the right hand side of (9) converges uniformly for all t in some interval around the origin. The interchange of summations (that is, formula (9)), then follows from

Weierstrass' classical theorem on double series. Since the right hand side of (9) is

$$\left\{\frac{1-e^{-\theta(tX^*)}}{\theta(tX^*)}(Y^*)f\right\}_{\mathrm{Exp}\,tX}$$

the theorem follows.

4. Examples and applications.

I. THE EXPONENTIAL MAPPING OF A LIE GROUP.

Consider the special case when M is a Lie group G, the point p is the identity element, and the affine connection ∇ is the left invariant affine connection on G given by $R=0$, $T(X,Y)=-[X,Y]$, X,Y being any left invariant vector fields on G. This is the $(-)$-connection on G in the notation of Cartan–Schouten [3] and Nomizu [6]. It has the property, that the geodesics γ through e are precisely the one-parameter subgroups of G, and the parallel translation along γ is given by left translations. Consequently, the Exponential mapping Exp_e reduces to the ordinary exponential mapping exp for G and X^* is the left invariant vector field on G such that ${X^*}_e=X$. Let $L(g)$ denote the left translation on G by the group element g and let ad denote the adjoint representation of the Lie algebra $\mathfrak{g}$ of G, $\mathrm{ad}\,X(Y)=[X,Y]$, $X,Y\in\mathfrak{g}$. Then the formula of the theorem reduces to

$$d\exp_X = dL(\exp X)\circ\frac{1-e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}$$

for sufficiently small $X\in\mathfrak{g}$. Using the analyticity, it is easy to show that the formula actually holds for all $X\in\mathfrak{g}$.

II. THE EXPONENTIAL MAPPING FOR A PSEUDO-RIEMANNIAN MANIFOLD.

THEOREM 2. *Let M be an analytic manifold with a pseudo-riemannian structure g. We assume that the tensor field g is analytic and that M is complete (in the pseudo-riemannian connection). Let p be a point in M and let* Exp *stand for* Exp_p. *Then*

$$g_p(X,Y) = g_{\mathrm{Exp}\,X}(d\,\mathrm{Exp}_X(X),\ d\,\mathrm{Exp}_X(Y))$$

for $X,Y\in M_p$.

PROOF. Let N_p be a normal neighborhood of p in M and if $Z\in M_p$, let Z^* denote the vector field on N_p adapted to Z. Let $X,Y\in M_p$. If t is sufficiently small, then $\mathrm{Exp}\,tX\in N_p$. It is obvious from Lemma 1, that

$$(\nabla_{X^*}(Z^*))_{\mathrm{Exp}\,tX} = 0\,. \tag{10}$$

Moreover, if A is any vector field on N_p we have from (5)

$$g(Z^*, \nabla_A Z^*) = \tfrac{1}{2} A\, g(Z^*, Z^*) = 0 \tag{11}$$

because $g(Z^*, Z^*)$ is constant on N_p. We shall now prove by induction that

$$g\big(X^*, \theta(X^*)^n(Y^*)\big)_{\operatorname{Exp} tX} = 0 \tag{12}$$

for each integer $n > 0$. For $n = 1$, the equation (12) follows from the fact that the torsion is 0, so

$$[X^*, Y^*]_{\operatorname{Exp} tX} = \big(\nabla_{X^*}(Y^*)\big)_{\operatorname{Exp} tX} - \big(\nabla_{Y^*}(X^*)\big)_{\operatorname{Exp} tX} = -\big(\nabla_{Y^*}(X^*)\big)_{\operatorname{Exp} tX};$$

hence
$$g\big(X^*, [X^*, Y^*]\big)_{\operatorname{Exp} tX} = -g\big(X^*, \nabla_{Y^*}(X^*)\big)_{\operatorname{Exp} tX} = 0\,.$$

Assuming (12) for an integer $n > 0$, we have

$$g_{\operatorname{Exp} tX}\big(X^*, [X^*, \theta(X^*)^n(Y^*)]\big) = g_{\operatorname{Exp} tX}\big(X^*, \nabla_{X^*}\big(\theta(X^*)^n(Y^*)\big)\big)$$

due to (11) and the vanishing of the torsion. Using (5) again, we obtain

$$\begin{aligned} & g\big(X^*, \nabla_{X^*}\big(\theta(X^*)^n(Y^*)\big)\big)_{\operatorname{Exp} tX} \\ &\quad = \big(X^* \cdot g\big(X^*, \theta(X^*)^n(Y^*)\big)\big)_{\operatorname{Exp} tX} - g\big(\nabla_{X^*}(X^*), \theta(X^*)^n(Y^*)\big)_{\operatorname{Exp} tX}\,. \end{aligned}$$

