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A PROBLEM FOR ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS DEFINED ON AN ARBITRARY OPEN SET

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1. Introduction.

Let Ω be an arbitrary open set in real n-space R^n and denote by $H=H(\Omega)$ the Hilbert space of all square integrable functions over Ω. Let

$$\Delta = (\partial/\partial x_1)^2 + \ldots + (\partial/\partial x_n)^2$$

be Laplace's operator and $V=V(x_1, \ldots x_n)$ a real potential defined in Ω. Consider the Schrödinger operator

$$A = -\Delta + V$$

with domain of definition consisting of all functions f in H such that Af in the distribution sense is also a function in H. If $\Omega=R^n$ it was shown by Carleman that A is self-adjoint provided that V is continuous and bounded from below [2]. Recently this was generalized by Browder [1], who in our terminology replaced $-\Delta$ by a differential operator a with uniformly elliptic and positive principal part and with suitably bounded coefficients. He proved that

$$(a+V)^* = (\bar{a}+V) \tag{1.1}$$

provided that V is continuous and bounded from below and $\Omega=R^n$. In (1.1) both operators have their maximal domains of definition. The asterisk denotes the adjoint in H and $\bar{a}$ is the formal adjoint of a. In this paper a similar problem is treated. We shall replace $-\Delta$ by a differential operator a which is defined in Ω and has an elliptic and positive principal part there. The coefficients of a are assumed to possess continuous derivatives up to a certain order but need not be bounded in Ω and the ellipticity of a may be non-uniform. Then we shall show the following theorem:

THEOREM. (1.1) *holds for every continuous potential V that increases rapidly enough at the boundary of Ω.*

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The rate of increase of V depends of course on the operator a. If $\Omega = R^n$ and the ellipticity of a is uniform and if suitable conditions of boundedness are imposed on the coefficients of a one can obtain Browder's result in a less general form. This case is not treated here. Our theorem also holds when Ω is a differentiable manifold. The proof is similar and will not be given.

The proof of our theorem proceeds in two steps. The order of a is necessarily even, say $2m$. Denote by $D_\alpha f$ the derivatives of f and let $|\alpha|$ be the order of $D_\alpha f$. Put

$$|D^j f(x)|^2 = \sum_{|\alpha| \leqq j} |D_\alpha f(x)|^2 .$$

In the first step we show that

$$\int_\Omega \left(\varepsilon(x)\,|D^m f(x)|^2 + r(x)\,|D^{m-1} f(x)|^2\right) dx < \infty \tag{1.2}$$

for any f in the domain of $a + V$ og $\bar{a} + V$. Here $\varepsilon = \varepsilon(x)$ and $r = r(x)$ are positive functions, ε depending on the ellipticity of a and r on the potential V. An essential point is that r can be taken arbitrarily large provided that V is large enough. To prove the theorem it is sufficient to show that

$$\big((a + V)f, g\big) = \big(f, (\bar{a} + V)g\big) \tag{1.3}$$

for every f and g in the domains of $a + V$ and $\bar{a} + V$, respectively (i.e. the inclusion $(\bar{a} + V) \subset (a + V)^*$). The scalar product in (1.3) is that of H. To prove (1.3) we multiply the integrands of the two sides by a suitable infinitely differentiable function $\psi = \psi(x)$ which is equal to 1 on a large compact part of Ω and vanishes in a neighbourhood of the boundary of Ω. Forming the difference between the two integrals and integrating by parts one gets an expression which by (1.2) tends to zero as ψ tends to 1 in all of Ω. This is the second step.

In the proof of (1.2) we use a number of well known inequalities plus a new one which may be useful in various situations. Consider an integral of the form

$$L(f) = L_{m, m-1}(f, f) = \int_\Omega \sum l_{\alpha\beta}(\varphi)\, D_\alpha f \overline{D_\beta f}\, dx , \tag{1.4}$$

where the summation extends over $|\alpha| \leqq m$ and $|\beta| \leqq m - 1$. The function f is assumed to possess locally square integrable weak derivatives $D_\alpha f$ for all α such that $|\alpha| \leqq m$. About φ we assume: φ is infinitely differentiable with compact support in Ω and the function

$$\varphi^* = \frac{D_{\alpha 1}\varphi \ldots D_{\alpha k}\varphi}{\varphi^{k-1}}$$

