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# CONVEX IDEALS AND POSITIVE MULTIPLICATIVE FORMS IN PARTIALLY ORDERED ALGEBRAS

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## Introduction.

Recently K. E. Aubert [1] has proved that the real ordered group algebra of a locally compact Abelian group contains one and only one convex regular maximal ideal. This fact is closely related to the uniqueness of the Haar measure. In the first part of this paper we shall give a correspondence between convex regular maximal ideals and positive multiplicative forms in real partially ordered Banach algebras (Theorem 1). As immediate consequences of this correspondence we derive the result of K. E. Aubert and a modified form of the Gelfand–Mazur theorem (Corollary 1).

In the second part of this paper we prove the existence of a positive multiplicative form on abstract (not necessarily topological) partially ordered algebras (Theorem 6). This theorem yields a proof for the existence of a positive multiplicative measure on certain generalized convolution algebras.

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## 1. Convex ideals in Banach algebras.

Let $B$ be a commutative Banach algebra over the real numbers and let $B_+ \subseteq B$ be a subset having the following properties:

1) If $f \in B_+$ and $g \in B_+$, then $f+g \in B_+$.
2) If $f \in B_+$ then $\alpha f \in B_+$ for any real number $\alpha \geqq 0$.
3) If $f \in B_+$ then $f^2 \in B_+$.

The set $B_+$ endows $B$ with a partial ordering $\geqslant$ if we define $f \geqslant g$ to mean that $f-g \in B_+$. We shall call any set $B_+$ having the above proper-

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ties 1), 2), and 3) a *positive wedge*. In the sequel we shall need the concepts of convex ideals and positive multiplicative forms.

DEFINITION. An ideal $I \subseteq B$ is called *convex* if $f \in I$, $g \in I$ and $f \leqslant h \leqslant g$ for $h \in B$ implies that $h \in I$.

DEFINITION. A complex-valued continuous linear functional $\varphi$ defined on $B$ will be called a *multiplicative form* if $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f,g \in B$. Such a form $\varphi$ will be called a *positive form* if $\varphi$ takes only real values on $B_+$ and if $\varphi(f) \geqq 0$ for $f \in B_+$.

DEFINITION. The positive wedge $B_+$ will be said to *generate* $B$ if for every $f \in B$ there exist $f_1, f_2 \in B_+$ such that $f = f_1 - f_2$.

THEOREM 1. *Let $B$ be a real commutative Banach algebra and let $B_+$ be a positive wedge in $B$. The kernel $\varphi^{-1}(0) = M_\varphi$ of any positive multiplicative form $\varphi$ is a convex regular maximal ideal. If $M$ is a convex regular maximal ideal, there exists a positive multiplicative form $\varphi$ such that $\varphi^{-1}(0) = M$.*

PROOF. If $\varphi$ is a positive multiplicative form on $B$ then the proof that $M_\varphi$ is a convex regular maximal ideal is immediate from basic Banach algebra theory (see, for instance, L. Loomis [9, p. 69]). Suppose now that $M$ is any convex regular maximal ideal in $B$. By the Gelfand–Mazur theorem (see S. Mazur [11], I. Gelfand [5], L. Tornheim [12]) the quotient space $B/M$ is either the real number field or the complex number field. Let $\varphi$ be a homomorphism of $B$ into the complex numbers having $M$ as its kernel. We shall show that $\varphi(B_+) \subseteq [0, \infty)$.

First we remark that $\varphi(B_+)$ is a positive wedge in the complex plane. In fact if $\varphi(f), \varphi(g) \in \varphi(B_+)$ then there exist elements $h, h' \in M$ and $k, k' \in B_+$ so that $f = k + h$ and $g = k' + h'$. Hence $\varphi(f) + \varphi(g) = \varphi(k + k') + \varphi(h + h') = \varphi(k + k') \in \varphi(B_+)$. The proofs of properties 2) and 3) of a positive wedge are similar. Using the fact that $M$ is convex we now show that

a) If $\varphi(f) \in \varphi(B_+)$ and $-\varphi(f) \in \varphi(B_+)$, then $\varphi(f) = 0$.

