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HYPOELLIPTIC CONVOLUTION EQUATIONS

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We shall study convolution equations of the form

$$S * u = f \tag{1}$$

where $S \in \mathscr{E}'(R^n)$, the space of distributions with compact support, u and $f \in \mathscr{D}'(R^n)$, the space of all distributions.

Definition. The convolution equation (1) (and the distribution S) are called hypoelliptic if all solutions $u \in \mathscr{D}'(R^n)$ of (1) are in fact in $C^\infty(R^n)$ when $f \in C^\infty(R^n)$.

It follows immediately from the definition that no $S \in C_0^\infty$ is hypoelliptic. More generally, if $S \in \mathscr{E}'$ and $\varphi \in C_0^\infty$, the distribution $S+\varphi$ is hypoelliptic if and only if S is hypoelliptic.

Ehrenpreis [1] has proved that S is hypoelliptic if and only if there are constants B_1 and M_1 such that

$$|\hat{S}(\xi)| \geqq |\xi|^{-B_1}, \qquad |\xi| \geqq M_1, \quad \xi \in R^n , \tag{2}$$

and

$$|\mathrm{Im}\,\zeta|/\log|\zeta| \to \infty \quad \text{if} \quad |\zeta| \to \infty \text{ in } C^n, \quad \hat{S}(\zeta) = 0 . \tag{3}$$

Here $\hat{S}$ denotes the Fourier–Laplace transform defined by

$$\hat{S}(\zeta) = S\big(e^{-i\langle x, \zeta\rangle}\big) ,$$

where the distribution S operates on the variable x, and

$$\langle x,\zeta\rangle = x_1\zeta_1 + \ldots + x_n\zeta_n .$$

To prove the sufficiency of this condition, Ehrenpreis constructed a fundamental solution and proved that it is infinitely differentiable outside a compact set. However, he did not make any detailed study of the size of the set of singularities of the fundamental solution. Since this is

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important if one wants to obtain local results, we shall supply such a study here.

In what follows we use the notation K for the support of S and K_s for the singular support of S, that is, the smallest closed set outside which S is an infinitely differentiable function. The convex hull of any set A will be denoted by $\tilde{A}$. If A and B are subsets of R^n we set

$$A \pm B = \{x \pm y;\ x \in A,\ y \in B\};$$

it is clear that these sets are open if either A or B is open.

Our main result is the following theorem which will be proved at the end of the paper.

THEOREM 1. *Let S be hypoelliptic and Ω be an open set in R^n. If $u \in \mathscr{D}'(\Omega - K + \tilde{K}_s)$ and $f = S * u \in C^\infty(\Omega + \tilde{K}_s)$, it follows that $u \in C^\infty(\Omega)$.*

The main point in the proof is a precise estimate of $1/\hat{S}$ when $\hat{S}$ satisfies (2) and (3). This estimate is obtained from the following lemma[1].

LEMMA 1. *Given positive constants A, B and ε, we can find a constant N such that if u is harmonic when $x^2+y^2 < R^2$ and satisfies the inequalities*

$$u(x,0) \leqq 0, \qquad u(x,y) \geqq -a|y| - Br, \qquad x^2+y^2 < R^2, \tag{4}$$

it follows that

$$u(x,y) \leqq a|y| + \varepsilon r, \qquad x^2+y^2 < r^2, \tag{5}$$

provided that $0 < a < A$ and $0 < r < R/N$.

PROOF. Assume that the statement were false. We can then find a sequence of numbers a_n, R_n and r_n with $0 < a_n < A$ and $R_n/r_n > n$, and a sequence of functions u_n harmonic when $x^2+y^2 < R_n{}^2$ so that

$$u_n(x,0) \leqq 0, \qquad u_n(x,y) \geqq -a_n|y| - Br_n, \qquad x^2+y^2 < R_n{}^2,$$
$$u_n(x_n,y_n) \geqq a_n|y_n| + \varepsilon r_n,$$

for some (x_n, y_n) with $x_n{}^2 + y_n{}^2 \leqq r_n{}^2$. We now change variables, setting

$$v_n(x,y) = u_n(r_n x, r_n y)/r_n, \qquad x'_n = x_n/r_n, \quad y'_n = y_n/r_n.$$

Recalling that $R_n/r_n \geqq n$, we obtain

$$v_n(x,0) \leqq 0, \qquad v_n(x,y) \geqq -a_n|y| - B, \qquad x^2+y^2 \leqq n^2,$$
$$v_n(x'_n, y'_n) \geqq a_n|y'_n| + \varepsilon.$$

From Harnack's inequality it follows that the sequence v_n is also bounded from above on every compact set. Hence we can find a subsequence

[1]) I wish to thank Professor P. Malliavin for a helpful discussion which led to this lemma.

