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ON UNIVERSAL MOMENT PROBLEMS

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1.

Let B be an arbitrary Banach space and let $\{x_\nu\}$ be a set of elements of B. We also consider a Banach space S of sequences $s=\{s_\nu\}$ with norm $\|s\|$. By a universal moment problem for the space B we mean the problem of finding conditions on the set $\{x_\nu\}$ and the space S so that for every $s \in S$ there exists a linear functional L on B such that

$$L(x_\nu) = s_\nu, \qquad \nu = 1,2,\ldots .$$

If this is true, there exists a constant M so that a solution exists with

$$\|L\|_{\bar{B}} \leqq M\,\|s\| .$$

In the following sections we shall give two examples of such problems in classical analysis.

2.

Let $w(x)=W(x)^{-1}$ be a continuous weight function defined on the real axis and with the following properties:

$$(2.1)\quad \begin{cases} \text{a)}\ W(x)=W(-x)\geqq 1. \\ \text{b)}\ \log W(x) \text{ is an increasing function, convex in } \log x,\ x>0, \\ \qquad W(x)\,x^{-n} \to \infty, \text{ all } n. \\ \text{c)}\ \displaystyle\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx < \infty. \end{cases}$$

Let B be the space of real measurable functions $f(x)$, $-\infty<x<\infty$, with the norm

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2\, w(x)\, dx .$$

As norm in our space of sequences we choose

$$\|s\|^2 = \sum_0^\infty |s_n|^2 \lambda_n{}^{-2} .$$

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We shall study the universal moment problem

$$L_n(f) = \int_{-\infty}^{\infty} f(x) x^n w(x)\, dx = s_n . \tag{2.2}$$

$L_n(f)$ is by assumption (2.1, b) a linear functional on B.

The function

$$u(x+iy) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|y| \log W(t)}{(x-t)^2+y^2} dt$$

is by (2.1, c) harmonic in $y>0$ and $=\frac{1}{2}\log W(x)$ on $y=0$. $u(e^{\zeta})=v(\zeta)$ is thus harmonic in $0<\eta<\pi$, $\zeta=\xi+i\eta$, and symmetric with respect to $\eta=\frac{1}{2}\pi$. Since $v_{\xi\xi}\geqq 0$ on the boundary, this inequality holds everywhere and thus $v_{\eta\eta}\leqq 0$. It follows that, on the line $\xi=$ constant, v takes its maximum at $\eta=\frac{1}{2}\pi$ which implies

$$u(x+iy) \leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{r}{r^2+t^2} \log W(t)\, dt = \mu(r) , \quad x^2+y^2 \leqq r^2 . \tag{2.3}$$

Let $Q(x)$ be a polynomial in B, $\|Q\|\leqq 1$. By the principle of the harmonic majorant we have for $y \neq 0$,

$$\log|Q(x+iy)| \leqq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|y| \log|Q(t)|}{(x-t)^2+y^2} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|y| \log\big(|Q(t)|^2 w(t)\big)}{(x-t)^2+y^2} dt + u(x+iy) .$$

In the integral we use the inequality $ab\leqq e^{a-1}+b\log b$, valid if $b>0$, and choose $a=\log(|Q|^2 w)$. We find

$$\log|Q(x+iy)| < \frac{1}{2\pi e} + \frac{1}{2}\log\frac{1}{|y|} + \mu(r) .$$

Hence by Cauchy's formula, $r>0$,

$$|Q^{(\nu)}(0)| \leqq e^{1/2\pi e}\, \nu!\, \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|\sin\theta|^{\frac{1}{2}}} \frac{e^{\mu(r)}}{r^{\nu+\frac{1}{2}}} \leqq K\, \nu!\, M_\nu^{-1} , \tag{2.4}$$

where K is a numerical constant and

$$M_\nu = \sup_{r>0} e^{(\nu+\frac{1}{2})\log r - \mu(r)} . \tag{2.5}$$

We now introduce the sequence of orthonormal polynomials belonging to the weight function $w(x)$:

$$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, w(x)\, dx = \delta_{nm}, \qquad m,n = 0,1,\ldots ,$$

where

$$P_n(x) = \sum_{\nu=0}^{n} \alpha_{\nu n} x^{\nu} .$$

We form the polynomial

$$Q(x) = \sum_{n=0}^{N} a_n P_n(x) ,$$

choosing for some ν

$$a_n = P_n^{(\nu)}(0) \left\{ \sum_{n=0}^{N} |P_n^{(\nu)}(0)|^2 \right\}^{-\frac{1}{2}} .$$