In fact if $\pm\varphi(f) \in \varphi(B_+)$ there exist then $h, h' \in M$ and $k, k' \in B_+$ such that $f = h + k$ and $-f = h' + k'$. Therefore $f - h \geqq 0$ and $-f - h' \geqq 0$ which implies that $f \in M$, that is, $\varphi(f) = 0$.

We show now that $\varphi(B_+)$ cannot contain a real number $-\varrho < 0$. (The simple argument which follows is due to O. Hustad, Oslo, and replaces a more complicated proof by the author.) In fact if $-\varrho \in \varphi(B_+)$ it follows respectively from properties 2) and 3) of a positive wedge that

$-\varrho^2 \in \varphi(B_+)$ and $\varrho^2 \in \varphi(B_+)$, and this contradicts a). This together with a) shows that $\varphi(B_+)$ is a wedge with an angle $<\pi$ situated in the complex plane as indicated in Fig. 1.

It is clear that by choosing the complex number $\lambda$ within $\varphi(B_+)$ but close to the edge of $\varphi(B_+)$ which forms the smaller angle with the negative real axis we obtain $\lambda^2 \notin \varphi(B_+)$ unless $\varphi(B_+) \subseteq [0,\infty)$. The assertion $\varphi(B_+) \subseteq [0,\infty)$ follows therefore from a) and so the theorem is proved.

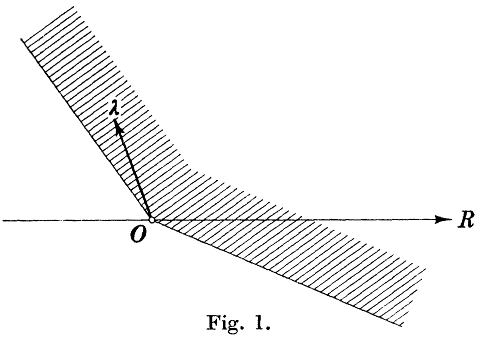

Fig. 1.

We note that if $B_+$ generates $B$, then $\varphi$ is obviously real-valued and hence as an immediate consequence of Theorem 1 we have the following special case of the Gelfand–Mazur result:

Corollary 1 (Gelfand–Mazur). *If the positive wedge $B_+$ generates $B$, then for any convex regular maximal ideal $M$, the quotient space $B/M$ is order isomorphic to the real number field.*

Corollary 2. *Let $A$ be any (not necessarily topological, nor commutative) algebra over the real numbers and let $A_+$ be a positive wedge in $A$. If $\varphi$ is a linear multiplicative mapping of $A$ into the complex numbers, and if $\varphi^{-1}(0)$ is a convex ideal, then $\varphi(A_+) \subseteq [0,\infty)$.*

## 2. Applications.

Let $G$ be a locally compact Abelian group with Haar measure $m$. The set of all real-valued, $m$-integrable functions on $G$ form an ordered ring $L_R{}^1$ with respect to the convolution product, pointwise addition and pointwise ordering (almost everywhere). The positive wedge $B_+ \subseteq L_R{}^1$ is defined by

$$B_+ = \{f \in L_R{}^1 \mid f \geqq 0 \text{ almost everywhere}\}.$$

K. E. Aubert [1] has shown that $L_R{}^1$ contains a unique convex regular maximal ideal, namely; the kernel of the Haar measure. The uniqueness of this convex regular maximal ideal is due to the fact that the Haar measure is unique as a positive multiplicative form on $L_R{}^1$. The proof which we shall give of this theorem is based on Theorem 1 and differs from that of K. E. Aubert. In particular it does not depend on the duality theory of locally compact Abelian groups, which is essentially used in K. E. Aubert's proof. As a result of this simplification we are able to prove a related result for certain generalized convolution algebras $\{Z, m_z, m, (f,g) \to f*g\}$ (for definitions and properties of such objects see C. Ionescu Tulcea and A. Simon [6], G. Maltese [10]).