$v_{n'}$ which has a limit v harmonic in the whole plane so that the sequences $x'_{n'}$, $y'_{n'}$ and $a_{n'}$ have limits x_0, y_0 and a_0. Then we have

$$(6) \qquad v(x,0) \leqq 0, \qquad v(x,y) \geqq -a_0|y|-B, \qquad v(x_0,y_0) \geqq a_0|y_0|+\varepsilon .$$

From the first two of these inequalities it follows that v is a linear function of y, for v must be a harmonic polynomial and be bounded when y is bounded. The slope of the linear function cannot exceed a_0 in absolute value. But then the first and the last of the inequalities in (6) contradict each other, which proves the lemma.

Let H be the supporting function of K, that is,

$$(7) \qquad H(\eta) = \sup_{x \in K} \langle x,\eta \rangle, \qquad \eta \in R_n .$$

In view of the generalized Paley–Wiener theorem there exist constants B_2 and M_2 such that

$$(8) \qquad |\hat{S}(\zeta)| \leqq |\zeta|^{B_2} e^{H(\operatorname{Im}\zeta)}, \qquad |\zeta| \geqq M_2, \quad \zeta \in C^n .$$

We shall now prove a corresponding estimate of $1/\hat{S}$.

THEOREM 2. *Let S be hypoelliptic. To every positive m one can then find a constant C_m such that*

$$(9) \quad |1/\hat{S}(\zeta)| \leqq |\zeta|^{B_1+1} e^{H(-\operatorname{Im}\zeta)}, \quad \text{if} \quad |\operatorname{Im}\zeta| \leqq m \log|\zeta| \text{ and } |\zeta| \geqq C_m .$$

Here B_1 is the constant in (2).

PROOF. Throughout the proof we reserve the notation $\zeta=\xi+i\eta$ for points in the set $\{\zeta;\ |\operatorname{Im}\zeta|<m\log|\zeta|\}$. Note that $|\zeta|/|\xi| \to 1$ if $|\zeta| \to \infty$. We shall study the analytic function of one complex variable

$$F_\zeta(z) = \hat{S}(\xi+z\eta/|\eta|)$$

when $|z|<M\log|\xi|$, where M is a number depending on m but not on ζ which we shall fix later. For given M we have in view of (2) when $|\zeta|$ is large

$$(10) \qquad |F_\zeta(z)| \geqq (2|\xi|)^{-B_1}, \quad \text{if} \quad z \text{ is real and } |z| < M\log|\xi| .$$

Further, if we write $H(\eta/|\eta|)=\alpha+\beta$ and $H(-\eta/|\eta|)=\alpha-\beta$, we have in view of (8)

$$(11) \qquad |F_\zeta(z)| \leqq |2\xi|^{B_2} e^{\beta \operatorname{Im} z+\alpha|\operatorname{Im} z|}, \qquad |z| < M\log|\xi| .$$

We now introduce the function

$$u_\zeta(z) = \log\{e^{\beta \operatorname{Im} z}|2\xi|^{-B_1}|F_\zeta(z)|^{-1}\}$$

which, in view of (3), is harmonic when $|z|<M\log|\xi|$ provided that $|\zeta|$ is large. Using (10) and (11) we obtain

(12) $u_\zeta(z) \leqq 0,\quad$ if $\quad z$ is real and $|z| < M \log|\xi|$,

(13) $u_\zeta(z) \geqq -\alpha|\mathrm{Im}\, z| - (B_1+B_2)\log|2\xi|, \qquad |z| < M\log|\xi|$.

In order to apply Lemma 1 we introduce the constants

$$B = (B_1+B_2+1)/(m+1), \qquad A = \sup_{|\eta|=1} \tfrac{1}{2}\big(H(\eta)+H(-\eta)\big), \qquad \varepsilon = 1/2(m+1)\,.$$

If N is the constant in Lemma 1 we put $M = N(m+1)$. With $r = (m+1)\log|\xi|$ we have

$$(B_1+B_2)\log|2\xi| \leqq (B_1+B_2+1)\log|\xi| = Br$$

provided that $|\zeta|$ is large enough. Further, we have $\alpha \leqq A$. Hence Lemma 1 gives

(14) $u_\zeta(z) \leqq \alpha|\mathrm{Im}\, z| + \varepsilon(m+1)\log|\xi|, \qquad |z| \leqq r = (m+1)\log|\xi|$.

