MATH. SCAND. 9 (1961), 243–257

SOME COMBINATORIAL THEOREMS FOR CONTINUOUS PARAMETER PROCESSES[1]

EDGAR REICH[2]

1. Introduction.

Let $x(t)$ be a given (non-random) function of t, $0 \leqq t \leqq T$, subject to certain restrictions which will be stated in section 2. We shall be interested in the following two quantities, determined by $x(\cdot)$. Firstly,

$$\text{(I)} \qquad m(t)-x(t), \quad \text{where} \quad m(t) = \sup_{0 \leqq \tau \leqq t} x(\tau) ,$$

and, secondly, the first passage time,

$$\text{(II)} \qquad \alpha = \begin{cases} \inf\{\tau \mid x(\tau)=0\}, & \text{if } x(\tau)=0 \text{ for some } \tau , \\ \infty, & \text{if } x(\tau) \neq 0 \text{ for all } \tau . \end{cases}$$

In connection with the study of (I), we let

$$\varphi(t;\xi) = \begin{cases} 1, & \text{if} \quad m(t)-x(t) \leqq \xi, \quad \xi \geqq 0 , \\ 0, & \text{if} \quad m(t)-x(t) > \xi \end{cases}$$

and in connection with (II) we put

$$\beta(t) = \begin{cases} 1, & \text{if} \quad \alpha \leqq t , \\ 0, & \text{if} \quad \alpha > t . \end{cases}$$

Our main combinatorial theorems, theorems 2.1, and 2.2, of section 2, state that $\varphi(t;0)$, and $\beta(t)$ satisfy certain Volterra integral equations of the second kind, whose respective kernels depend, roughly speaking, on the amount of oscillation of $x(\cdot)$ in the interval $(0,t)$. The quantity $\varphi(t;\xi)$, $\xi>0$, is expressible by a quadrature in terms of $\varphi(t;0)$. Although (I) and (II) appear, on the surface, to bear little relationship to each other, it is interesting that the respective integral equations are quite closely related.

In spite of the elementary nature of theorems 2.1 and 2.2 they can be used to obtain the non-trivial *probabilistic* theorems 3.1, 3.2, of section 3, in the case when $x(\cdot)$ is a *stochastic process with independent* (not

Received April 19, 1961.

[1] Work done with partial support through P. L. 584, 79^{th} U. S. Congress.

[2] John Simon Guggenheim Fellow, on leave from University of Minnesota.

necessarily stationary) *increments* and $x(\cdot)$ also satisfies certain additional hypotheses which correspond to the hypotheses made in theorems 2.1 and 2.2. Namely, if

$$F(t;\,\xi) = \Pr\{m(t)-x(t) \leqq \xi\}, \qquad \xi \geqq 0\,,$$

and

$$B(t) = \Pr\{\alpha \leqq t\}\,,$$

then $F(t;\,0)$ and $B(t)$ will turn out to satisfy Volterra equations of the second kind, (3.6), and (3.8), while $F(t;\,\xi)$, $\xi > 0$, is expressible by a quadrature in terms of $F(t;\,0)$. The restrictions on the process $x(\cdot)$ insure that $x(\cdot)$ does not oscillate locally too rapidly, and exclude such processes as Brownian motion for which $F(t;\,0)$ would be identically zero.

The purpose of theorems 3.3 and 3.4 is to show that, without any further essential restrictions, the Volterra equations (3.6), (3.8) have unique solutions, in spite of the fact that the kernels involved are not necessarily sufficiently regular for the classical Volterra equation theory to apply.

When $x(\cdot)$ is *stationary* the integrals occurring in (3.6) and (3.8) are of the convolution type, and $F(t;\,0)$ and $B(t)$ can be obtained by Laplace transforms. This fact illustrates what seems a fairly general heuristic principle: When a problem involving additive processes with stationary increments can be solved by Laplace transforms the more general case of non-stationary increments leads to a Volterra equation (or correspondingly, a triangular matrix, in the discrete case).

2. Combinatorial results.

In this section $x(t)$ shall be a (non-random) function defined for $0 \leqq t \leqq T$, $T > 0$.

In connection with our consideration of (I) we suppose that the following assumptions hold.

(2.1) $$x(0+) = x(0), \qquad x(t \pm 0) \text{ exists}, \qquad 0 \leqq t \leqq T$$

(2.2) $$x(t-0) = x(t) \geqq x(t+0), \qquad 0 \leqq t \leqq T$$

(2.3) For each k there are at most a finite number of t's, $0 \leqq t \leqq T$, such that $x(t-0) \geqq k \geqq x(t+0)$.

In connection with our consideration of (II) we suppose, in addition, that

(2.4) If $x(t_0-0) = x(t_0+0)$ then $x(t)$ is *increasing* in some neighborhood of t_0.