THEOREM 2 (K. E. Aubert). *The only (proper) convex regular maximal ideal in $L_R{}^1$ is the kernel of the Haar measure. In other words, if $\mu$ is an order preserving ring homomorphism of $L_R{}^1$ onto an ordered field $\mathscr{F}$ then $\mathscr{F}$ is order isomorphic to the reals and $\mu$ is the Haar measure of the group $G$.*

PROOF. Let $(0) \neq M_R$ be a convex regular maximal ideal in $L_R{}^1$ and let $\varphi$ be a positive multiplicative form which corresponds to $M_R$ as in Theorem 1. The form $\varphi$ is real-valued (and unique) since the wedge $B_+$ generates $L_R{}^1$. Define a complex-valued functional $\tilde{\varphi}$ on $L_R{}^1 + iL_R{}^1 = L_K{}^1$ (the group algebra of all complex-valued $m$-integrable functions) by the relation

$$\tilde{\varphi}(f+ig) = \varphi(f) + i\varphi(g), \qquad f,g \in L_R{}^1 .$$

It is immediate that $\tilde{\varphi}$ is a continuous, multiplicative, linear functional on $L_K{}^1$. But such a functional has the well-known representation (L. Loomis [9, p. 136]):

$$\tilde{\varphi}(F) = \int F(s)\chi(s)\, dm(s), \qquad F \in L_K{}^1 ,$$

for some continuous character $\chi$ of the group $G$. Hence we have

$$0 \leqq \varphi(f) = \tilde{\varphi}(f) = \int f(s)\chi(s)\, dm(s) \qquad \text{for all} \quad f \in B_+ .$$

From this we conclude that $\chi(s) \geqq 0$ everywhere on $G$ and hence $\chi(s) \equiv 1$. Finally we have

$$\left\{f \in L_R{}^1 \;\middle|\; \int f(s)\, dm(s) = 0\right\} = \{f \in L_R{}^1 \mid \varphi(f) = 0\} = \varphi^{-1}(0) = M_R$$

to conclude the proof of the theorem.

Using essentially the same methods we can prove also the following theorem.

THEOREM 3. *Every convex regular maximal ideal $M_R$ in the real generalized convolution algebra $L_R{}^1 = \{Z, m_z(\text{real}), m, (f,g) \to f*g\}$ can be expressed by the relation*

$$M_R = \{f \in L_R{}^1 \mid \textstyle\int f(s)\,\chi(s)\, dm(s) = 0\}$$

*for some unique non-negative character* $\chi$ (see C. Ionescu Tulcea and A. Simon [6, p. 1765]) for the definition of a character of a generalized convolution algebra).

## 3. Partially ordered algebras.

In this section we shall study abstract partially ordered algebras. Making use of a basic result due to R. Kadison [7] we prove the existence of a non-zero, real-valued, positive, multiplicative, linear functional on such partially ordered algebras. As an immediate corollary of this result, which may be of independent interest, we obtain an existence proof for a positive multiplicative measure on certain generalized convolution algebras.

Let $V$ be a vector space over the real number field and let $V_+$ be a subset of $V$ satisfying the following properties.

V1) If $f,g \in V_+$ then $f+g \in V_+$.
V2) If $f \in V_+$ then $\alpha f \in V_+$ for any real number $\alpha \geqq 0$.
V3) If $f \in V_+$ and $-f \in V_+$ then $f=0$.

As usual we shall call $V_+$ the *positive cone* of $V$ and the elements of $V_+$ will be called the positive elements. In the sequel we shall always suppose that the partially ordered vector space $V$ has an *order unit* 1, that is;

V4) There exists an element $1 \in V_+$, $1 \neq 0$, with the property that for every $f \in V$ there is a real number $\lambda_f > 0$ such that $-\lambda_f 1 \leqslant f \leqslant \lambda_f 1$.

It follows from the above properties that the cone $V_+$ generates $V$ (in fact we may take $f_1 = \lambda_f 1$ and $f_2 = \lambda_f 1 - f$.)

