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ON THE RATE OF GROWTH OF THE PARTIAL MAXIMA OF A SEQUENCE OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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1. Introduction.

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent, identically distributed random variables defined on a probability field $(\Omega, \mathscr{A}, P)$ and let F denote their (common) distribution function so that

$$F(x) = P(\{X_n \leqq x\}) \tag{1.1}$$

for all $x \in (-\infty, \infty)$ and $n = 1, 2, \ldots$.

Our main result (Theorem 1) is the following. Let $\{\lambda_n\}$ be a nondecreasing sequence of real numbers, such that the sequence $\{(F(\lambda_n))^n\}$ is nonincreasing. Then

$$P\left(\left\{\max_{1\leqq k\leqq n} X_k \leqq \lambda_n \text{ for infinitely many } n\right\}\right) = \begin{cases} 0 \\ 1 \end{cases}$$

if

$$\sum_{n=3}^{\infty} (F(\lambda_n))^n \frac{\log\log n}{n} \quad \begin{cases} < \infty \\ = \infty . \end{cases}$$

The result is established by means of a generalization of the convergence part of the Borel–Cantelli lemmas.

Let $A_1, A_2, \ldots, A_n, \ldots$ be an arbitrary sequence of events (measurable subsets of Ω). Because of its intuitive appeal, we shall use the notation $\{A_n \text{ i.o.}\}$ for $\limsup A_n$, that is,

$$\{A_n \text{ i.o.}\} = \limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n , \tag{1.2}$$

i.o. being an abbreviation for "infinitely often". Furthermore, we denote the complement of an event A by A^c.

Finally, let

$$X_{(n)} = \max_{1\leqq k\leqq n} X_k, \qquad n = 1, 2, \ldots , \tag{1.3}$$

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let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be a nondecreasing sequence of real numbers and let

$$E_n = \{X_{(n)} \leqq \lambda_n\}, \qquad n = 1, 2, \ldots . \tag{1.4}$$

The following two theorems describe to some extent the rate of growth of $X_{(n)}$ as $n \to \infty$.

THEOREM 1. *If the sequence $(F(\lambda_n))^n$, $n = 1, 2, \ldots$, is nonincreasing, then*

$$P(\{E_n \text{ i.o.}\}) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{if} \quad \sum_{n=3}^{\infty} (F(\lambda_n))^n \frac{\log\log n}{n} \quad \begin{cases} < \infty \\ = \infty \end{cases}. \tag{1.5}$$

REMARKS. It follows in particular that if for some δ and some n_0

$$F(\lambda_n) = 1 - (1+\delta)\frac{\log\log n}{n} \qquad \text{for} \quad n > n_0 \tag{1.6}$$

then

$$P(\{E_n \text{ i.o.}\}) = \begin{cases} 0 & \text{if} \quad \delta > 0 \\ 1 & \text{if} \quad \delta \leqq 0 . \end{cases} \tag{1.7}$$

The proof of Theorem 1 is given in Section 3. In the course of the proof we shall use a generalization of the convergence part of the Borel–Cantelli lemmas. This generalization is discussed in Section 2. The theorem (in the divergence case) is not true without the condition that $(F(\lambda_n))^n$ be nonincreasing, as will be shown by an example in Section 4.

THEOREM 2.

$$P(\{E_n{}^c \text{ i.o.}\}) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{if} \quad \sum_{n=1}^{\infty} (1 - F(\lambda_n)) \quad \begin{cases} < \infty \\ = \infty \end{cases}. \tag{1.8}$$

REMARKS. This is an elementary and known result, see [3, p. 51]. It follows immediately from the Borel–Cantelli lemmas (stated as Lemmas 1 and 2 in Section 2) and the fact that

$$P(\{E_n{}^c \text{ i.o.}\}) = P(\{X_n > \lambda_n \text{ i.o.}\}) . \tag{1.9}$$

I welcome this opportunity to thank my friend H. Brøns for introducing me to the problem considered here.