(2.5) If $x(t_0) = 0$ then $x(t_0+0) = x(t_0-0) = 0\,.$

We say that $x(\cdot)$ has a relative maximum at t_0 if

$$x(t_0-\varepsilon) \leqq x(t_0) \geqq x(t_0+\varepsilon)$$

for all sufficiently small $\varepsilon>0$. For $0 \leqq a \leqq b \leqq T$, $\xi \geqq 0$, let $\mathscr{Z}(a,b;\,\xi)$ be the number of elements of the set

$$(2.6) \qquad \{t \mid a \leqq t < b,\ x(t)=x(b)+\xi,\ x(\cdot) \text{ does not have a relative maximum at } t\},$$

and let $\mathscr{J}(a,b)$ be the number of the elements of the set

$$(2.7) \qquad \{t \mid a \leqq t < b,\ x(t+0) < x(b) < x(t-0)\}.$$

By (2.3), $\mathscr{Z}$ and $\mathscr{J}$ are finite.

THEOREM 2.1. *Suppose* $x(t)$ *satisfies* (2.1)–(2.3). *Then*

$$(2.8) \qquad \varphi(t;\,\xi)-\int_0^t \varphi(u;\,0)\, d_u\mathscr{Z}(u,t;\,\xi) = \begin{cases} 1, & \text{if } x(t) \geqq x(0)-\xi \\ 0, & \text{if } x(t) < x(0)-\xi \end{cases}, \quad \xi \geqq 0, \quad 0 \leqq t < T.$$

NOTE. If we consider the kernel $d_u\mathscr{Z}(u,t;\xi)$ and the right-hand side as known, then setting $\xi=0$ in (2.8) gives a Volterra equation for $\varphi(t;\,0)$, while for $\xi>0$, (2.8) expresses $\varphi(t;\,\xi)$ in terms of $\varphi(t;\,0)$ by a quadrature. A typical $x(\cdot)$ satisfying (2.1)–(2.3) is shown in Fig. 1.

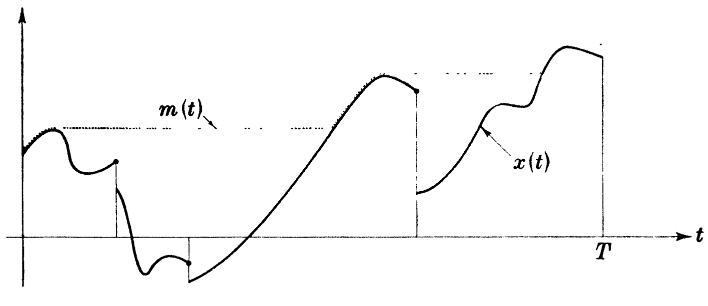

Fig. 1.

PROOF OF THEOREM 2.1. By (2.1), (2.2),

$$m(t) = \sup_{0 \leqq \tau \leqq t} x(\tau) = \max_{0 \leqq \tau \leqq t} x(\tau)$$

exists and is a continuous non-decreasing function of t, $0 \leqq t \leqq T$. Let $\mathscr{M}(t;\,\xi)$, $\xi \geqq 0$, $0 \leqq t \leqq T$ be the number of elements of the finite set

$$\{\tau \mid 0 \leqq \tau < t,\ m(\tau)=x(\tau)=x(t)+\xi,\ x(\cdot) \text{ does not have a relative maximum at } \tau\}.$$

To derive the theorem we shall obtain two different expressions for $\mathscr{M}(t;\xi)$. The fact that both expressions equal $\mathscr{M}(t;\xi)$ will yield (2.8).

If $x(t)<x(0)-\xi$, $\xi \geqq 0$, then $m(\tau)\geqq x(0)>x(t)+\xi$, $0\leqq\tau\leqq t$. Hence, in this case $\mathscr{M}(t;\xi)=0$. On the other hand, we shall now see that

$$(2.9)\qquad x(t)\geqq x(0)-\xi \;\Rightarrow\; \mathscr{M}(t;\xi) = \begin{cases} 1, & \text{if } m(t)-x(t)>\xi \\ 0, & \text{if } m(t)-x(t)\leqq\xi\,. \end{cases}$$

If $m(t)-x(t)<\xi$, then $m(\tau)<x(t)+\xi$, $0\leqq\tau\leqq t$. Hence

$$(2.10)\qquad x(t)\geqq x(0)-\xi,\; m(t)-x(t) < \xi \Rightarrow \mathscr{M}(t;\xi) = 0\,.$$

Next suppose $m(t)-x(t)>\xi$. Since $m(t)$ is continuous, $m(0)=x(0)\leqq x(t)+\xi$, $m(t)>x(t)+\xi$, the set

$$\{\tau \mid 0\leqq\tau\leqq t,\; m(\tau)=x(t)+\xi\},\qquad 0<t\leqq T\,,$$

is non-empty and closed, and therefore has a largest element $s=s(t)$. Clearly,

$$m(s) = x(t)+\xi, \qquad \text{and} \qquad s<t\,.$$

Hence, by (2.3), $x(s-\varepsilon)<m(s)$ for sufficiently small $\varepsilon>0$. But $x(s+\varepsilon)>m(s)$, for arbitrarily small $\varepsilon>0$. For, suppose, on the contrary, $x(s+\varepsilon)<m(s)$ for all sufficiently small $\varepsilon>0$. This would imply $m(s+\varepsilon)=m(s)=x(t)+\xi$, contradicting the definition of s. Hence $x(\cdot)$ does not have a relative maximum at s, and, using (2.1), (2.2),

$$x(s) = x(s-0) \leqq m(s) \leqq x(s+0) \leqq x(s)\,,$$

implying that $x(s)=m(s)=x(t)+\xi$. Thus we have shown that

$$(2.11)\qquad x(t)\geqq x(0)-\xi,\; m(t)-x(t) > \xi \Rightarrow \mathscr{M}(t;\xi) \geqq 1\,.$$