has the same property for every integer $k \geqq 1$. The coefficients $l_{\alpha\beta}(\varphi)$ are assumed to be quadratic forms in the functions φ^* with coefficients in $C^{|\alpha|+|\beta|}(\Omega)$ where $C^k(\Omega)$ is the set of functions with continuous derivatives $D_\alpha f$ for all α such that $|\alpha| \leqq k$. Then, denoting by $\mathrm{St}\,\varphi$ the support of φ,

$$|L(f)| \leqq s\int |D^m \varphi f|^2\, dx + t\int_{\mathrm{St}\,\varphi} |f|^2\, dx + \tau\int |\varphi f|^2\, dx\,,$$

where s and t are positive numbers which may be choosen arbitrarily small provided that $\tau = \tau(s,t,\varphi,(l_{\alpha\beta}))$ is taken large enough.

2. Notations and remarks.

We consider a linear elliptic differential operator a of order $2m$ defined for every x in Ω by the expression

$$a = a(x,D) = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D_\alpha\,,$$

where $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_j)$, with $1 \leqq \alpha_k \leqq n$ and $|\alpha| = j \leqq 2m$. Furthermore

$$D_{\alpha_1} = \partial/i\partial x_{\alpha_1}, \qquad D_\alpha = D_{\alpha_1} \ldots D_{\alpha_j}\,.$$

The formal adjoint $\bar{a}$ of a is by definition

$$\bar{a} = \bar{a}(x,D) = \sum_{|\alpha| \leqq 2m} D_\alpha \overline{a_\alpha(x)}\,.$$

Since the operator is supposed to be elliptic

$$\mathrm{Re}\, a_0(x,\xi) = \mathrm{Re} \sum_{|\alpha|=2m} a_\alpha(x)\xi_\alpha, \qquad \xi_\alpha = \xi_{\alpha_1} \ldots \xi_{\alpha_{2m}}\,,$$

is a definite form in ξ for all x in Ω. Let it be positive. Then

$$\mathrm{Re}\, a_0(x,\xi) \geqq \varrho(x) \sum_{|\alpha|=m} \xi_\alpha^{\,2}\,,$$

where $\varrho(x) > 0$ on Ω. If the coefficients of a_0 are continuous, $\varrho(x)$ may be choosen to be continuous. Since $\mathrm{Re}\,\overline{a_0}(x,\xi) = \mathrm{Re}\, a_0(x,\xi)$ the same is true for $\bar{a}$.

By $C^k(\Omega)$ we shall denote, as before, the space of k times continuously differentiable functions in Ω. The space of infinitely differentiable functions in Ω is denoted by $C^\infty(\Omega)$, and $C_0^\infty(\Omega)$ is the set of functions in $C^\infty(\Omega)$ with compact supports. As in the introduction $H = H(\Omega)$ is the Hilbert space of all square integrable functions in Ω. The set of functions

f such that $D_\alpha f$ in the weak sense is square integrable on every compact subset of Ω for all $|\alpha| \leqq k$ is denoted by $\mathscr{H}^k(\Omega)$. The corresponding set of functions with compact supports is $\mathscr{H}_0{}^k(\Omega)$. It is well known that for elliptic operators A of order $2m$ with sufficiently differentiable coefficients in Ω, $Af=g$ implies that f is in $\mathscr{H}^{2m}(\Omega)$ if f and g are in $\mathscr{H}(\Omega)$, cf. [4]. We shall assume that a and $\bar{a}$ fulfill this regularity condition.

3. Some inequalities.

Let us introduce the following notation:

$$|D_\alpha f,S|^2 = \int_S |D_\alpha f(x)|^2\, dx ,$$

$$|D^k f(x)|^2 = \sum_{|\alpha|\leqq k} |D_\alpha f(x)|^2 ,$$

$$|D^k f, S|^2 = \int_S |D^k f(x)|^2\, dx .$$

If $S=\Omega$ we write simply $|D^k f|^2$ instead of $|D^k f,\Omega|^2$ etc.

We shall show that with a suitable countable locally finite covering of Ω by closed spheres (S_k) and a suitable partition of unity $1=\sum \varphi_k{}^2$ belonging to that covering the following inequalities are valid,

<u>(i)</u>:

$$\operatorname{Re}(af,f) \geqq \varepsilon_k |D^m f|^2 - \tau_k |f|^2$$

for all $f \in \mathscr{H}_0{}^{2m}(S_k)$. Here ε_k and τ_k are positive numbers that depend on the ellipticity and the coefficients of the operator a in S_k.