Denote by $V^*$ the algebraic dual of $V$ (N. Bourbaki [3, p. 49]). In $V^*$ we define a set $K^*$ called the (*normalized*) *dual cone* as follows:

$$K^* = \{x \in V^* \mid x(f) \geqslant 0 \text{ for } f \in V_+,\ x(1) = 1\}\,.$$

An element $x \in K^*$ is called a *positive normalized linear functional* (we use throughout essentially the terminology of R. Kadison [7]). An extension theorem due to R. Kadison [7, Corollary 2.1, pp. 5–6] shows that $K^*$ is non-void. The set $K^*$ is obviously geometrically convex, that is; if $x,y \in K^*$ and $\beta \in [0,1]$ then $\beta x + (1-\beta)y$ is also an element of $K^*$. The set $K^*$ is also compact for the topology induced by the locally convex topology $\sigma(V^*, V)$ (see N. Bourbaki [3, p. 50 and p. 60]).

We suppose now that there exists a multiplication "$\star$" in $V$ with the following properties:

V5) $V$ is a commutative algebra under "$\star$".
V6) For each $f \in V_+$, $f \neq 0$, there exists a number $\alpha_f > 0$ such that $1 \star f \geqslant \alpha_f 1$.
V7) If $f \in V_+$ and $g \in V_+$ then $f \star g \in V_+$.

We shall give now several results concerning $V$ and $V^*$. They lead to the proof of the existence of a non-identically zero, positive multiplicative element in $V^*$.

PROPOSITION 1. *For every $f \in V_+$, $f \neq 0$ there exists $x_f \in K^*$ such that for all $g \in V$ we have $x_f(g \star f) = x_f(g) x_f(1 \star f)$.*

PROOF. Define a mapping $S$ on $K^*$ by the equations

$$(Sx)(g) = x(g \star f)/x(1 \star f), \qquad x \in K^*,\ g \in V\,.$$

Since $1 \star f \geqslant \alpha_f 1$ for some $\alpha_f > 0$, we have $x(1 \star f) \geqslant \alpha_f > 0$ for any $x \in K^*$, so that the mapping $S$ is well-defined. We can easily show that $S$ maps $K^*$ into $K^*$. In fact if $g \in V_+$ then $g \star f \in V_+$ and $x(g \star f) \geqslant 0$, so that $(Sx)(g) \geqslant 0$. The mapping $S$ is also seen to be continuous in the $\sigma(V^*, V)$ topology. We have therefore a continuous mapping, $S$, of the convex, compact $\big($for $\sigma(V^*, V)\big)$ set, $K^*$, into itself. By the Schauder–Tychonoff fixed point theorem (N. Dunford and J. Schwartz [4, p. 456]) there exists $x_f \in K^*$ such that $Sx_f = x_f$. In other words we have

$$x_f(g \star f) = x_f(g) x_f(1 \star f)$$

for all $g \in V$. Hence the proposition is proved.

REMARK. For future applications of Proposition 1 let us note that if we define the set $M_f = \{g \in V \mid x_f(g) = 0\}$, then $f \star M_f \subseteq M_f$. (For any set $M \subseteq V$ and $f \in V$ we define $f \star M = \{f \star g \mid g \in M\}$.)

DEFINITION. A linear space $I \subseteq V$ is called an *order ideal* if $f \in I$ whenever $-h \leqslant f \leqslant h$ for some $h \in I$.

Let $I$ be an order ideal in $V$, $V/I$ the quotient space, and $\varphi$ the canonical mapping of $V$ onto $V/I$; we shall sometimes write $\varphi(f) = \dot{f}$. Let $V_+$ be the positive cone in $V$ and $\dot{V}_+ = \varphi(V_+)$. Then $\dot{V}_+$ satisfies conditions V1), V2), V3), V4). The order unit in $V/I$ is $\dot{1}$. In the sequel we shall make strong use of the following fundamental result due to R. Kadison [7, pp. 3–6]:

THEOREM 4. *If $V$ is a partially ordered vector space over the reals with order unit* 1, *the quotient space $V/M$ for any maximal order ideal $M$ is order isomorphic to the real line (considered as an ordered vector space). There exists a one-to-one correspondence between maximal order ideals $M$ and positive normalized linear functionals $x$ given by $M = \{g \in V \mid x(g) = 0\}$.*

PROPOSITION 2. *For every $f \in V_+$, $f \neq 0$, define the following sets:*

$$M_f = \{g \in V \mid x_f(g)=0\}\,,$$
$$N_f = \{g\star f - x_f(1\star f)g \mid g \in V\}\,,$$
$$A_f = \{h \in V \mid h_1 \leqslant h \leqslant h_2 \textit{ for } h_1, h_2 \in N_f\}\,.$$

*Then the following statements are valid*: 1) $M_f$ *is a maximal order ideal.* 2) $N_f \subseteq A_f \subseteq M_f$. 3) $A_f$ *is an order ideal.* 4) *For every* $k \in V_+$ *we have* $k\star A_f \subseteq A_f$.