Both terms on the right hand side vanish due to (10) and the induction hypothesis (12). This proves that

$$g\big(X^*, \theta(X^*)^{n+1}(Y^*)\big)_{\operatorname{Exp} tX} = 0\,,$$

so (12) holds for all $n > 0$. Using Theorem 1 and (12) it follows that

$$g_{\operatorname{Exp} tX}\big(d \operatorname{Exp}_{tX}(X), d \operatorname{Exp}_{tX}(Y)\big) = g(X^*, Y^*)_{\operatorname{Exp} tX} = g_p(X, Y) \tag{13}$$

if t is sufficiently small. Now if M is an analytic manifold with an analytic, complete affine connection, the mapping Exp_p is an analytic mapping of M_p into M. It follows that the left hand side of (13) is analytic in t, hence constant, so Theorem 2 is proved.

Corollary 1. *If $Y \in M_p$ is a null vector of $d \operatorname{Exp}_X$, then $g_p(X, Y) = 0$.*

Corollary 2. *Let S_r denote the „sphere" $g_p(X, X) = r^2$ in the tangent space M_p (g_p is not necessarily positive definite). Let $\mathbf{S}_r = \operatorname{Exp}_p S_r$ and assume that the mapping Exp_p is regular on S_r. Then each geodesic emanating from p intersects $\mathbf{S}_r$ orthogonally.*

In fact, each straight line in M_p through p intersects the sphere S_r orthogonally with respect to the inner product g_p. Since the tangent space to S_r at X is mapped *onto* the tangent space to $\mathbf{S}_r$ at $\operatorname{Exp}_p X$, the corollary follows from Theorem 2.

III. Gaussian curvature and the Riemannian curvature tensor.

Let M be a C^∞ manifold with an affine connection ∇. The curvature tensor R is obviously skew symmetric:

$$R(X,Y) = -R(Y,X), \qquad X,Y \in \mathfrak{D}^1(M). \tag{14}$$

If ∇ is torsion-free, the Bianchi identity

$$R(X,Y)\cdot Z + R(Y,Z)\cdot X + R(Z,X)\cdot Y = 0, \qquad X,Y,Z \in \mathfrak{D}^1(M),$$

is easily verified. If ∇ is the affine connection arising from a pseudo-riemannian structure g, then

$$g\big(R(X,Y)Z,T\big) = -g\big(R(X,Y)T,Z\big) \tag{15}$$

for $X,Y,Z,T \in \mathfrak{D}^1(M)$. This well known skew symmetry is verified as follows: It suffices to verify (15) at each point $p \in M$; hence we can assume that M is a normal neighborhood of p and that the vector fields are adapted to their value at p. From (6) we find in this case

$$g\big(\nabla_X(Z),Z\big) = 0\,.$$

Using (5) and the relation $\big(\nabla_T(Z)\big)_p = 0$ it follows that

$$\begin{aligned} g_p\big(R_p(X,Y)Z,Z\big) &= g_p(\nabla_X\nabla_Y Z,Z) - g_p(\nabla_Y\nabla_X Z,Z) \\ &= X_p g(\nabla_Y Z,Z) - Y_p g(\nabla_X Z,Z) = 0 \end{aligned}$$

which is equivalent to (15).

Let F be a Riemannian manifold of dimension 2 and let p be a point in F. Let $B_r(0)$ denote the open disk in the tangent plane F_p with center 0 and radius r. If r is sufficiently small, the mapping Exp_p is a diffeomorphism of $B_r(0)$ onto the open disk $\boldsymbol{B}_r(p)$ in F, consisting of all points $q \in F$ whose distance from p is less than r. Let $A_0(r)$ and $A(r)$ denote the areas of $B_r(0)$ and $\boldsymbol{B}_r(p)$ respectively. The Gaussian curvature of F at p is defined as the limit

$$K = \lim_{r\to 0} 12\,\frac{A_0(r) - A(r)}{r^2 A_0(r)}\,. \tag{16}$$

As is well known from the differential geometry of surfaces, this is equivalent to Gauss' original definition in terms of principal curvatures. The existence of the limit above is contained in the following lemma which at the same time facilitates the computation of K.