To complete the proof of (2.9) it suffices to show that, if $x(t)\geqq x(0)-\xi$,

$$\mathscr{M}(t;\xi) \leqq \begin{cases} 1, & \text{if } \; m(t)-x(t)>\xi \\ 0, & \text{if } \; m(t)-x(t)=\xi \end{cases}, \qquad 0\leqq t<T, \quad \xi\geqq 0\,.$$

Suppose the contrary. Then there exist τ_1, τ_2, $0\leqq\tau_1<\tau_2\leqq t$, such that $x(\tau_1)=x(\tau_2)=m(\tau_1)=m(\tau_2)=x(t)+\xi$, and such that $x(\cdot)$ does not have relative maximum at τ_1. Since $m(\cdot)$ is non-decreasing,

$$(2.12)\qquad m(\tau) = m(\tau_1) = x(\tau_1), \qquad \tau_1\leqq\tau\leqq\tau_2\,.$$

Since $m(\tau_1)=x(\tau_1)$ it follows that $x(\tau_1-\varepsilon)<x(\tau_1)$ for sufficiently small $\varepsilon>0$. But $x(\cdot)$ does not have a relative maximum at τ_1. Therefore

$$x(\tau_1+\varepsilon) > x(\tau_1)$$

for arbitrarily small $\varepsilon > 0$. This contradicts (2.12) for $\tau = \tau_1 + \varepsilon$. Thus we have established (2.9).

In view of (2.9) and the lines preceding it,

$$\varphi(t;\xi) + \mathscr{M}(t;\xi) = \begin{cases} 1, & \text{if} \quad x(t) \geqq x(0) - \xi \\ 0, & \text{if} \quad x(t) < x(0) - \xi \end{cases}, \quad \xi \geqq 0 . \tag{2.13}$$

On the other hand, from the definitions of $\varphi(t;\xi)$, $\mathscr{M}(t;\xi)$, and $\mathscr{X}(a,b;\xi)$ it is clear that

$$\mathscr{M}(t;\xi) = -\int_0^t \varphi(u;0)\, d_u \mathscr{X}(u,t;\xi) .$$

If the above is combined with (2.13) we obtain (2.8), completing the proof of the theorem.

THEOREM 2.2. *Suppose $x(t)$ satisfies* (2.1)–(2.5), $x(0) < 0$. *Then*

$$\beta(t) - \int_0^t \beta(u)\, d_u[\mathscr{X}(u,t;0) - \mathscr{J}(u,t)] = \begin{cases} 1, & \textit{if} \quad x(t) \geqq 0 \\ 0, & \textit{if} \quad x(t) < 0 \end{cases}, \quad 0 \leqq t < T. \tag{2.14}$$

PROOF. It is not difficult to verify, using (2.1)–(2.5), that the intervals on which $x(\cdot)$ is increasing are separated by the downward jumps of $x(\cdot)$; more precisely, between any two points of either the set (2.6) or (2.7) there is exactly one point of the other set. Thus,

$$\mathscr{X}(a,t;0) - \mathscr{J}(a,t;0) = 0 \text{ or } \pm 1, \qquad t > a .$$

In fact, if $\alpha < \infty$, it follows, with the help of (2.5) (See Fig. 2) that

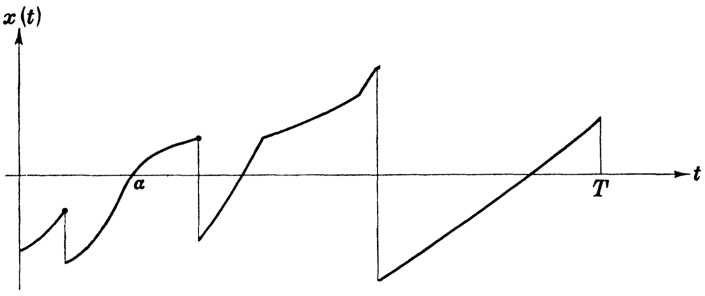

Fig. 2.

$$\mathscr{X}(\alpha,t;0) - \mathscr{J}(\alpha,t;0) = \begin{cases} 0, & x(t) \geqq 0 \\ -1, & x(t) < 0 \end{cases}, \quad t > \alpha .$$

Therefore, if $t > \alpha$,

$$(2.15) \qquad \beta(t)+\mathscr{L}(\alpha,t;\,0)-\mathscr{J}(\alpha,t;\,0) = \begin{cases} 1, & x(t)\geqq 0 \\ 0, & x(t)<0 \end{cases}.$$

Let us define

$$\mathscr{L}(a,b) = \mathscr{J}(a,b) = 0, \qquad b\leqq a\,.$$

Equation (2.15) then holds at $t=\alpha$, also, assuming the form $1+0+0=1$, and also for $t<\alpha$, where it assumes the form $0+0-0=0$. (Note that $x(0)<0$, by hypothesis.) Thus (2.15) holds for $0\leqq t\leqq T$. On the other hand, since,

$$\beta(u) = \begin{cases} 1, & u \geqq \alpha \\ 0, & u < \alpha \end{cases},$$

it follows, from the definitions of $\mathscr{L}(u,t;\,0)$, $\mathscr{J}(u,t)$, that

$$(2.16) \quad \mathscr{L}(\alpha,t) = -\int_0^t \beta(u)\, d_u\mathscr{L}(u,t;\,0), \qquad \mathscr{J}(u,t) = -\int_0^t \beta(u)\, d_u\mathscr{J}(u,t)\,.$$

The above are easily checked both in the case $\alpha<t$, and $\alpha\geqq t$. On substituting (2.16) into (2.15) we obtain (2.14).