<u>(ii)</u>:

Let $f \in \mathscr{H}^m(\Omega)$ and denote by φ an arbitrary φ_k. If

$$L_{m,m-1}(f,f) = \int \sum_{\substack{|\alpha|\leqq m \\ |\beta|\leqq m-1}} l_{\alpha\beta}(\varphi)\, D_\alpha f\, \overline{D_\beta f}\, dx ,$$

where the functions $l_{\alpha\beta}(\varphi)$ are the same as in (1.4), then

$$|L_{m,m-1}(f,f)| \leqq s\,|D^m \varphi f|^2 + \tau(s,t,\varphi,l)\,|\varphi f|^2 + t|f,\mathrm{St}\,\varphi|^2 .$$

Here s and t are arbitrary positive numbers and $\tau(s,t,\varphi,l)$ is a positive number that depends on s, t, φ and l where l stands for the collection $(l_{\alpha\beta}(\varphi))$. $\mathrm{St}\,\varphi$ denotes the support of φ

<u>(iii)</u>:

For $f \in \mathscr{H}_0{}^m(\Omega)$ we have

$$|D^{m-1}f|^2 \leqq \varepsilon|D^m f|^2 + \tau(\varepsilon)|f|^2 ,$$

where $\varepsilon > 0$ is arbitrary and $\tau(\varepsilon)$ is a positive number such that $\tau(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

(iv):

Let $f \in \mathscr{H}^{2m}(\Omega)$. Then

$$\operatorname{Re}(\varphi_k{}^2 af,f) \geqq \varepsilon_k|\varphi_k D^m f|^2 + r_k|\varphi_k D^{m-1}f|^2 - t_k|f,\operatorname{St}\varphi_k|^2 - \tau(r_k,t_k)|\varphi_k f|^2 .$$

In this inequality ε_k is a positive number that depends on the ellipticity of the operator in S_k, r_k and t_k are arbitrary positive numbers and $\tau(r_k,t_k)$ is a positive number that depends on r_k and t_k.

(v):

If (τ_k) is a sequence of positive numbers there exists a function $\tau = \tau(x)$ in e.g. $C^\infty(\Omega)$ such that

$$\min_{x \in S_k} \tau(x) \geqq \tau_k .$$

Now we take $t_k = 2^{-k}$ in (iv) and majorize $\tau_k = \tau(r_k, 2^{-k})$ by $V_r = V(r,x)$ according to (v). The number r indicates the dependence on the sequence (r_k). This results in

(iv′):

$$\operatorname{Re}\big(\varphi_k{}^2(a+V_r)f,f\big) + 2^{-k}|f,\operatorname{St}\varphi_k|^2 \geqq \varepsilon_k|\varphi_k D^m f|^2 + r_k|\varphi_k D^{m-1}f|^2 .$$

If $(a+V_r)f \in H$ and $f \in H$ then, since $V_r \in C^0(\Omega)$, $f \in \mathscr{H}^{2m}(\Omega)$. This is a consequence of the regularity theorem for elliptic operators. From (iv′) and the fact that $|f,\operatorname{St}\varphi_k| \leqq |f|$ and $\sum_1^\infty 2^{-k} = 1$ it follows by Lebesgue's theorem that

(vi):

$$\operatorname{Re}\big((a+V_r)f,f\big) + |f|^2 \geqq \int_\Omega \varepsilon(x)\,|D^m f(x)|^2\,dx + \int_\Omega r(x)\,|D^{m-1}f(x)|^2\,dx ,$$

i.e.

$$\int_\Omega \varepsilon(x)\,|D^m f(x)|^2\,dx < \infty$$

and

$$\int_\Omega r(x)\,|D^{m-1}f(x)|^2\,dx < \infty ,$$

where $0 < \varepsilon(x) = \sum \varepsilon_k \varphi_k{}^2(x)$ and $0 < r(x) = \sum r_k \varphi_k{}^2(x)$. It is easy to see that the inequality (vi) is also true with the same $\varepsilon(x)$, $r(x)$ and V_r if $a + V_r$

is replaced by $\bar{a}+V_r$ provided that V_r is any continuous function $\geqq \max\big(V_r(a,x)$, where $V_r(\bar{a},x)\big)$ is, for example $V_r=V_r(a,x)+V_r(\bar{a},x)$. Here $V_r(a,x)$ refers to the potential constructed in (iv′).