The estimate (2.4) yields, letting $N \to \infty$,

$$\sum_{n=0}^{\infty} |P_n^{(\nu)}(0)|^2 \leqq K^2 (\nu!)^2 M_\nu^{-2} . \tag{2.6}$$

We assume that our interpolation problem has a solution $f(x)$ of the form

$$f(x) = \sum_{0}^{\infty} b_n P_n(x)$$

and find

$$b_n = \int_{-\infty}^{\infty} \left(\sum_{\nu=0}^{n} \alpha_{\nu n} x^{\nu} \right) f(x)\, w(x)\, dx = \sum_{\nu=0}^{n} \alpha_{\nu n} s_\nu .$$

Conversely, if $\{b_n\}$ defined as above satisfies the condition $\sum |b_n|^2 < \infty$ our choice of $f(x)$ is a solution. By (2.6) the following estimates hold

$$\begin{aligned}
\sum_{0}^{\infty} |b_n|^2 &\leqq \sum_{n=0}^{\infty} \left\{ \sum_{\nu=0}^{n} |\alpha_{\nu n}|^2 \lambda_\nu^2 \sum_{\nu=0}^{n} |s_\nu|^2 \lambda_\nu^{-2} \right\} \\
&\leqq \|s\|^2 \sum_{\nu=0}^{\infty} \lambda_\nu^2 \sum_{n=0}^{\infty} |\alpha_{\nu n}|^2 \\
&= \|s\|^2 \sum_{\nu=0}^{\infty} \lambda_\nu^2 (\nu!)^{-2} \sum_{n=0}^{\infty} |P_n^{(\nu)}(0)|^2 \\
&\leqq K^2 \|s\|^2 \sum_{\nu=0}^{\infty} (\lambda_\nu / M_\nu)^2 .
\end{aligned}$$

We summarize our result in a theorem.

THEOREM 1. *The universal moment problem* (2.2) *can be solved if the spaces B and S satisfy the condition*

$$\sum_{\nu=0}^{\infty} (\lambda_\nu/M_\nu)^2 < \infty ,$$

where $\{M_\nu\}$ was defined in (2.5).

3.

In this section we shall apply the result of Theorem 1 to non-quasianalytic classes of functions. For the general theory of infinitely differentiable functions we refer to [3].

Let $\{A_\nu\}_0^\infty$ be an increasing sequence of positive numbers and assume that $\log A_\nu$ is a convex function of ν and that $A_0=1$. The class $C\{A_\nu\}$ is non-quasianalytic if and only if

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\, dx < \infty ,$$

where

$$\log W(x) = 2 \sup_{\nu\geqq 0} (\nu \log|x| - \log A_\nu) + \log(1+x^2) .$$

Associate with $W(x)$ the function $\mu(r)$ by means of the relation (2.3) and with $\mu(r)$ the sequence $\{M_\nu\}$, defined by formula (2.5). Then the following theorem holds.

THEOREM 2. *Given a sequence c_n in the class $C\{M_n\}$, i.e. satisfying the inequalities $|c_n| \leqq b^n M_n$, there exists a function $\varphi(t) \in C\{A_n\}$ such that*

$$\varphi^{(n)}(0) = c_n .$$

Using previous notations, we define $s_n=(3bi)^{-n}c_n$ and $\lambda_n=2^{-n}M_n$ so that $\{s_n\} \in S$. The condition of Theorem 2 is fulfilled which means that $f \in B$ exists with