Lemma 2. *Let f denote the Radon–Nikodym derivative of* Exp_p *on* $B_r(0)$ *(that is, f is the ratio of the corresponding surface elements on* $\boldsymbol{B}_r(p)$ *and* $B_r(0)$*). Then*

$$K = -\tfrac{3}{2}(\Delta f)_0\,,$$

where Δ is the Laplacian on the metric vector space F_p, $\Delta = \partial^2/\partial x_1{}^2 + \partial^2/\partial x_2{}^2$, x_1 and x_2 being coordinates with respect to some orthonormal basis.

The proof is an immediate application of Taylor's formula, see [4].

Let M be a Riemannian manifold with Riemannian structure g and curvature tensor R. Let p be a point in M and let Exp stand for Exp_p. Let N_0 denote a normal neighborhood of 0 in M_p and let $N_p = \mathrm{Exp}\, N_0$. Let S be a two-dimensional vector subspace of M_p. Then $\mathrm{Exp}\,(N_0 \cap S)$ is a connected submanifold M_S of M of dimension 2 and has a Riemannian structure induced by that of M. The Gaussian curvature $K(S)$ of M_S at p is called the *sectional curvature* of M at p along the *plane section* S. Using Theorem 1 we shall now prove the classical formula

$$K(S) = -\frac{g_p\big(R_p(Y,Z)Y,Z\big)}{|Y \vee Z|^2}, \tag{17}$$

where Y and Z are any linearly independent vectors in S; $Y \vee Z$ denotes the parallelogram spanned by these vectors and $|Y \vee Z|$ denotes the area. In order to apply Theorem 1 we shall first assume that M and g are analytic. We also assume temporarily that the vectors Y and Z in S are orthonormal. Let $X_1, \ldots, X_m$ be an orthonormal basis of M_p such that $X_1 = Y$ and $X_2 = Z$. Then each $X \in S$ can be written $X = x_1X_1 + x_2X_2$, $x_1, x_2 \in \boldsymbol{R}$, and the Laplacian on S is $\Delta = \partial^2/\partial x_1{}^2 + \partial^2/\partial x_2{}^2$. A curve in the manifold M_S has the same length regardless whether it is measured by means of the Riemannian structure on M or by means of the induced structure on M_S. If $q \in M_S$, the unique geodesic in N_p from p to q is the shortest curve in M_S joining p and q. It follows that the Exponential mappings at p for M and M_S respectively coincide on $S \cap N_0$. Let $X_1{}^*, \ldots, X_m{}^*, X^*$ denote the vector fields on N_p adapted to the tangent vectors $X_1, \ldots, X_m$. If $X \in S \cap N_0$, we put

$$v_1 = \mathrm{Exp}_X(X_1), \qquad v_2 = \mathrm{Exp}_X(X_2)$$

and define the functions $c_{ij}{}^k$ on N_p by

$$[X_i{}^*, X_j{}^*] = \sum_{k=1}^{m} c_{ij}{}^k X_k{}^*, \qquad 1 \leqq i,\, j \leqq m\,. \tag{18}$$

The mapping $\mathrm{Exp}\,(x_1X_1 + x_2X_2) \to (x_1, x_2)$ is a system of coordinates on the manifold M_S and the vectors v_1 and v_2 are tangent vectors to M_S. They can therefore be expressed

$$v_1 = \sum_{i=1}^{m} f_i X_i{}^*, \qquad v_2 = \sum_{j=1}^{m} g_j X_j{}^*\,,$$

where f_i, g_j are analytic functions of x_1 and x_2. For sufficiently small x_1 and x_2, these functions can be determined by Theorem 1. In fact, we have

$$v_1 = X_1^* - \tfrac{1}{2}[X^*, X_1^*] + \tfrac{1}{6}[X^*, [X^*, X_1^*]] - \ldots, \tag{19}$$

$$v_2 = X_2^* - \tfrac{1}{2}[X^*, X_1^*] + \tfrac{1}{6}[X^*, [X^*, X_2^*]] - \ldots. \tag{20}$$