4. Proof of the theorem.

We are now in a position to prove the inclusion $(a+V_r)^*\supset\bar{a}+V_r$ (the reverse one is trivial) for a suitable V_r. Therefore let (Ω_i) be a compact covering of Ω such that $\Omega_{i-1}\subset\operatorname{int}(\Omega_i)$ and introduce the function $\psi_i\in C_0^\infty(\Omega_i)$, $0\leqq\psi_i\leqq 1$ with $\psi_i=1$ on Ω_{i-1}. By the definition of $\bar{a}$,

$$\big(\psi_i(a+V_r)f,g\big) = \big(f,(\bar{a}+V_r)\psi_i g\big)$$

for every f and g in the domains of $a+V_r$ and $\bar{a}+V_r$ respectively. Since

$$\bar{a}\psi_i g = \psi_i\bar{a}g + \sum_{|\alpha|\leqq 2m-1} b_\alpha(\psi_i)\,D_\alpha g\ ,$$

where $b_\alpha(\psi_i)$ is a linear combination of derivatives of ψ_i, it follows that

$$\big(\psi_i(a+V_r)f,g\big) = \big(\psi_i f,(\bar{a}+V_r)g\big) + \sum_{|\alpha|\leqq 2m-1}\big(f,b_\alpha(\psi_i)D_\alpha g\big)\,.$$

The second term on the right hand side may be transformed by partial integrations into

$$L^{(i)}_{m,m-1}(f,g) = \sum_{\substack{|\alpha|\leqq m\\ |\beta|\leqq m-1}}\int_{\omega_i} l_{\alpha\beta}(\psi_i)\,D_\alpha f\,\overline{D_\beta g}\,dx$$

the integrations being performed over $\omega_i=\Omega_i-\Omega_{i-1}$. With

$$\sup_{\alpha,\beta,x}|l_{\alpha\beta}(\psi_i)| = C_i$$

we obtain

$$\begin{aligned}
|L^{(i)}_{m,m-1}(f,g)| &\leqq \sum_{\substack{|\alpha|\leqq m\\ |\beta|\leqq m-1}}\int_{\omega_i} C_i\,|D_\alpha f(x)|\,|\overline{D_\beta g(x)}|\,dx\\
&\leqq \sum_{\substack{|\alpha|\leqq m\\ |\beta|\leqq m-1}}\int_{\omega_i} \varepsilon(x)^{\frac12}\,|D_\alpha f(x)|\,\frac{C_i}{\varepsilon(x)^{\frac12}}\,|D_\beta g(x)|\,dx\\
&\leqq \sum_{\substack{|\alpha|\leqq m\\ |\beta|\leqq m-1}}\int_{\omega_i}\Big(\varepsilon(x)\,|D_\alpha f(x)|^2+\frac{C_i^2}{\varepsilon(x)}\,|D_\beta g(x)|^2\Big)\,dx\\
&\leqq d_{m-1}\int_{\omega_i}\varepsilon(x)\,|D^m f(x)|^2\,dx + d_m\int_{\omega_i}\frac{C_i^2}{\varepsilon'_i}\,|D^{m-1}g(x)|^2\,dx\ ,
\end{aligned}$$

where $\varepsilon'_i=\inf_{\omega_i}\varepsilon(x)>0$ and d_k is the number of derivatives of order

$\leqq k$. Let $\chi_i(x) = 1$ on ω_i and zero elsewhere be the characteristic function of ω_i and put

$$\chi(x) = \sum \frac{C_i^2}{\varepsilon'_i} \chi_i(x) .$$

This expression is well defined for every $x \in \Omega$ since the covering (ω_i) where $\omega_1 = \Omega_1$ is locally finite. The compactness of S_k then implies that only a finite number of the ω_i will meet S_k so that

$$\max_{x \in S_k} \chi(x) = C'_k < \infty .$$

If $r_k \geqq C'_k$ it follows that

$$\chi(x) = \sum \varphi_k^2(x)\, \chi(x) \leqq \sum \varphi_k^2(x)\, C'_k \leqq \sum \varphi_k^2(x)\, r_k = r(x) .$$