Since $|\eta| \leqq m\log|\zeta| \leqq (m+1)\log|\xi| = r$ if $|\zeta|$ is large enough, we may take $z = i|\eta|$ in (14), which gives

$$\tfrac{1}{2}(\log|\xi|) + \alpha|\eta| \geqq u_\zeta(i|\eta|) = \log[e^{\beta|\eta|}|2\xi|^{-B_1}|\hat{S}(\zeta)|^{-1}]\,,$$

hence

$$|1/\hat{S}(\zeta)| \leqq e^{(\alpha-\beta)|\eta|}|2\xi|^{B_1+\frac{1}{2}} \leqq e^{H(-\eta)}|\xi|^{B_1+1}$$

when $|\zeta|$ is large. This completes the proof of Theorem 2.

We now proceed to construct and study a fundamental solution of S, or rather a parametrix, which is sufficient for our purposes. Our arguments are essentially the same as those of Ehrenpreis [1] so we shall only indicate them briefly.

Let C be a constant $\geqq 1$ such that $\hat{S}(\xi) \neq 0$ if ξ is real and $|\xi| \geqq C$. According to (2), the function which is $= 1/\hat{S}(\xi)$ when $|\xi| \geqq C$ and $= 0$ elsewhere is temperate, hence is the Fourier transform of a temperate distribution F. The Fourier transform of $S * F$ is 1 when $|\xi| \geqq C$ and 0 elsewhere. Let ψ be the analytic function

$$\psi(x) = (2\pi)^{-n} \int_{|\xi| \leqq C} e^{i\langle x,\xi\rangle}\, d\xi\,.$$

We then have

(15) $$S * F + \psi = \delta\,,$$

where δ denotes the Dirac measure at the origin.

Theorem 3. *If S is hypoelliptic, the distribution F is in C^∞ outside $-\tilde{K}$.*

PROOF. Let μ be a positive integer so large that $2\mu > B_1 + 1 + n$, where n is the dimension and B_1 the constant in Theorem 2. Then we have with the differentiation made in the distribution sense

$$F = (-\Delta)^\mu (2\pi)^{-n} \int_{|\xi| \leqq C} e^{i\langle x, \xi\rangle} / [\hat{S}(\xi) |\xi|^{2\mu}] \, d\xi \,, \tag{16}$$

where the integral is absolutely convergent in virtue of Theorem 2. If $x_0 \notin -\tilde{K}$ we have to prove that F is infinitely differentiable in some neighborhood of x_0. Since $-x_0 \notin \tilde{K}$ we can find a real η so that $\langle -x_0, \eta\rangle > H(\eta)$. Multiplying η by a constant we may assume that $H(\eta) + \langle x_0, \eta\rangle < -2$. We shall now study the integral (16) when x is in the neighborhood U of x_0 defined by $U = \{\eta;\ H(\eta) + \langle x, \eta\rangle < -1\}$. In what follows we keep η fixed.

Let m be an arbitrary positive number. For every real ξ with $|\xi| \geqq \max(1, C_m)$ we denote by $t(\xi)$ the smallest positive number such that $t|\eta| = m \log|\xi - it\eta|$ and let Γ_m be the set of all points $\zeta = \xi - it(\xi)\eta$ thus obtained. Then we have when $x \in U$

$$F(x) \equiv (-\Delta)^\mu (2\pi)^{-n} \int_{\Gamma_m} e^{i\langle x, \zeta\rangle} \hat{S}(\zeta)^{-1} \langle \zeta, \zeta\rangle^{-\mu} \, d\zeta \,, \tag{17}$$

where the symbol $\equiv$ means that the two sides only differ by a Laplace integral over a compact set, hence an analytic function of x. In fact, it follows from Theorem 2 that the shift of the integration contour is legitimate. Now we have on Γ_m

$$|e^{i\langle x, \zeta\rangle} \hat{S}(\zeta)^{-1}| \leqq e^{t\langle x, \eta\rangle} |\zeta|^{B_1+1} e^{tH(\eta)} \leqq e^{-t} |\zeta|^{B_1+1} = |\zeta|^{B_1+1-m/|\eta|} \,.$$

Choosing m so large that $m/|\eta| > B_1 + 1 + n$ we can thus differentiate (17) under the integral sign and obtain

$$F(x) \equiv (2\pi)^{-n} \int_{\Gamma_m} e^{i\langle x, \zeta\rangle} \hat{S}(\zeta)^{-1} \, d\zeta, \qquad x \in U \,, \tag{18}$$

and by repeated differentiations we find that $F \in C^k(U)$ if $m/|\eta| > B_1 + 1 + n + k$. Since m may be chosen arbitrarily large, this shows that $F \in C^\infty(U)$. The proof is complete.