The function f is the ratio of the surface elements in $S \cap N_0$ and M_S. Hence

$$f(X) = \frac{|v_1 \vee v_2|}{|X_1 \vee X_2|} = |v_1 \vee v_2| .$$

The projection of the parallelogram $v_1 \vee v_2$ into the $(X_i^*, X_j^*)_{\mathrm{Exp}\, X}$-plane has area

$$|f_i g_j - f_j g_i| \, |X_i^* \vee X_j^*| = |f_i g_j - f_j g_i| .$$

It follows that

$$|v_1 \vee v_2|^2 = \sum_{i>j} (f_i g_j - f_j g_i)^2 .$$

Let this quantity be denoted by F. The relation $f = F^{\frac{1}{2}}$ implies

$$2f\Delta f = \Delta F - \frac{1}{2f^2}\left\{\left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2\right\}.$$

This expression has to be evaluated for $(x_1, x_2) = (0,0)$. Since the torsion vanishes, we have

$$[X_i^*, X_j^*] = \nabla_{X_i^*}(X_j^*) - \nabla_{X_j^*}(X_i^*)$$

so, by (10), the functions c_{ij}^k vanish at p. From (19) and (20) we obtain expansions for the functions f_i and g_j

$$f_i = \delta_{1i} - \tfrac{1}{2}x_2 c_{21}^{\,i} + \tfrac{1}{6}x_1 x_2 (X_1^* c_{21}^{\,i}) + \tfrac{1}{6}x_2^2 (X_2^* c_{21}^{\,i}) + \ldots,$$
$$g_j = \delta_{2j} - \tfrac{1}{2}x_1 c_{12}^{\,j} + \tfrac{1}{6}x_1 x_2 (X_2^* c_{12}^{\,j}) + \tfrac{1}{6}x_1^2 (X_1^* c_{12}^{\,j}) + \ldots,$$

where δ_{ij} is Kronecker's delta and the terms which are not written vanish for $(x_1, x_2) = (0,0)$ of higher than second order. It follows that $\partial F/\partial x_1$ and $\partial F/\partial x_2$ vanish for $(x_1, x_2) = (0,0)$ so

$$2(\Delta f)_0 = (\Delta F)_0 = \left(\Delta (f_1 g_2)^2\right)_0 .$$

Omitting again terms of higher than second order we have

$$(f_1 g_2)^2 = 1 - x_1 c_{12}^{\,2} - x_2 c_{21}^{\,1} + \tfrac{1}{3}x_1^2 (X_1^* c_{12}^{\,2}) + \tfrac{1}{3}x_2^2 (X_2^* c_{21}^{\,1}) + \\ + \tfrac{1}{3}x_1 x_2 (X_1^* c_{21}^{\,1} + X_2^* c_{12}^{\,2}) .$$

Since

$$X_1 c_{12}^{\,2} = (X_1^* c_{12}^{\,2})_0 = (\partial/\partial x_1 \, c_{12}^{\,2})_0 \quad \text{etc.},$$

we obtain

$$2(\Delta f)_0 = -\tfrac{4}{3}(X_1 c_{12}^{\,2} + X_2 c_{21}^{\,1}) ,$$

so due to Lemma 2

$$K(S) = X_1 g([X_1^*, X_2^*], X_2^*) + X_2 g([X_2^*, X_1^*], X_1^*) \,. \tag{21}$$

On the other hand, using the relation $[X_i^*, X_j^*]_p = 0$ together with (5) and (10), we find

$$\begin{aligned} -g_p\big(R_p(Y,Z)Y,Z\big) &= g_p(\nabla_{X_2^*}\nabla_{X_1^*}X_1^*, X_2^*) - g_p(\nabla_{X_1^*}\nabla_{X_2^*}X_1^*, X_2^*) \\ &= X_2 g(\nabla_{X_1^*}X_1^*, X_2^*) - X_1 g(\nabla_{X_2^*}X_1^*, X_2^*) \,, \end{aligned}$$

which by (6) equals the right hand side of (21). This proves (17) in the analytic case when Y and Z are orthonormal. If Y and Z are any linearly independent vectors in S we can write $A = y_1 Y + z_1 Z$, $B = y_2 Y + z_2 Z$ where A and B are orthonormal vectors in S. Using the skew symmetries (14) and (15) we find

$$\begin{aligned} K(S) &= -g_p\big(R_p(A,B)A,\, B\big) \\ &= -g_p\big(R_p(y_1 Y + z_1 Z, y_2 Y + z_2 Z)(y_1 Y + z_1 Z), y_2 Y + z_2 Z\big) \\ &= -(y_1 z_2 - y_2 z_1)^2 g_p\big(R_p(Y,Z)Y,\, Z\big) = -\frac{g_p\big(R_p(Y,Z)Y,\, Z\big)}{|Y \vee Z|^2} . \end{aligned}$$

Finally, since both sides of the formula (17) only depend on g in an arbitrary small neighborhood of p, one can derive (17) in the C^∞ case by approximating the C^∞ Riemannian structure g by analytic Riemannian structures g_n for which (17) holds.