3. Stochastic processes with independent increments.

The results of section 2 will now be applied when $x(t)$, $0\leqq t\leqq T$, is a stochastic process with independent increments. We actually are considering a function, $x=x(\omega,t)$, where ω is a point in an appropriate probability sample space, as well as other functions of ω, but the ω will usually be surpressed for briefness.

The following conditions will be used in connection with the formulations of theorems 3.1, 3.2. They correspond to conditions (2.1)–(2.5) of the last section.

(3.1) Condition (2.1) holds for almost all ω.

(3.2) Condition (2.2) holds for almost all ω.

(3.3) Let $N(t)$ denote the expected number of extremal points (relative maxima or minima of $x(\tau)$) in the interval $0\leqq\tau\leqq t$ $(0\leqq t\leqq T)$. Then $N(T)<\infty$.

(3.4) Condition (2.4) holds for almost all ω.

(3.5) Condition (2.5) holds for almost all ω.

If (3.1)–(3.3) hold then $\alpha=\alpha(\omega)$, $m(t)=m(\omega,t)$, $\varphi(t;\,\xi)$, $\beta(t)$, $\mathscr{L}(u,t;\,\xi)$, and $\mathscr{J}(u,\,t)$ are random variables. We shall consider the following *expectations*:

$$F(t;\xi) = E\varphi(t;\xi) = \Pr\{m(t)-x(t) \leqq \xi\},$$
$$B(t) = E\beta(t) = \Pr\{\alpha \leqq t\},$$
$$Z(a,b;\xi) = E\mathscr{Z}(a,b;\xi), \qquad J(a,b) = E\mathscr{J}(a,b).$$

THEOREM 3.1. *If* $x(t)$, $0 \leqq t < T$, *is a stochastic process with independent increments for which* (3.1)–(3.3) *hold, then*

$$(3.6)\quad F(t;\xi) - \int_0^t F(u;0)\, d_u Z(u,t;\xi) = \Pr\{x(t) \geqq x(0) - \xi\}, \qquad 0 \leqq t < T.$$

PROOF. For fixed ξ and t the function $Z(u,t;\xi)$ is a decreasing function of u, $0 \leqq u \leqq t$. Furthermore, since any two zeroes of $x(t)=k$ are separated by at least one extremal point of $x(\cdot)$,

$$(3.7)\qquad Z(a,b;\xi) \leqq N(b) - N(a).$$

Taking the expectation of both sides of (2.8) we obtain

$$F(t;\xi) - \int_0^t E[\varphi(u;0)\, d_u \mathscr{Z}(u,t;\xi)] = \Pr\{x(t) \geqq x(0) - \xi\}.$$

It is now that we make use of the fact that $x(t)$ is a process with independent increments. The random variable $\varphi(u;0)$ is completely determined by the history of $x(\tau)$, $0 \leqq \tau < u$. On the other hand, $\mathscr{Z}(u,t;\xi)$ is determined by the values $x(t)+\xi-x(\tau)$, $u \leqq \tau < t$, i.e. by the history of $x(t)-x(\tau)$, $u \leqq \tau < t$. Therefore, $\varphi(u;0)$ and $\mathscr{Z}(u,t;\xi)$ are *independent*, and we can write

$$E[\varphi(u;0)\, d_u \mathscr{Z}(u,t;\xi)] = E\varphi(u;0)\, E d_u \mathscr{Z}(u,t;\xi) = F(u;0)\, d_u Z(u,t;\xi),$$

thus obtaining (3.6).

In a similar way, starting with (2.14), we prove the following theorem:

THEOREM 3.2. *If* $x(t)$, $0 \leqq t < T$, *is a stochastic process with independent increments for which* (3.1)–(3.5) *hold, then*

$$(3.8)\qquad B(t) - \int_0^t B(u)\, d_u[Z(u,t;0) - J(u,t)] = \Pr\{x(t) \geqq 0\}.$$

Note that $J(u,t)$ is like $Z(u,t;\xi)$ a decreasing function of u, for fixed t, and clearly satisfies a corresponding inequality,

$$(3.9)\qquad J(a,b) \leqq N(b) - N(a).$$

Both integral equations (3.6) and (3.8) are of the form

$$f(t) - \int_0^t f(u)\, d_u K(u,t) = r(t), \qquad 0 \leqq t < T.$$

If it were true that $K(u,t)$ were an absolutely continuous function of u, and $\frac{\partial}{\partial u}K(u,t)$ sufficiently smooth we could merely refer to the classical Volterra-equation theory to prove that there would be a unique solution. However, it may be that $x(\cdot)$ has fixed discontinuities. This would produce discontinuities in $K(u,t)$ as a function of u. The lemma which follows, fortunately, enables us to avoid any unessential restrictions on the smoothness of $K(u,t)$ in our special case.