$$\int_{-\infty}^{\infty} x^n f(x)\, w(x)\, dx = s_n .$$

Define

$$\varphi(t) = \int_{-\infty}^{\infty} e^{3bixt} f(x)\, w(x)\, dx .$$

Then $\varphi^{(n)}(0)=c_n$ and

$$|\varphi^{(n)}(t)| \leqq (3b)^n \int_{-\infty}^{\infty} |x|^n \, |f(x)| \, w(x) \, dx$$
$$\leqq (3b)^n \, \|f\| \left\{ \int_{-\infty}^{\infty} \frac{x^{2n}}{W(x)} \, dx \right\}^{\frac{1}{2}}$$
$$\leqq (3b)^n \, \|f\| \, A_n \left\{ \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \right\}^{\frac{1}{2}},$$

so that $\varphi(t) \in C\{A_n\}$.

EXAMPLE 1. If $C\{A_n\}$ is non-quasianalytic, $C\{M_n\}$ contains the analytic class.

The definition of $\mu(r)$ shows that $\mu(r) = o(r)$ which implies

$$n! = O(\sup e^{(n+\frac{1}{2})\log r - r}) = O(M_n) \, .$$

The result is of course well known.

EXAMPLE 2. If $M_n = (n!)^a$, $a > 1$, $\varphi(t)$ can be chosen in the same class.

EXAMPLE 3. If $M_n = (n \log n)^n$ (a quasianalytic class), $\varphi(t)$ can be chosen in $C\{(n(\log n)^2)^n\}$.

4.

When $\bar{B}$ is a Banach algebra, the case when $L(x_\nu)$ are multiplicative functionals and S is the space of bounded sequences is particularly important. We shall give an elementary result and illustrate the situation for Fourier integrals.

THEOREM 3. *Assume that B is a *Banach algebra and $L_\nu(x)$ are multiplicative. Let N be an arbitrary integer. Assume that for every choice of a subset U of $(1, 2, \ldots, N)$ there exists an element $x \in B$ such that*

$$|L_\nu(x)| \geqq 1, \quad \nu \in U \, , \tag{4.1}$$
$$|L_\nu(x)| \leqq \delta, \quad \nu \notin U \, , \tag{4.2}$$
$$\|x\| \leqq M \, , \tag{4.3}$$

where $0 < \delta < 1$ and M are independent of U. Then, for any sequence $\{c_\nu\}_1^N$, $|c_\nu| \leqq 1$, $x \in B$ exists so that

$$L_\nu(x) = c_\nu, \qquad \nu = 1, 2, \ldots, N \, ,$$

and $\|x\| \leqq A$, A depending only on δ and M.

PROOF. Since B is a * algebra we may assume $L_\nu(x)$ real, $L_\nu(x) \geqq 0$, and also, replacing x by x^k, $\delta < \frac{1}{2}$. We shall first show that $x = x(U)$ exists so that $\|x\|$ is uniformly bounded and

$$\begin{aligned} L_\nu(x) &\geqq 1, \quad \nu \in U , \\ L_\nu(x) &= 0, \quad \nu \notin U . \end{aligned}$$

Denote the solution of (4.1–3) by $y(U)$ and the complement of U by U_1. We form

$$y_1(U) = y(U) - a_1 y(U_1)$$

where $a_1 > 0$ is the largest number so that $L_\nu(y_1) \geqq 0$, all ν. Clearly $a_1 \leqq \delta$ and $L_\nu(y_1) = 0$ for some $\nu \in U_1$. Delete this (or these) index from U_1 and call the remainder U_2. We now form

$$y_2(U) = y(U) - a_1 y(U_1) - a_2 y(U_2)$$

with $a_2 > 0$ as the largest number such that $L_\nu(y_2) \geqq 0$ for $\nu \in U_2$. This implies $a_1 + a_2 \leqq \delta$. We now delete one or more indices from U_2 as above and continue the process. We finally get

$$y_k(U) = y(U) - \sum_1^k a_i y(U_i) .$$

Here

$$a_i > 0, \qquad \sum_1^k a_i \leqq \delta \tag{4.4}$$

and

$$\begin{aligned} L_\nu(y_k) &\geqq 1 - \delta^2, \quad \nu \in U , \\ 0 \geqq L_\nu(y_k) &\geqq -\delta^2, \quad \nu \notin U . \end{aligned} \tag{4.5}$$