5. Pseudo-riemannian manifolds of constant curvature.

The definition (16) has no meaning in the general pseudo-riemannian case when g is no longer positive definite. However, using the formula (17) as a motivation, we can give a definition of sectional curvature which applies to all cases.

DEFINITION.. Let M be a C^∞ manifold with pseudo-riemannian structure g and curvature tensor R. Let p be a point in M. Let S be a two-dimensional subspace of the tangent space M_p such that the restriction of g_p to S is non-degenerate. The sectional curvature $K(S)$ of M at p along the section S is defined by

$$K(S) = -\frac{g_p\big(R_p(X,Y)X,\, Y\big)}{g_p(X,X)\, g_p(Y,Y) - g_p(X,Y)^2} \tag{22}$$

X and Y being any linearly independent vectors in S.

Some remarks are necessary in order to make the definition legitimate. We first observe that in the Riemannian case,

$$|X \vee Y|^2 = g_p(X,X)\, g_p(Y,Y) - g_p(X,Y)^2 ,$$

so the definition is then equivalent to the previous one. Secondly, the fact that g_p is non-degenerate on S insures that the denominator in (22) is not zero. Finally, the skew symmetries (14) and (15) imply that the right hand side of (22) is independent of the choice of X and Y in S.

Suppose for a moment, that, for each $p \in M$, $K(S)$ is a constant K independent of p and S (assuming of course that g_p is non-degenerate on S). Then the relation

$$g_p\big(R_p(X,Y)X,Y\big) = K\big(g_p(X,Y)^2 - g_p(X,X)g_p(Y,Y)\big) \tag{23}$$

holds for all p and all pairs $(X,Y) \in M_p$ which span a two-dimensional subspace on which g_p is non-degenerate. But then, by continuity, (23) holds for all $X,Y \in M_p$. In this case we say that the pseudo-riemannian manifold has constant curvature.

Consider the quadri-linear function on $M_p \times M_p \times M_p \times M_p$,

$$B(X,Y,Z,T) = g_p\big(R_p(X,Y)Z,T\big) - K\{g_p(X,T)g_p(Y,Z) - g_p(X,Z)g_p(Y,T)\} .$$

The function B satisfies the identities

$$B(X,Y,Z,T) = -B(Y,X,Z,T), \quad B(X,Y,Z,T) = -B(X,Y,T,Z)$$
$$B(X,Y,Z,T) + B(Y,Z,X,T) + B(Z,X,Y,T) = 0$$
$$B(X,Y,X,Y) = 0$$

It is elementary to verify that these four identities imply that B is identically 0. Consequently, the relation

$$g\big(R(X,Y)Z,T\big) = K\{g(X,T)\, g(Y,Z) - g(X,Z)\, g(Y,T)\} , \tag{24}$$

for $X,Y,Z,T \in \mathfrak{D}^1(M)$, is a necessary and sufficient condition for the pseudo-riemannian manifold M to have constant sectional curvature K.

Now let K be any constant and let p and q be two non-negative integers such that $p+q>1$. We shall construct a pseudo-riemannian manifold whose pseudo-riemannian structure has signature (p,q) (that is p plus signs, q minus signs) and sectional curvature K. The special case $p=1$ is considered in detail in [5]. Since the general case[1] offers no additional difficulties, the proofs involved are only outlined.