PROOF. 1) This is a consequence of the R. Kadison result (Theorem 4). 2) This is obvious by definition (to show that $A_f \subseteq M_f$ we use Proposition 1). 3) Let $h \in A_f$ and suppose that $-h \leqslant k \leqslant h$ for some $k \in V$. There exists $h_2 \in N_f$ such that $h \leqslant h_2$ and hence $-h_2 \leqslant -h \leqslant k \leqslant h \leqslant h_2$ so that $k \in A_f$. (It is immediate that $A_f$ is a vector space). 4) Using the fact that the multiplication "$\star$" is commutative we show first that $k\star N_f \subseteq N_f$ for $k \in V_+$. For this purpose let $g\star f - x_f(1\star f)g \in N_f$. Then

$$k\star[g\star f - x_f(1\star f)g] = (g\star k)\star f - x_f(1\star f)g\star k \in N_f\,,$$

hence $k\star N_f \subseteq N_f$. Finally if $h \in A_f$ then $h_1 \leqslant h \leqslant h_2$ for some $h_1, h_2 \in N_f$. Hence $k\star h_1 \leqslant k\star h \leqslant k\star h_2$ and since $k\star h_1, k\star h_2 \in N_f$ we conclude that $k\star h \in A_f$, so that the proposition is completely proved.

Now let $\mathfrak{H}$ be the set of all order ideals $H$ which have the property that for all $k \in V_+$, $k\star H \subseteq H$. By Zorn's lemma the family $\mathfrak{H}$ has a maximal element $I$.

THEOREM 5. *The maximal element $I$ of the family $\mathfrak{H}$ is a maximal order ideal.*

PROOF. Suppose that the quotient space $V/I$ is not of dimension one, that is; $I$ is not a maximal order ideal. We denote by $\varphi$ the canonical mapping $V \to V/I$. The normalized dual cone $\dot{K}^*$ in $(V/I)^*$ is defined as follows:

$$\dot{K}^* = \{X \in (V/I)^* \mid X(\dot{f}) \geqslant 0 \text{ for } \dot{f} \in \dot{V}_+,\ X(\dot{1})=1\}\,.$$

We recall here that $\dot{K}^*$ is geometrically convex and compact (for the topology $\sigma((V/I)^*, (V/I))$. Define a multiplication "$\dot{\star}$" in $V/I$ as follows:

$$\dot{f}\dot{\star}\dot{g} = \dot{\widehat{f\star g}}\,.$$

The multiplication "$\dot{\star}$" is well-defined on $V/I$ since $k\star I \subseteq I$ for all $k \in V_+$ implies that $I$ is also an ideal in the algebra $V$. Now if we take $k \in V_+$ we obtain

$$\dot{k}\dot{\star}\dot{1} = \dot{\widehat{k\star 1}} \geqslant \dot{\widehat{\alpha_k 1}} = \alpha_k \dot{1}\,.$$

By Proposition 1 for $\dot{f} \in \dot{V}_+$, $\dot{f} \neq 0$, there exists $X_{\dot{f}} \in \dot{K}^*$ such that

$$X_{\dot{f}}(\dot{g} \star \dot{f}) = X_{\dot{f}}(\dot{g}) X_{\dot{f}}(\dot{\mathrm{i}} \star \dot{f})$$

for all $\dot{g} \in V/I$. We may and do suppose that $\dot{f}/\lambda$ is not an identity no matter how the scaler $\lambda \neq 0$ is chosen. This is possible since we have assumed that $V/I$ has dimension strictly greater than one. Define now the sets

$$N_{\dot{f}} = \{\dot{g} \star \dot{f} - X_{\dot{f}}(\dot{\mathrm{i}} \star \dot{f})\dot{g} \mid \dot{g} \in V/I\},$$

$$A_{\dot{f}} = \{\dot{h} \mid \dot{h}_1 \leqslant \dot{h} \leqslant \dot{h}_2 \quad \text{for} \quad \dot{h}_1, \dot{h}_2 \in N_{\dot{f}}\}.$$