PROOF OF THEOREM 1. It is clearly enough to prove the theorem when $\tilde{K}_s$ is replaced by an arbitrary open convex set $\omega \supset \tilde{K}_s$. Let $\varphi \in C_0^\infty(\omega)$ be equal to 1 in a neighborhood of $\tilde{K}_s$ and write $S = S_1 + S_2$ where $S_1 = \varphi S$ has its support in ω and $S_2 = (1-\varphi)S \in C_0^\infty$. From a remark made after the definition of hypoellipticity it follows that S_1 is also hypoelliptic. Hence we can according to Theorem 3 find a distribution

F_1 which is in C^∞ outside a compact subset of $-\omega$ and a function $\psi_1 \in C^\infty$ so that $S_1 * F_1 + \psi_1 = \delta$. Let $\chi \in C_0^\infty(-\omega)$ be equal to 1 in a neighborhood of the singular support of F_1 and set $F'_1 = \chi F_1$. Then $F'_1 - F_1$ is in C^∞, hence $S_1 * F'_1 + \psi'_1 = \delta$ for some ψ'_1 in $C_0^\infty(K-\omega)$, and F_1' has its support in $-\omega$.

Now if $u \in \mathscr{D}'(\Omega - K + \omega)$ and $f = S * u \in C^\infty(\Omega + \omega)$, it follows that $f_1 = S_1 * u \in C^\infty(\Omega + \omega)$. Hence $u = \delta * u = (S_1 * F'_1 + \psi'_1) * u = F'_1 * f_1 + \psi'_1 * u$ is in $C^\infty(\Omega)$, which proves Theorem 1.

In particular it follows from Theorem 1 that Theorem 3 can be improved by replacing $-\tilde{K}$ by $-\tilde{K}_s$. In general, it is easy to see that the set $\tilde{K}_s$ in Theorem 1 can be replaced by a compact set K' if and only if the singular support of the parametrix F is contained in $-K'$. It would therefore be interesting to improve further the description of the singularities of F given by Theorem 3. However, the following theorem shows that our result is not less precise than the usual theorem of supports.

THEOREM 4. *Let S be hypoelliptic and F be a parametrix, that is, $S * F - \delta \in C^\infty(R_n)$. Then it follows that* $\operatorname{sing\,supp} S$ *and* $-\operatorname{sing\,supp} F$ *have the same convex hull.*

PROOF. Let as before K_s be the singular support of S and denote the singular support of F by L_s. From Theorem 1 it then follows that $\tilde{L}_s \subset -\tilde{K}_s$. Now take $\varphi \in C_0^\infty(R_n)$ equal to 1 in a neighborhood of L_s and set $F' = \varphi F$. Since $F' - F \in C_0^\infty$ we have $\operatorname{sing\,supp} F' = L_s$ and $S * F' - \delta = \psi \in C_0^\infty$. Hence

$$\hat{S}\hat{F}' = 1 + \hat{\psi}\,. \tag{19}$$

If A is an upper bound for $|x|$ when $x \in \operatorname{supp}\psi$, we have for an arbitrary integer N in virtue of the Paley–Wiener theorem

$$\hat{\psi}(\zeta) = O\big((1+|\zeta|)^{-N}\, e^{A|\operatorname{Im}\zeta|}\big)\,.$$

Hence $\hat{\psi}(\zeta) \to 0$ if $\zeta \to \infty$ while $|\operatorname{Im}\zeta|/\log|\zeta|$ remains bounded. Thus it follows from (19) that (2) and (3) are valid with $\hat{S}$ replaced by $\hat{F}'$, that is, F' is hypoelliptic. From the equation $S * F' - \delta = \psi \in C_0^\infty(R_n)$ we thus obtain by applying Theorem 1 to the hypoelliptic convolution operator F' that $\tilde{K}_s \subset -\tilde{L}_s$, which completes the proof.

REMARK. Note that the proof of Theorem 4 shows at the same time that the conditions (2) and (3) are in fact necessary for the existence of a parametrix of S with compact singular support.

REFERENCE

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