Consider the quadratic form

$$Q(X) = Q(x_1, \ldots, x_{p+q+1}) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q+1} x_j^2$$

[1] Added in proof: The more general question was raised in a letter by Dr. J. Wolf; in his forthcoming paper [7] the global classification problem is studied.

and let $\boldsymbol{O}(p,q+1)$ denote the group of linear transformations leaving Q invariant. The subgroup of $\boldsymbol{O}(p,q+1)$, which leaves the point $p_0=(0,\ldots,0,1)\in\boldsymbol{R}^{p+q+1}$ fixed, can then be identified with the group $\boldsymbol{O}(p,q)$. The coset space $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$ is diffeomorphic, by means of the mapping $\psi:x\boldsymbol{O}(p,q)\to x\cdot p_0$, $x\in\boldsymbol{O}(p,q+1)$, with the orbit of the point p_0 under $\boldsymbol{O}(p,q+1)$. This orbit is the quadric $Q(X)+1=0$. The Lie algebra $\mathfrak{o}(p,q+1)$ of the group $\boldsymbol{O}(p,q+1)$ can be identified with the Lie algebra of all real square matrices of the form

$$\begin{pmatrix} X_1 & X_2 \\ {}^tX_2 & X_3 \end{pmatrix}, \tag{25}$$

where X_2 is an arbitrary matrix of p rows and $q+1$ columns, ${}^tX_2$ denotes the transpose of X_2 and X_1 and X_3 are skew symmetric matrices of orders p and $q+1$ respectively. The Lie algebra $\mathfrak{o}(p,q)$ of the subgroup $\boldsymbol{O}(p,q)$ is obtained by replacing all elements in the last row and the last column in (25) by zeros. The Killing form $B(X,Y)=\mathrm{Tr}(\mathrm{ad}\,X\cdot\mathrm{ad}\,Y)$ of $\mathfrak{o}(p,q+1)$ is given by

$$B(X,Y) = (p+q-1)\,\mathrm{Tr}(XY)\,.$$

It follows that B is non-degenerate, so the Lie algebra of $\mathfrak{o}(p,q+1)$ is semi-simple.

Let s_0 denote the linear transformation of $\boldsymbol{R}^{p+q+1}$ given by

$$s_0\colon\ (x_1,\ldots,x_{p+q},x_{p+q+1}) \to (-x_1,\ldots,-x_{p+q},x_{p+q+1})\,.$$

Then the mapping $\sigma\colon x\to s_0xs_0$, $x\in\boldsymbol{O}(p,q+1)$, is an involutive automorphism of $\boldsymbol{O}(p,q+1)$. The corresponding automorphism of the Lie algebra $\mathfrak{o}(p,q+1)$ is $d\sigma\colon X\to s_0Xs_0$, and it is easy to see that $\mathfrak{o}(p,q)$ is the set of all fixed points of $d\sigma$. Let $\mathfrak{m}$ denote the eigenspace for the eigenvalue -1 of $d\sigma$. Let as usual E_{kl} denote the matrix

$$E_{kl} = (\delta_{mk}\delta_{nl})_{1\leqq m\leqq p+q+1,\ 1\leqq n\leqq p+q+1}\,.$$

Then $\mathfrak{m}$ has a basis consisting of the matrices

$$\begin{aligned} Z_i &= E_{ip+q+1}+E_{p+q+1i}, & 1 &\leqq i \leqq p; \\ Z_j &= E_{jp+q+1}-E_{p+q+1j}, & p+1 &\leqq j \leqq p+q\,. \end{aligned}$$

In the formulas below, the range of the indices i, j is $1\leqq i\leqq p$, $p+1\leqq j\leqq p+q$. Then

$$\begin{aligned} B(Z_i,Z_i) &= 2(p+q-1)\,, \\ B(Z_j,Z_j) &= -2(p+q-1)\,. \end{aligned}$$

Therefore, the bilinear form $B^*=B/2(p+q-1)$ on $\mathfrak{m}\times\mathfrak{m}$ is non-degenerate and has signature (p,q). Let π denote the natural projection of

$\boldsymbol{O}(p,q+1)$ onto $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$. The differential $(d\pi)_e$ induces an isomorphism of $\mathfrak{m}$ onto the tangent space to $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$ at $\pi(e)$; for simplicity in notation we identify $\mathfrak{m}$ with this tangent space. There exists a unique pseudo-riemannian structure g on $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$ invariant under the action of $\boldsymbol{O}(p,q+1)$ such that $g_{\pi(e)}=B^*$.