LEMMA. *Suppose there exists a set of numbers* $\{t_k\}$, $0=t_0<t_1<\ldots<t_n=T$, *such that*

$$\int_{t_{k-1}}^{t} |d_uK(u,t)| < 1, \qquad t_{k-1}\leqq t\leqq t_k, \qquad k = 1,2,\ldots,n\,. \tag{3.10}$$

Then the integral equation

$$f(t)-\int_0^t f(u)\, d_uK(u,t) = r(t), \qquad 0\leqq t<T \tag{3.11}$$

has at most one bounded solution $f(t)$.

PROOF. Suppose (3.11) has solutions $f(t)$, $g(t)$, $0\leqq t<T$. Then

$$f(t)-g(t) = \int_0^t [f(u)-g(u)]\, d_uK(u,t), \qquad 0\leqq t<T\,. \tag{3.12}$$

Hence

$$|f(t)-g(t)| \leqq \sup_{0\leqq u\leqq t} |f(u)-g(u)| \int_0^t |d_uK(u,t)|, \qquad 0\leqq t<T\,.$$

In particular, using (3.10), with $0=t_0\leqq t\leqq t_1$,

$$f(t) \equiv g(t), \qquad 0=t_0\leqq t\leqq t_1\,.$$

Once we have established that $f(t)\equiv g(t)$, $0\leqq t\leqq t_{k-1}$, we have, by (3.12),

$$f(t)-g(t) = \int_{t_{k-1}}^{t} [f(u)-g(u)]\, d_uK(u,t), \qquad t_{k-1}\leqq t<T\,.$$

In particular,

$$|f(t)-g(t)| \leqq \sup |f(u)-g(u)| \int_{t_{k-1}}^{t} |d_uK(u,t)|, \qquad t_{k-1}\leqq t\leqq t_k\,.$$

Therefore, by (3.10), $f(t)\equiv g(t)$, $t_{k-1}\leqq t\leqq t_k$, also.

While the solution of (3.11) may not be expressible by a *single* Neumann series converging over the entire range, $0 \leqq t < T$, the proof indicates that the solution can be found by means of at most a finite number of *continuations* of Neumann series.

Denote by $\mathscr{E}(t)$ the set of all functions $x(\cdot)$ having an extremum at t. We then may state the following

THEOREM 3.3. *If* (3.1)–(3.3) *hold, and if* $\Pr\{x(\cdot) \in \mathscr{E}(t)\} < 1$, $0 \leqq t \leqq T$, *then* $F(t;\,0)$ *is the unique bounded solution of*

$$F(t;\,0) - \int_0^t F(u;\,0)\, d_u Z(u,t;\,0) \;=\; \Pr\{x(t) \geqq x(0)\}\,.$$

REMARK. An interesting consequence of theorems 3.1 and 3.3 is that, under the hypotheses stated, a knowledge of $\Pr\{x(t) \geqq x(0) - \xi\}$, and $d_u Z(u,t;\,\xi)$ suffices for the calculation of $F(t,\xi) = \Pr\{m(t) - x(t) \leqq \xi\}$. Now, as $h \to 0+$,

$$Z(u+h,t;\,\xi) - Z(u,t;\,\xi) \;\approx\; \Pr\{x(\sigma) = x(t) + \xi\} \quad \text{for some } \sigma \in (u,u+h)\,,$$

assuming that we can neglect the probability of a relative maximum in $(t,t+h)$. Thus $d_u Z(u,t;\,\xi)$ depends on the transition probabilities of $x(\cdot)$ in a rather simple way. A phenomenon of a similar nature is known to occur in the case when the basic process $x(\cdot)$ is the sum S_n, $n = 1,2,\ldots$, of n independent identically distributed random variables (the limiting case is then a process with independent *stationary* increments). From the work of Sparre Andersen [2] it is known that a knowledge of the sequence $\{a_n{}^*\}$, $a_n{}^* = \Pr\{S_n \leqq 0\}$, suffices for a determination of the probability that the maximal term in $\{S_0,S_1,\ldots,S_n\}$ occurs at the n^{th} plase.

PROOF OF THEOREM 3.3. Since $Z(u,t;\,\xi)$ is monotonic in u,

$$\text{(3.13)} \quad \int_a^t |d_u Z(u,t;\,0)| \;=\; -\int_a^t d_u Z(u,t;\,0)$$
$$= Z(a,t;\,0) \;\leqq\; N(t) - N(a) \;\leqq\; N(b) - N(a), \qquad a \leqq t \leqq b\,.$$

By (3.3), $N(t)$, $0 \leqq t \leqq T$, is an increasing function, $N(0) = 0$, $N(T) < \infty$. It also follows from the definition of $N(t)$ that

$$N(\tau+0) - N(\tau-0) \;=\; \Pr\{x(\cdot) \in \mathscr{E}(\tau)\}\,.$$

In view of the hypothesis, and the fact that $N(T) < \infty$, there exists a number $\theta < 1$ such that

$$N(t+0) - N(t-0) \;\leqq\; \theta, \qquad 0 \leqq t \leqq T\,.$$

Hence there exist numbers, $0=t_0<t_1<\ldots<t_n=T$, such that, say,

$$N(t_k)-N(t_{k-1}) \leqq \tfrac{1}{2}(\theta+1) < 1, \qquad k = 1,2,\ldots,n.$$

In view of (3.13) we can choose these $\{t_k\}$ as the $\{t_k\}$ of the lemma, and this completes the proof.