In combination with the fact that

$$\chi(x) = \frac{C_i^2}{\varepsilon'_i} \qquad \text{on} \quad \omega_i ,$$

this choice of $r(x)$ gives

$$|L^{(i)}_{m,m-1}(f,g)| \leqq d_{m-1} \int_{\omega_i} \varepsilon(x)\, |D^m f(x)|^2\, dx + d_m \int_{\omega_i} r(x)\, |D^{m-1} g(x)|^2\, dx .$$

If we let $i \to \infty$ it follows from (vi) that

$$\int_{\omega_i} \varepsilon(x)\, |D^m f(x)|^2\, dx \to 0 \qquad \text{and} \qquad \int_{\omega_i} r(x)\, |D^{m-1} g(x)|^2\, dx \to 0 ,$$

that is $L^{(i)}_{m,m-1}(f,g) \to 0$. Further $\psi_i \nearrow 1$ on all of Ω, and we get by Lebesgue's theorem

$$\big((a + V_r)f, g\big) = \big(f, (\bar{a} + V_r)g\big) .$$

This proves our assertion.

5. Partition of unity.

Let (S_k) be a countable locally finite covering of Ω by closed spheres S_k with radius r_k and center z_k such that $S_k \subset \Omega$ for all k. We shall assume that if S'_k is the closed sphere with center z_k and radius $\frac{1}{2} r_k$ the collection (S'_k) is also a covering of Ω. It is not difficult but somewhat tedious to show that such coverings exist and we do not go into the details. We are now going to construct a partition of unity belonging to the covering (S_k), that is, functions $\varphi_i(x)$ with the following properties:

$$(5.1) \qquad 0 \leqq \varphi_i(x) \in C_0^\infty(S_i)\,,$$

$$(5.2) \qquad 1 = \sum \varphi_i^2(x) \quad \text{on} \quad \Omega\,.$$

To do this put

$$\gamma_i(x) = \begin{cases} \exp\left\{-\dfrac{r_i^2}{r_i^2 - |x-z_i|^2}\right\} & \text{for} \quad |x-z_i| < r_i\,, \\ 0 & \text{for} \quad |x-z_i| \geqq r_i\,. \end{cases}$$

Then $\gamma_i(x) \in C_0^\infty(S_i)$ and since $\gamma_i > 0$ on S'_i and (S'_k) is a covering of Ω we have

$$\sum \gamma_i^2(x) > 0 \quad \text{on} \quad \Omega\,.$$

From this it follows that

$$(5.3) \qquad \varphi_i(x) = \frac{\gamma_i(x)}{\left(\sum \gamma_k^2(x)\right)^{\frac{1}{2}}} \in C_0^\infty(S_i)$$

and

$$1 = \sum \varphi_i^2(x) \quad \text{on} \quad \Omega\,.$$

It is easily seen that the function

$$(5.4) \qquad \begin{cases} \dfrac{D_{\alpha^1}\gamma_i\, D_{\alpha^2}\gamma_i \,\ldots\, D_{\alpha^k}\gamma_i}{\gamma_i^{k-1}} & \text{for} \quad |x-z_i| < r_i\,, \\ 0 & \text{for} \quad |x-z_i| \geqq r_i\,, \end{cases}$$

is in $C_0^\infty(S_i)$ for every integer $k \geqq 1$. Here $\alpha^1, \alpha^2, \alpha^3, \ldots \alpha^k$ are multi-indices. Because of (5.3) the same will be true if we replace γ_i by φ_i, i.e.

$$(5.5) \qquad \begin{cases} \dfrac{D_{\alpha^1}\varphi_i\, D_{\alpha^2}\varphi_i \,\ldots\, D_{\alpha^k}\varphi_i}{\varphi_i^{k-1}} & \text{for} \quad |x-z_i| < r_i\,, \\ 0 & \text{for} \quad |x-z_i| \geqq r_i\,. \end{cases}$$

is in $C_0^\infty(S_i)$. At last we point out that by squaring (5.5) and dividing by φ_i we obtain a new function of the same kind. We also remark that differentiation of (5.5) again leads to a function of the same kind.

6. Proof of the inequalities.

The inequalities in section 3 will now be proved one by one.

(i):

Since $\varrho(x)$ is a continuous function in Ω and > 0 it follows that $\inf_{S_k} \varrho(x) = \varrho_k > 0$, that is, the ellipticity is uniform in S_k. Gårding's inequality [3] then implies (i).