Since the space $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$ is symmetric, the curvature tensor R at the point $\pi(e)$ is given by (Nomizu [6])

$$R(X,Y)\cdot Z = -[[X,Y],Z]$$

for $X,Y,Z \in \mathfrak{m}$. Let $X=\sum x_i Z_i+\sum x_j Z_j$, $Y=\sum y_i Z_i+\sum y_j Z_j$. Then

$$\begin{aligned} B^*\big(R(X,Y)X,Y\big) &= -\tfrac{1}{2}\operatorname{Tr}\big(([[X,Y],X])\,Y\big) \\ &= -\tfrac{1}{2}\operatorname{Tr}(XYXY-YXXY-XXYY+XYXY) \\ &= \operatorname{Tr}(XXYY-XYXY)\,. \end{aligned}$$

A simple computation shows that

$$\begin{aligned} \operatorname{Tr}(XYXY) &= \sum (x_i y_i)^2+\sum (x_j y_j)^2+(\sum x_i y_i-\sum x_j y_j)^2\,, \\ \operatorname{Tr}(XXYY) &= \sum (x_i y_i)^2+\sum (x_j y_j)^2+\big(\sum x_i^2-\sum x_j^2\big)\big(\sum y_i^2-\sum y_j^2\big)\,, \end{aligned}$$

so

$$B^*\big(R(X,Y)X,Y\big) = B^*(X,X)B^*(Y,Y)-B^*(X,Y)^2\,.$$

In view of (23) this shows that the space $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$ has sectional curvature -1 for all sections through the point $\pi(e)$. Since the space is homogeneous, it has constant curvature -1.

On the other hand, the mapping ψ above is a diffeomorphism of $\boldsymbol{O}(p,q+1)/\boldsymbol{O}(p,q)$ onto the quadric $Q(X)+1=0$. The quadratic form Q on $\boldsymbol{R}^{p+q+1}$ induces a pseudo-riemannian structure on this quadric, invariant under the action of the transitive group $\boldsymbol{O}(p,q+1)$. We shall now prove that the mapping ψ is an isometry; due to the homogeneity, it suffices to verify that the differential $d\psi_{\pi(e)}$ is an isometry of $\mathfrak{m}$ onto the tangent plane $x_{p+q+1}=1$ to the quadric $Q(X)+1=0$ at the point p_0. Since $\psi\big((\exp tX)\,\boldsymbol{O}(p,q)\big)=\exp tX\cdot p_0$, it follows that $d\psi_{\pi(e)}(X)=X\cdot p_0$ for $X \in \mathfrak{m}$. Consequently

$$d\psi_{\pi(e)}(Z_n) = (\delta_{1n},\ldots,\delta_{p+q+1n}) \qquad \text{for} \quad 1 \leqq n \leqq p+q\,,$$

which shows that

$$Q\big(d\psi_{\pi(e)}(X)\big) = B^*(X,X)$$

and ψ is an isometry.

The preceding discussion is summarized in the next theorem.

THEOREM 3. *Let p and q be two non-negative integers such that $p+q>1$. Let V be a vector space over $\boldsymbol{R}$ of dimension $p+q+1$.*

I. *Let* Q^- *be a quadratic form on* V *with signature* $(p, q+1)$ *and let* X_- *be some fixed vector in* V *for which* $Q^-(X_-) < 0$. *The quadric*

$$Q^-(X) = Q^-(X_-)$$

has a pseudo-riemannian structure of signature (p, q) *induced by* Q^-. *This pseudo-riemannian manifold is complete, symmetric and has constant sectional curvature* < 0.

II. *Let* Q^+ *be a quadratic form on* V *with signature* $(p+1, q)$ *and let* X_+ *be some fixed vector in* V *for which* $Q^+(X_+) > 0$. *The quadric*

$$Q^+(X) = Q^+(X_+)$$

has a pseudo-riemannian structure of signature (p, q) *induced by* Q^+. *This pseudo-riemannian manifold is complete, symmetric and has constant sectional curvature* > 0.

III. *The quadrics in* I *and* II *exhaust the class of pseudo-riemannian manifolds of constant curvature* $\neq 0$ *up to local isometry.*

The first part is already proved. The second follows from I applied to the quadratic form $-Q^+$. Concerning III, let M be a pseudo-riemannian manifold of constant curvature. It follows from (24) that the curvature tensor R is invariant under parallelism, so M is a locally symmetric space. The pseudo-riemannian connection on M is therefore determined locally by the value of R at a given point (Cartan [2, p. 237], Nomizu [6, p. 63]). A diffeomorphism φ leaving a pseudo-riemannian connection invariant is an isometry, provided $d\varphi_p$ is an isometry for some point p. It follows that for each $p \in M$ there exists an open neighborhood N_p, isometric to an open set of a quadric as defined in I or II. This concludes the proof.

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