In a similar way we now form

$$z(U) = y_k(U) + \sum_1^l b_i y(V_i) ,$$

where

$$b_i > 0, \qquad \sum_1^l b_i \leqq \delta^2 , \tag{4.6}$$

and

$$\begin{aligned} L_\nu(z) &\geqq 1 - \delta^2, \quad \nu \in U , \\ 0 \leqq L_\nu(z) &\leqq \delta^3, \quad \nu \notin U . \end{aligned} \tag{4.7}$$

In the limit we obtain an element $x_1 = \sum_V \mu(V)\, y(V)$, where $\sum |\mu(V)| \leqq \sum_1^\infty \delta^n < 1$ and

$$L_\nu(x) \geqq 1-\delta^2-\delta^4-\ldots > \tfrac{1}{2}, \quad \nu \in U ,$$
$$L_\nu(x) = 0, \quad \nu \notin U .$$

$x(U)=2x_1$ has the desired properties.

Now let $\{c_\nu\}_1^N$ be an arbitrary sequence, $|c_\nu| \leqq 1$, and $c_\nu=\alpha_\nu+i\beta_\nu$. Let U_0 be the set of indices ν, $\alpha_\nu>0$. Define $\lambda(U_0)>0$ to be the largest number such that $\lambda(U_0)\, L_\nu(x(U_0)) \leqq \alpha_\nu$, $\nu \in U_0$. We have equality for some $\nu \in U_0$. The rest of U_0 is denoted U_1 and $\lambda(U_1)$ is defined similarly. Clearly

$$x_1 = \sum_i \lambda(U_i)\, x(U_i)$$

has the property $L_\nu(x_1)=\alpha_\nu$, $\nu \in U_0$, $L_\nu(x_0)=0$, $\nu \notin U_0$. Arguing in the same way for $\alpha_\nu<0$ and for β_ν we have proved the theorem.

Remark. If B is the space of bounded analytic functions in the unit circle it was proved in [1], that the interpolations $f(z_\nu)=c_\nu$, $|c_\nu| \leqq 1$, $f \in B$, are possible if (and only if)

$$\prod_{\nu \neq \mu} \left| \frac{z_\nu - z_\mu}{1-z_\nu \bar{z}_\mu} \right| \geqq \delta > 0 .$$

By deleting certain factors in the above product we immediately see that the inequalities

$$|f(z_\nu)| \geqq 1, \quad \nu \in U , \qquad f(z_\nu) = 0, \quad \nu \notin U , \tag{4.8}$$

have uniformly bounded solutions. Hence also for this algebra, which is not a *algebra, (4.8) implies that all bounded interpolations are possible.

As an illustration we shall prove the following theorem. An argument somewhat similar to the one below was used by Edwards [2].

Theorem 4. *A sufficient condition that the moment problems*

$$\int_0^{2\pi} e^{in_\nu x}\, d\mu(x) = c_\nu, \quad |c_\nu| \leqq 1, \quad n_\nu > 0 , \tag{4.9}$$

can be solved is that every interval $(2^k, 2^{k+1})$ *contains a bounded number if* n_ν*'s* (Sidon [4]). *Another sufficient condition is that the number* $P_s(n)$ *of solutions of*

$$n = \pm n_{k_1} \pm n_{k_2} \pm \ldots \pm n_{k_s}, \qquad k_1 < k_2 < \ldots < k_s ,$$

satisfies the inequalities

$$(4.10) \qquad P_s(n) \leqq K^s$$

for a fixed number K or that $\{n_\nu\}$ *is a finite union of such sets* (Stečkin [5]). *A necessary condition is that every interval of length* λ *contains* $O(\log \lambda)$ *numbers* n_ν.