From Proposition 2 we conclude that $A_{\dot{f}}$ is an order ideal and that $\dot{k} \star A_{\dot{f}} \subseteq A_{\dot{f}}$ for every $\dot{k} \in \dot{V}_+$. We remark now that we have

$$V \supseteq \varphi^{-1}(A_{\dot{f}}) \supseteq I, \qquad I \neq \varphi^{-1}(A_{\dot{f}}), \qquad V \neq \varphi^{-1}(A_{\dot{f}}).$$

In fact if $\varphi^{-1}(A_{\dot{f}}) = I$, then $A_{\dot{f}} = \{0\}$, hence $N_{\dot{f}} = \{0\}$ so that $\dot{g} \star \dot{f} = X_{\dot{f}}(\dot{\mathrm{i}} \star \dot{f})\dot{g}$ for all $\dot{g} \in V/I$. This implies that $\dot{g} \star (\dot{f}/\lambda) = \dot{g}$ where $\lambda = X_{\dot{f}}(\dot{\mathrm{i}} \star \dot{f})$, that is; $\dot{f}/\lambda$ is an identity for $V/I$ contradicting our choice of $\dot{f}$. On the other hand if $\varphi^{-1}(A_{\dot{f}}) = V$, then $\varphi^{-1}(M_{\dot{f}}) = V$, and hence $X_{\dot{f}} \equiv 0$, which is impossible. The inclusions $V \supseteq \varphi^{-1}(A_{\dot{f}}) \supseteq I$ being obvious, the assertion is proved.

We now show that $\varphi^{-1}(A_{\dot{f}})$ is an order ideal. For this purpose let $g \in V$ and suppose that $-h \leqslant g \leqslant h$ for some $h \in \varphi^{-1}(A_{\dot{f}})$. Then $-\dot{h} \leqslant \dot{g} \leqslant \dot{h}$, so $\dot{g} \in A_f$ and hence $g \in \varphi^{-1}(A_{\dot{f}})$. Therefore we conclude that $\varphi^{-1}(A_{\dot{f}})$ is indeed an order ideal. Next we assert that

$$k \star \varphi^{-1}(A_{\dot{f}}) \subseteq \varphi^{-1}(A_{\dot{f}})$$

for every $k \in V_+$. To see this let $h \in \varphi^{-1}(A_{\dot{f}})$, then $\dot{h} \in A_{\dot{f}}$ and $\widehat{k \star h} = \dot{k} \star \dot{h} \in A_{\dot{f}}$ which implies that $k \star h \in \varphi^{-1}(A_{\dot{f}})$ and hence the assertion is proved. But these statements contradict the assumption that $I$ was a maximal element of $\mathfrak{H}$. Therefore we conclude that $V/I$ is one dimensional, that is; $I$ is a maximal order ideal.

THEOREM 6. *There exists a non-identically zero linear functional $m$ defined on $V$ such that $m(g) \geqslant 0$ for $g \in V_+$, and $m(f \star g) = m(f)m(g)$ for all $f, g \in V$.*

PROOF. By the cited result (Theorem 4) of R. Kadison [7] there is a positive normalized linear functional $x$ corresponding to the maximal order ideal $I$ of Theorem 5, so that

$$I = \{f \in V \mid x(f)=0\} .$$

For any $g \in V$, $k \in V_+$ we have $g - x(g)1 \in I$ and $k\star g - x(g)1\star k \in I$. From these two facts it follows that

$$x(k\star g) - x(g)x(1\star k) = 0 .$$

Since this is true for all $k \in V_+$, taking $k=1$ we conclude

$$x(1\star g) = x(g)x(1\star 1) .$$

We now let $m = x(1\star 1)x$ to obtain

$$\begin{aligned} m(k\star g) &= x(1\star 1)x(k\star g) = x(1\star 1)x(g)x(1\star k) \\ &= x(1\star 1)x(g)x(1\star 1)x(k) = m(g)m(k) . \end{aligned}$$

Since the cone $V_+$ generates $V$ the conclusion of the theorem is valid.