In connection with the discussion of the uniqueness of (3.8) we assume that (3.1)–(3.5) hold. In view of (3.4), the only extrema of $x(t)$ in this case are the relative maxima of $x(t)$ which occur at the points where there are jumps. The result is then as follows.

Let $\mathscr{D}(t)$ denote the set of all functions $x(\cdot)$ having a jump at t.

THEOREM 3.4. *If* (3.1) *to* (3.5) *hold, and if* $\Pr\{x(\cdot)\in\mathscr{D}(t)\}<1$, $0\leqq t\leqq T$, *then* $B(t)$ *is the unique bounded solution of* (3.8).

PROOF. Again choose $\theta<1$ so that $N(t+0)-N(t-0)\leqq\theta$, $0\leqq t\leqq T$. There exist at most a finite number of points $\tau_1,\tau_2,\ldots,\tau_m$, $0\leqq\tau_i\leqq T$, such that

$$N(\tau_i+0)-N(\tau_i-0) \geqq \tfrac{1}{4}, \qquad i = 1,2,\ldots,m.$$

Choose $\varepsilon>0$ so small so that the intervals $[\tau_i-\varepsilon,\ \tau_i+\varepsilon]$ are non-overlapping, and so that

$$N(\tau_i+\varepsilon)-N(\tau_i-\varepsilon) \leqq \tfrac{1}{2}(\theta+1), \qquad i = 1,2,\ldots,m.$$

This guarantees that, as in the last proof,

$$\int_{\tau_i-\varepsilon}^{t} |d_uJ(u,t)| \leqq \tfrac{1}{2}(\theta+1), \qquad \tau_i-\varepsilon\leqq t\leqq\tau_i+\varepsilon, \quad i = 1,2,\ldots,m.$$

Condition (3.4) guarantees that we can make

$$\int_{\tau_i-\varepsilon}^{t} |d_uZ(u,t;0)|, \qquad \tau_i-\varepsilon\leqq t\leqq\tau_i+\varepsilon,$$

arbitrarily small by taking $\varepsilon>0$ sufficiently small. Let us therefore choose $\varepsilon>0$ so small so that actually

$$(3.14)\qquad \int_{\tau_i-\varepsilon}^{t} [|d_uZ(u,t;0)|+|d_uJ(u,t)|] < 1, \qquad \tau_i-\varepsilon\leqq t\leqq\tau_i+\varepsilon, \quad i = 1,2,\ldots,m.$$

Let $R=[a,b]$ be a component of $[0,T]-\bigcup_{i=1}^{m}(\tau_i-\varepsilon,\tau_i+\varepsilon)$. If $t\in R$ then

$$N(t+0)-N(t-0) < \tfrac{1}{4}.$$

Therefore we can find points $a=s_1^R<s_2^R<\ldots<s_p^R=b$ such that, say,

$$N(s_k^R)-N(s_{k-1}^R) \leqq \tfrac{3}{8}, \qquad k = 1,2,\ldots,p.$$

Hence

$$\int_{s_{k-1}^R}^{t} [|d_u Z(u,t;\,0)| + |d_u J(u,t)|] \leqq \tfrac{3}{8} + \tfrac{3}{8} = \tfrac{3}{4}, \qquad s_{k-1}^R \leqq t \leqq s_k^R \,.$$

Thus if we choose as the points $\{t_i\}$ the points $\{\tau_i \pm \varepsilon\}$ and $\{s_k^R\}$, we shall have

$$\int_{t_{i-1}}^{t} [|d_u Z(u,t;\,0)| + |d_u J(u,t)|] < 1, \qquad t_{i-1} \leqq t \leqq t_i, \qquad i = 1, 2, \ldots \,.$$

The theorem then follows from the lemma.

4. Example.

We shall illustrate the above results by an application to a stochastic process occurring in the theory of queues.

Consider a single server queue, serving customers arriving at $0 < t_1 < t_2 < \ldots$ on a first-come, first-served basis. Let the service period of the k^{th} customer be $\chi_k \geqq 0$. Following Takács [4] let the so-called virtual waiting time $\eta(t)$, $t \geqq 0$, be defined as follows:

(i) $\eta(0) = \eta(0+) \geqq 0$ is given (the waiting time of an "inspector" if he joined the queue at $t = 0$);

(ii) $\eta(t_k + 0) - \eta(t_k - 0) = \chi_k$, $k = 1, 2, \ldots$;

(iii) $\eta(t) = \max[\eta(t_k + 0) - (t - t_k), 0]$, $t_k < t \leqq t_{k+1}$, $k = 1, 2, \ldots$ (Cf. Fig. 3).