PROOF. *Sidon's result.* $\{n_\nu\}$ can be decomposed, $\{n_\nu\} = \bigcup_{i=1}^{p} \{n_{\nu i}\}$, where $n_{\nu i}/n_{\nu+1,i} < 3^{-1}$. Choose an arbitrary finite subset U_i of each $\{n_{\nu i}\}$ and a small positive number ϱ. The trigonometrical polynomial

$$T(x) = \sum_{i=1}^{p} \prod_{\nu \in U_i} (1 + 2\varrho \cos n_{\nu i} x) = \sum_{-A}^{A} a_n e^{inx}$$

has the following properties:

$$T(x) \geqq 0, \qquad \frac{1}{2\pi} \int_0^{2\pi} T(x)\,dx = p\,,$$

$$a_\nu \geqq \varrho, \quad \nu \in \bigcup U_i\,,$$

$$a_\nu = O(\varrho^2), \ \varrho \to 0, \quad \nu \notin \bigcup U_i\,.$$

In Theorem 3 we now choose $B = L^1(0, 2\pi)$ under convolution and $x(U) = \varrho^{-1}T$. Since $e^{in_\nu x}$ define the linear functionals, $f_N(x)$ exists such that

$$\int_0^{2\pi} e^{in_\nu x} f_N(x)\,dx = c_\nu, \quad \nu \leqq N, \qquad \int_0^{2\pi} |f_N(x)|\,dx \leqq M\,.$$

We let $N \to \infty$ and select a weakly convergent subsequence of $\{f_N(x)dx\}$.

Stečkin's theorem. Suppose that (4.10) holds, and form for some subset of U of $(1, 2, \ldots, N)$

$$T(x) = \prod_{\nu \in U} (1 + \varrho \cos n_\nu x) = \sum_{-A}^{A} a_n e^{inx}\,.$$

The following estimates hold:

$$0 < a_0 \leqq 1 + \sum_{s=1}^{\infty} (\tfrac{1}{2}\varrho)^s P_s(0) \leqq \text{Const.}$$

if $K\varrho < 2$;

$$a_{n_\nu} \geqq \tfrac{1}{2}\varrho, \quad \nu \in U;$$

$$a_n \leqq \sum_{s=2}^{\infty} (\tfrac{1}{2}\varrho)^s P_s(n) \leqq \text{Const.}\ \varrho^2, \quad n \neq n_\nu$$

if $K\varrho < 2$.

As above we can now conclude that the interpolation is possible for a finite union of sets with the property (4.10).

Now assume that all moment problems (4.9) have solutions μ with

$$\int |d\mu| \leqq M .$$

Let

$$P(x) = \sum_1^N a_\nu e^{in_\nu x}$$

be a polynomial with $|P| \leqq 1$. Then, if $\varepsilon_\nu = \text{sign}(a_\nu)$,

$$\sum_1^N |a_\nu| = \sum_1^N \varepsilon_\nu a_\nu = \int P\, d\mu \leqq M . \tag{4.11}$$

Let $\varphi_\nu(t)$ be the Rademacher system and consider

$$\psi_t(x) = \frac{\gamma}{N^{\frac{1}{2}}} \sum_{\nu=1}^N \varphi_\nu(t) \cos n_\nu x .$$

If $\gamma < 1/4e$ it follows from a well-known inequality ([6, p. 214]) that

$$\int_0^1 e^{\psi_t(x)^2}\, dt \leqq \text{Constant} = C .$$

Hence t exists with

$$\int_0^{2\pi} e^{\psi_t(x)^2}\, dx \leqq 2\pi C . \tag{4.12}$$

Now, by (4.11)

$$\max_x |\psi_t(x)| = |\psi_t(x_0)| > 2kN^{\frac{1}{2}}$$

with k independent of N. Using a trivial estimate of $|\psi_t'(x)|$, we find

$$|\psi_t(x)| > kN^{\frac{1}{2}}, \qquad |x - x_0| < \frac{k}{N^{\frac{1}{2}} \cdot n_N}$$

which by (4.12) yields

$$k\, e^{k^2 N}\, N^{-\frac{1}{2}}\, n_N^{-1} \leqq 2\pi C$$

or

$$N = O(\log n_N) .$$

Since we can as well consider a sequence $n_{\mu+1} - n_\mu, \ldots, n_{\mu+N} - n_\mu$, we have proved the theorem.

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