(ii):

From the considerations in section 5 it follows that if φ is any φ_i then the function

$$\begin{cases} \dfrac{l_{\alpha\beta}(\varphi)}{\varphi} & \text{for} \quad |x-z| < r, \\ 0 & \text{for} \quad |x-z| \geqq r, \end{cases}$$

is in $C_0^{|\alpha|+|\beta|}(S)$. This implies

$$L_{m,m-1}(f,f) = \int \sum_{\substack{|\alpha| \leqq m \\ |\beta| \leqq m-1}} \frac{l_{\alpha\beta}(\varphi)}{\varphi} \varphi D_\alpha f \overline{D_\beta f}\, dx$$

$$= \int \sum_{\substack{|\alpha| \leqq m \\ |\beta| \leqq m-1}} \frac{l_{\alpha\beta}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f}\, dx + \int \sum_{\substack{|\alpha| \leqq m-1 \\ |\beta| \leqq m-1}} l_{\alpha\beta}^{(1)}(\varphi)\, D_\alpha f \overline{D_\beta f}\, dx .$$

Since the function $l_{\alpha\beta}(\varphi)$ are quadratic forms in the functions (5.5) with coefficients in $C^{|\alpha|+|\beta|}(\Omega)$ the same is true for $l_{\alpha\beta}^{(1)}(\varphi)$ (this is easy to check). This implies that

$$\frac{l_{\alpha\beta}^{(1)}(\varphi)}{\varphi} \in C_0^{|\alpha|+|\beta|}(S) .$$

By iterating the procedure we obtain, writing $L_{m,m-1}$ for $L_{m,m-1}(f,f)$,

$$L_{m,m-1} = \int \sum_{\substack{|\alpha| \leqq m \\ |\beta| \leqq m-1}} \frac{l_{\alpha\beta}^{(0)}(\varphi)}{\varphi} D_\alpha \varphi f\, \overline{D_\beta f}\, dx + \int \sum_{\substack{|\alpha| \leqq m-1 \\ |\beta| \leqq m-1}} \frac{l_{\alpha\beta}^{(1)}(\varphi)}{\varphi} D_\alpha \varphi f\, \overline{D_\beta f}\, dx +$$

$$+ \ldots + \int \sum_{\substack{|\alpha| \leqq 0 \\ |\beta| \leqq m-1}} \frac{l_{\alpha\beta}^{(m)}(\varphi)}{\varphi} D_\alpha \varphi f\, \overline{D_\beta f}\, dx ,$$

where we have put $l_{\alpha\beta}(\varphi) = l_{\alpha\beta}^{(0)}(\varphi)$. From

$$\left| \frac{l_{\alpha\beta}^{(k)}(\varphi)}{\varphi} D_\alpha \varphi f \overline{D_\beta f} \right| \leqq \tfrac{1}{2} r |D_\alpha \varphi f|^2 + r^{-1} \left| \frac{l_{\alpha\beta}^{(k)}(\varphi)}{\varphi} D_\beta f \right|^2$$

for every $r > 0$ it follows that

$$|L_{m,m-1}| \leqq \tfrac{1}{2} r d_{m-1} |D^m \varphi f|^2 + r^{-1} \int \sum_{\substack{|\alpha| \leqq m \\ |\beta| \leqq m-1}} \left| \frac{l_{\alpha\beta}^{(0)}(\varphi)}{\varphi} \right|^2 |D_\beta f|^2\, dx +$$

$$+ \ldots + \int \sum_{\substack{|\alpha| \leqq 0 \\ |\beta| \leqq m-1}} \left| \frac{l_{\alpha\beta}^{(m)}(\varphi)}{\varphi} \right|^2 |D_\beta f|^2\, dx ,$$

that is,

$$(6.1)\qquad |L_{m,m-1}| \leqq \tfrac{1}{2} r d_{m-1} |D^m \varphi f|^2 + r^{-1} \int \sum_{|\beta| \leqq m-1} l_\beta(\varphi) |D_\beta f|^2 \, dx ,$$

where

$$l_\beta(\varphi) = \sum_{|\alpha| \leqq m} \left| \frac{l^{(0)}_{\alpha\beta}(\varphi)}{\varphi} \right|^2 + \sum_{|\alpha| \leqq m-1} \left| \frac{l^{(1)}_{\alpha\beta}(\varphi)}{\varphi} \right|^2 + \ldots + \sum_{|\alpha| \leqq 0} \left| \frac{l^{(m)}_{\alpha\beta}(\varphi)}{\varphi} \right|^2 .$$