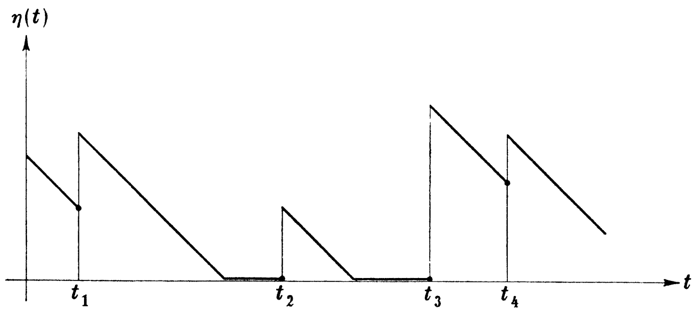

Fig. 3.

The sequences $\{t_k\}$, $\{\chi_k\}$ are random, and shall be assumed to be distributed as follows:

(4.1) The process $\nu(t) = \max\{k \mid t_k < t\}$ is a process with independent increments; $E\nu(t) < \infty$, $t \geqq 0$.

(4.2) The variables χ_k are mutually independent; each χ_k is independent of $\{t_j\}_{j \neq k}$.

(4.3) $\Pr\{\chi_k \leqq y\} = H(y, t_k)$, where $H(y,t)$ is a distribution function in y for each $t \geqq 0$.

It follows from the above that $\eta(t)$ is a Markov process, but we shall not use this fact. (As a special case $\eta(t)$ may be non-random. This occurs when $\{t_k\}$ and $\{\chi_k\}$ are non-random, and hence $H(y,t_k)$ is a one-step function in y.) The following two quantities are of interest [3][4]:

(4.4) $\Pr\{\eta(t) \leqq \xi\}$, if $\eta(0) = 0$,

(4.5) $\Pr\{\beta \leqq t\}$, where $\beta = \inf\{t \mid \eta(t) = 0\}$, $\eta(0) > 0$.

Let us define an auxiliary process, $x(t)$, as follows.

(i) $x(0) = x(0+) = -\eta(0)$;
(ii) $x(t_k - 0) - x(t_k + 0) = \chi_k$, $k = 1, 2, \ldots$,
(iii) $x(t) = x(t_k + 0) + t - t_k$, $t_k < t \leqq t_{k+1}$.

It follows from (4.1)–(4.3) that $x(t)$ is a process with independent increments satisfying (3.1)–(3.5).

From the definitions of $\eta(t)$ and $x(t)$ it is clear that their first zeros coincide, that is, in terms of the notation introduced in section 1, $\alpha = \beta$, and, hence,

(4.6) $$\Pr\{\beta \leqq t\} = B(t) .$$

Also, it turns out that, if $\eta(0) = 0$,

$$\eta(t) = m(t) - x(t), \qquad m(t) = \max_{0 \leqq \tau \leqq t} x(\tau) .$$

This can be seen most readily if $x(t)$ and $m(t)$ are graphed on the same set of coordinate axes. It is also a consequence of Lemma 5.1 of [3]. Hence, in the notation of the previous sections, if $\eta(0) = 0$,

(4.7) $$\Pr\{\eta(t) \leqq \xi\} = F(t, \xi), \qquad \xi \geqq 0 .$$

(The question of determining $F(t,\xi)$ when $\eta(0) > 0$ can also be handled by the methods of this paper, if $m(t)$ is defined slightly differently.)

We shall now consider the right-hand sides and kernels of (3.6) and (3.8) in the present situation, in order to obtain Volterra equations of the second type for $\Pr\{\beta \leqq t\}$ and $\Pr\{\eta(t) = 0\}$. Such equations have previously been obtained by completely different methods in [3] for the case when $\{t_k\}$ are the instants of a variable Poisson process, and $H(x,t) = H(x)$ satisfies certain regularity conditions. A Volterra equation of the first type for $\Pr\{\eta(t) = 0\}$, and expression for $\Pr\{\eta(t) \leqq \xi\}$, $\xi > 0$, by quadratures, was obtained by Beneš [1] by means of a combinatorial argument based on the special geometric properties of the sample functions $\eta(t)$ and $x(t)$.

Since

$$x(t)-x(0) = -\sum_{0<t_k<t} \chi_k + t, \tag{4.8}$$

we find, for the right-hand side of (3.6),

$$\Pr\{x(t) \geqq x(0)-\xi\} = \Pr\left\{\sum_{0<t_k<t} \chi_k \leqq t+\xi\right\}. \tag{4.9}$$

Similarly, if we set $\eta(0) = -x(0) = \chi_0$, the right-hand side of (3.8) becomes

$$\Pr\{x(t) \geqq 0\} = \Pr\left\{\sum_{0\leqq t_k<t} \chi_k \leqq t\right\}. \tag{4.10}$$

By (4.8), $Z(a,b;\xi)$, $a<b$, is the expected number of elements of the set

$$\left\{ t \;\middle|\; a \leqq t < b,\ \sum_{t<t_k<b} \chi_k = b-t+\xi \right\}, \tag{4.11}$$

and $J(a,b)$, $a<b$, is the expected number of elements of the set

$$\left\{ t \;\middle|\; a \leqq t < b,\ \sum_{t<t_k<b} \chi_k < b-t,\ \sum_{t\leqq t_k<b} \chi_k > b-t \right\}. \tag{4.12}$$