**4. Applications and examples.**

Let $Z$ be a compact Hausdorff space and let $C(Z)$ denote the vector space of continuous complex-valued functions $f$ defined on $Z$. For each $z \in Z$ let $m_z$ be a positive Radon measure defined on the cartesian product $Z \times Z$. For every $f,g \in C(Z)$ denote by $f\star g$ the function

$$z \to \iint_{Z\times Z} f(x)g(y)\, dm_z(x,y) .$$

We shall suppose that the following conditions (AI) and (AII) of generalized convolution algebras hold:

(AI) The operation $(f,g) \to f\star g$ is a mapping of $C(Z) \times C(Z)$ into $C(Z)$.

(AII) The multiplication $(f,g) \to f\star g$ defines on the vector space $C(Z)$ the structure of a commutative algebra.

(See for instance, G. Maltese [10], C. Ionescu Tulcea and A. Simon [6, conditions (3.1) and (3.2)]). In addition we shall assume that the convolution "$\star$" satisfies the following positivity condition:

(P) If $f$ is a real-valued function in $C(Z)$ and $f \not\equiv 0, f \geqq 0$, then $1\star f(z) > 0$ for all $z \in Z$.

The space $C_R(Z)$ (the real-valued) functions in $C(Z)$) is a partially ordered space over the real numbers with the ordering given by the definition $f \geqslant g$ if $f(z) \geqslant g(z)$ for all $z \in Z$. The partially ordered vector space $C_R(Z)$ has an order unit, namely; the function 1. With this definition of order and with convolution as multiplication, it is immediately seen that $C_R(Z)$ satisfies conditions V1)–V7) of Section 3. Hence we may apply Theorem 6 to find a positive linear functional $m$ defined on $C_R(Z)$ such that

$$(1) \qquad m(f^*g) = m(f)m(g) \qquad \text{for all} \quad f,g \in C_R(Z)\,.$$

$m$ is called a *positive multiplicative measure* on $Z$. (For a discussion concerning generalized convolution algebras and positive multiplicative measures see also the reviews of R. Godement [Math. Reviews 12 (1951), pp. 188–189].) We may therefore restate our existence result in the following form:

THEOREM 7. *There exists a non-identically zero, positive, multiplicative measure defined on every compact Hausdorff space $Z$ which satisfies the above conditions.*

REMARKS. It is clear that equation (1) is also satisfied for complex-valued functions. If a positive multiplicative measure $m$ satisfies equation (1) then $m=0$ if and only if $m(1)=0$. In fact if $m(1)=0$ and $f \in C_R(Z)$, then there exists $\lambda_f > 0$ such that $-\lambda_f 1 \leqslant f \leqslant \lambda_f 1$ and this implies that $m(f)=0$.

Using certain results of M. Krein and M. Rutman [8] and imposing supplementary conditions, Yu. Berezanski and S. Krein [2] have given proofs for the existence and uniqueness of $m$.

We mention briefly a non-trivial example of a system which satisfies conditions V1)–V7). Let $Z=[-1,+1]$, $V=C_R(Z)$ and let $V_+$ be the set of positive (pointwise ordering) functions in $V$. For $f \in V$ choose $\lambda_f = \|f\| = \sup\{|f(z)| \mid z \in Z\}$. The multiplication "$\star$" is defined by

$$f\star g(z) = \tfrac{1}{2}\int_{-1}^{+1}\left[f\left(zt+(1-t^2)^{\frac12}(1-z^2)^{\frac12}\right)+f\left(zt-(1-t^2)^{\frac12}(1-z^2)^{\frac12}\right)\right](1-t^2)^{-\frac12}\, g(t)\, dt.$$

Finally for $f \in V_+$, $f \not\equiv 0$ we let

$$\alpha_f = \int_{-1}^{+1} (1-t^2)^{-\frac12} f(t)\, dt\,.$$

Hence $V$ and $V_+$ satisfy all the properties V1)–V7).

For more details concerning this example and other examples of systems satisfying V1)–V7) we refer to the paper of Yu. Berezanski and S. Krein [2].

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