We notice that $l_\beta(\varphi)$ is a quadratic form in the functions (5.5) with coefficients in $C^{|\beta|}(\Omega)$, and that the second term on the right hand side of (6.1) considered as a quadratic form in the derivatives $D_\beta f$ is of order $(m-1, m-1)$. We denote it by $L_{m-1,m-1}$. We have thus

$$(6.2)\qquad |L_{m,m-1}| \leqq \tfrac{1}{2} r d_{m-1} |D^m \varphi f|^2 + r^{-1} |L_{m-1,m-1}| ,$$

where d_k as before denotes the number of derivatives of order k. It is now natural to proceed in the same way with $L_{m-1,m-1}$. By partial integrations $L_{m-1,m-1}$ is transformed into $L_{m,m-2}$, all boundary terms disappearing since the coefficients in $L_{m-1,m-1}$ have compact supports. Then

$$|L_{m,m-2}| \leqq \tfrac{1}{2} r^2 d_{m-1} |D^m \varphi f|^2 + r^{-2} |L_{m-2,m-2}| ,$$

that is,

$$(6.3)\qquad |L_{m-1,m-1}| \leqq \tfrac{1}{2} r^2 d_{m-1} |D^m \varphi f|^2 + r^{-2} |L_{m-2,m-2}|$$

for all $r > 0$. Analogously

$$\begin{aligned} |L_{m-2,m-2}| &\leqq d_{m-2} |D^{m-1} \varphi f|^2 + |L_{m-3,m-3}| , \\ |L_{m-3,m-3}| &\leqq d_{m-3} |D^{m-2} \varphi f|^2 + |L_{m-4,m-4}| , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ |L_{1,1}| &\leqq d_1 |D^2 \varphi f|^2 + |L_{0,0}| . \end{aligned}$$

By adding these inequalities we obtain

$$(6.4)\qquad |L_{m-2,m-2}| \leqq (m-2) d_{m-2} |D^{m-1} \varphi f|^2 + |L_{0,0}| .$$

Inserting (6.3) into (6.2) and using (6.4) we obtain after trivial simplifications

$$(6.5)\qquad |L_{m,m-1}| \leqq m d_m (r |D^m \varphi f|^2 + r^{-3} |D^{m-1} \varphi f|^2) + r^{-3} |L_{0,0}| .$$

Here

$$L_{0,0} = \int m_0(\varphi) f \bar{f} \, dx = \int (m_0(\varphi)/\varphi) \, \varphi f \bar{f} \, dx ,$$

so that

$$(6.6)\qquad |L_{0,0}| \leqq \int (s^{-1} |\varphi f|^2 + s |m_0(\varphi)/\varphi|^2 \, |f|^2) \, dx \leqq s^{-1} |\varphi f|^2 + s C_\varphi |f, \mathrm{St}\,\varphi|^2 ,$$

where $C_\varphi = \max_{\mathrm{St}\,\varphi} |m_0(\varphi)/\varphi|^2$ and s arbitrary > 0. Taking $s = t r^3$ where t is arbitrary > 0 we get from (6.5) and (6.6)

(6.7) $|L_{m,m-1}| \leqq md_m(r|D^m\varphi f|^2+r^{-3}|D^{m-1}\varphi f|^2) + r^{-6}t^{-1}|\varphi f|^2 + tC\varphi|f,\mathrm{St}\varphi|^2 .$

According to (iii)

$$|D^{m-1}\varphi f|^2 \leqq r^4|D^m\varphi f|^2 + \tau(r^4)\,|\varphi f|^2 .$$

This inequality and (6.7) yield

(6.8) $|L_{m,m-1}| \leqq 2md_m r|D^m\varphi f|^2 + \big(r^{-3}\tau(r^4)md_m+r^{-6}t^{-1}\big)|\varphi f|^2 + tC\varphi|f,\mathrm{St}\varphi|^2 .$

The substitution

$$r \to (2md_m)^{-1}r, \qquad t \to C^{-1}t ,$$

in (6.8) gives (ii) and we are finished.

(iii):

This is a classical inequality and is proved by partial integrations and Schwarz's inequality or by a Fourier transformation (e.g. [3]).