What we actually need, however, is $d_uZ(u,t)$, and $d_uJ(u,t)$. In order to simplify the situation we shall henceforth suppose that the probability that t_k has any particular value is zero; that is $\eta(t)$ and $x(t)$ have no fixed discontinuities. We also assume that $H(y,t)$ is absolutely continuous in y and continuous as a function of t. The general case could easily be handled with only slight modifications. Set

$$S(a,b;y) = \Pr\left\{\sum_{a<t_k<b} \chi_k \leqq y\right\}, \qquad a < b.$$

If the expected number of "events" of some sort in a given interval is finite, and if at most one event can occur at one time, then the expected number of such events in a small interval is essentially the probability that exactly one event occurs in that small interval. Applying this remark to the case when the "events" are the occurrences of an element of (4.12), we find

$$-d_uJ(u,t) = \Pr\{\nu(\cdot) \text{ has a jump in } (u, u+du)\}\,.$$

$$\cdot\Pr\left\{\sum_{u<t_k<t} \chi_k \leqq t-u,\ \chi_{(u)} + \sum_{u<t_k<t} \chi_k > t-u\right\}.$$

Here $\chi_{(u)}$ is a random variable with distribution $H(y,u)$, and which is independent of $\{\chi_k\}$, $u<t_k<t$.

Thus, defining $S'(a,b;y)$ as the convolution (with respect to y),

$$S'(a,b;y) = \int_0^y S(a,b;\lambda)\,dH(y-\lambda,a)\,,$$

we find

$$(4.13) \qquad -d_u J(u,t) = \Pr\{\nu(\cdot) \text{ has a jump in } (u,u+du)\} \cdot [S(u,t;t-u) - S'(u,t;t-u)]\,.$$

Similarly, since $x(t)$ is with high probability linear with slope $+1$ in the interval $(u,u+du)$

$$-d_u Z(u,t) = \Pr\left\{t-u+\xi-du < \sum_{u<t_k<t} \chi_k \leqq t-u+\xi\right\}.$$

Hence,

$$(4.14) \qquad -d_u Z(u,t) = \left[\frac{\partial S(u,t;y)}{\partial y}\right]_{y=t-u+\xi} du\,.$$

The formulas (4.9), (4.10), (4.13), (4.14) show how the kernels and right-hand sides of the equations (3.6) and (3.8) for $F(t;\xi)$ and $B(t)$ are determined from a knowledge of the function $S(a,b;y)$. To illustrate this more specifically, suppose the following is true.

(4.15) The instants $\{t_k\}$ are the events of a non-stationary Poisson process with density $\lambda(t)$, $\lambda(t) \geqq 0$,

$$\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau < \infty, \quad t \geqq 0\,.$$

(4.16) The random variables $\chi_1, \chi_2, \ldots$ are equidistributed with distribution

$$H(y) = \int_0^y h(z)\,dz\,.$$

Let $h^{*n}(y)$ denote the n-fold convolution of $h(y)$, and let

$$H^{*n}(y) = \int_0^y h^{*n}(z)\,dz\,.$$

Then

$$S(a,b;y) = \sum_{k=0}^{\infty} e^{-[\Lambda(b)-\Lambda(a)]} \frac{[\Lambda(b)-\Lambda(a)]^k}{k!} H^{*k}(y)\,.$$

Therefore, if (4.15)–(4.16) hold, (4.9) becomes

$$(4.17) \qquad \Pr\{x(t) \geqq x(0)-\xi\} = \sum_{k=0}^{\infty} e^{-\Lambda(t)} \frac{[\Lambda(t)]^k}{k!} H^{*k}(t+\xi)\,.$$

To obtain (4.10) we must convolute $S(0,t;\, y)$ with the distribution of χ_0, and then set $y=t$. Similarly, by (4.14),

$$(4.18)\qquad -d_u Z(u,t) = \left[\frac{\partial}{\partial y}\sum_{k=1}^{\infty} e^{-[\Lambda(t)-\Lambda(t-u)]}\frac{[\Lambda(t)-\Lambda(t-u)]^k}{k!} H^{*k}(y)\right]_{y=t-u+\xi} du, \qquad u < t .$$

Also, by (4.13),

$$(4.19)\qquad -d_u J(u,t) = \lambda(u)du \cdot \sum_{k=0}^{\infty} e^{-[\Lambda(t)-\Lambda(u)]}\frac{[\Lambda(t)-\Lambda(u)]^k}{k!}[H^{*k}(t-u)-H^{*(k+1)}(t-u)], \qquad u < t .$$

If the expressions (4.17)–(4.19) are used in (3.6) and (3.8) we obtain Volterra equations of the second kind for $F(t;\, 0)$ and $B(t)$ under the assumptions (4.15)–(4.16). A computation shows that these integral equations actually agree with the ones in theorems 1 and 2 of [3] in case $H(y)$ and $\Lambda(t)$ satisfy the further restrictions imposed in [3].

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UNIVERSITY OF AARHUS, DENMARK