(iv):

$$(\varphi_k{}^2af,f) = (a\varphi_kf,\varphi_kf) + L^{(k)}_{m,m-1}(f,f) .$$

Here $L^{(k)}_{m,m-1}$ is of the form (ii). By (i)

(6.9) $$\mathrm{Re}(a\varphi_kf,\varphi_kf) \geqq \varepsilon_k|D^m\varphi_kf|^2 - \tau_k|\varphi_kf|^2$$

and according to (ii)

$$|L^{(k)}_{m,m-1}| \leqq \tfrac{1}{2}\varepsilon_k|D^m\varphi_kf|^2 + \tau(t_k)\,|\varphi_kf|^2 + t_k|f,\mathrm{St}\varphi_k|^2$$

which in combination with (6.9) gives

(6.10) $$\mathrm{Re}(\varphi_k{}^2af,f) \geqq \tfrac{1}{2}\varepsilon_k|D^m\varphi_kf|^2 - \tau'(t_k)\,|\varphi_kf|^2 - t_k|f,\mathrm{St}\varphi_k|^2$$
$$\geqq \tfrac{1}{4}\varepsilon_k|D^m\varphi_kf|^2 + 2r_k|D^{m-1}\varphi_kf|^2 - \tau''(t_k,r_k)\,|\varphi_kf|^2 - t_k|f,\mathrm{St}\varphi_k|^2.$$

Here r_k and t_k are arbitrary positive numbers. Now

$$|D^m\varphi_kf|^2 = |\varphi_kD^mf|^2 + L^{(k)}_{m,m-1} ,$$
$$|D^{m-1}\varphi_kf|^2 = |\varphi_kD^{m-1}f|^2 + L^{(k)}_{m-1,m-2} .$$

Here $L^{(k)}_{m,m-1}$ is, of course, not the same form as above but also of type (ii). By (ii)

$$|L^{(k)}_{m,m-1}| \leqq |D^m\varphi_kf|^2 + \tau_0(t)\,|\varphi_kf|^2 + t|f,\mathrm{St}\varphi_k|^2 ,$$
$$|L^{(k)}_{m-1,m-2}| \leqq |D^{m-1}\varphi_kf|^2 + \tau_1(s)\,|\varphi_kf|^2 + s|f,\mathrm{St}\varphi_k|^2 ,$$

where t and s are arbitrary >0, so that

(6.11) $$2|D^m\varphi_kf|^2 \geqq |\varphi_kD^mf|^2 - \tau_0(t)\,|\varphi_kf|^2 - t|f,\mathrm{St}\varphi_k|^2 ,$$

(6.12) $$2|D^{m-1}\varphi_kf|^2 \geqq |\varphi_kD^{m-1}f|^2 - \tau_1(s)\,|\varphi_kf|^2 - s|f,\mathrm{St}\varphi_k|^2 .$$

Choose $t = 8\varepsilon_k^{-1} t_k$, $s = r_k^{-1} t_k$. Insertion of (6.11) and (6.12) into (6.10) gives

$$(6.13)\quad \operatorname{Re}(\varphi_k^2 af, f) \geqq \tfrac{1}{8}\varepsilon_k |\varphi_k D^m f|^2 + r_k |\varphi_k D^{m-1} f|^2 - \\ - \tau(r_k, t_k)\, |\varphi_k f|^2 - 3t_k |f, \operatorname{St}\varphi_k|^2 .$$

The substitution

$$\varepsilon_k \to 8\,\varepsilon_k, \qquad 3t_k \to t_k ,$$

in (6.13) gives (iv).

(v):

Let $1 = \sum \varphi_k^2(x)$ be our partition of unity belonging to the covering (S_k). Set

$$m_k = \max_{S_v \cap S_k \neq \emptyset} \tau_v .$$

Define

$$\tau(x) = \sum m_i \varphi_i^2(x) .$$

For $x \in S_k$ we have

$$\tau(x) = \sum_{S_v \cap S_k \neq \emptyset} m_v \varphi_v^2(x) \geqq \sum_{S_v \cap S_k \neq \emptyset} \tau_k \varphi_v^2(x) = \tau_k ,$$

that is, $\inf_{S_k} \tau(x) \geqq \tau_k$. Notice that $\tau(x) \in C^\infty(\Omega)$.

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