2. Generalization of the Borel–Cantelli lemmas.

The celebrated Borel–Cantelli lemmas state that

LEMMA 1. *For any sequence $A_1, A_2, \ldots, A_n, \ldots$ of events satisfying*

$$\sum_{n=1}^{\infty} P(A_n) < \infty , \tag{2.1}$$

we have

(2.2) $$P(\{A_n \text{ i.o.}\}) = 0 .$$

LEMMA 2. *For any sequence $A_1, A_2, \ldots, A_n, \ldots$ of independent events satisfying*

(2.3) $$\sum_{n=1}^{\infty} P(A_n) = \infty ,$$

we have

(2.4) $$P(\{A_n \text{ i.o.}\}) = 1 .$$

For a proof of Lemmas 1 and 2, see e.g. [4, p. 228]. Lemma 1 may be generalized to

LEMMA 1*. *For any sequence $A_1, A_2, \ldots, A_n, \ldots$ of events satisfying*

(2.5) $$P(A_n) \to 0 \quad \textit{as} \quad n \to \infty$$

and

(2.6) $$\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}{}^{c}) < \infty ,$$

we have

(2.7) $$P(\{A_n \text{ i.o.}\}) = 0 .$$

PROOF. Since

(2.8) $$P(\{A_n{}^{c} \text{ i.o.}\}) = \lim_{n\to\infty} P\left(\bigcup_{\nu=n}^{\infty} A_\nu{}^{c}\right) \geqq \lim_{n\to\infty} P(A_n{}^{c}) = 1$$

we have, in consequence of (2.6) and lemma 1,

(2.9) $$P(\{A_n \text{ i.o.}\}) = P(\{A_n \cap A_{n+1}{}^{c} \text{ i.o.}\}) = 0 .$$

Although we shall not use it in the sequel we mention here that the last result is included as a special case in

LEMMA 1**. *For any sequence $A_1, A_2, \ldots, A_n, \ldots$ of events satisfying*

(2.10) $$P(A_n) \to 0 \quad \textit{as} \quad n \to \infty$$

and, for some sequence $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ of positive integers,

(2.11) $$\sum_{n=1}^{\infty} P(A_n \cap A_{n+\nu_n}{}^{c}) < \infty$$

we have

(2.12) $$P(\{A_n \text{ i.o.}\}) = 0 .$$

PROOF. To every k, $k=1,2,\ldots$, let us define a sequence of integers $i_{k1}, i_{k2}, \ldots, i_{kn}, \ldots$ as follows:

(2.13) $$i_{kn} = \begin{cases} k & \text{for} \quad n=1 \\ i_{k,n-1} + \nu_{i_{k,n-1}} & \text{for} \quad n>1 . \end{cases}$$

We have in consequence of (2.11) and Lemma 1

$$P(A_n \cap A_{n+\nu_n}{}^c \text{ i.o.}) = 0 \tag{2.14}$$

and, in consequence of (2.11) and Lemma 1*, for every k,

$$P\left(\bigcap_{n=1}^{\infty} A_{i_{kn}}\right) \leqq P\left(\limsup_{n\to\infty} A_{i_{kn}}\right) = 0 . \tag{2.15}$$

Hence

$$P(\{A_n \text{ i.o.}\}) = P\left(\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} A_{i_{kn}}\right) \leqq \sum_{k=1}^{\infty} P\left(\bigcap_{n=1}^{\infty} A_{i_{kn}}\right) = 0 . \tag{2.16}$$

3. Proof of Theorem 1.

LEMMA 3. *Without loss of generality it may be assumed that the distribution function F is continuous.*

PROOF. Let $X_1^*, X_2^*, \ldots, X_n^*, \ldots$ denote a sequence of independent, identically distributed random variables defined on a probability field $(\Omega^*, \mathscr{A}^*, P^*)$ and having a distribution function F^*, such that F^* is continuous and

$$F^*(\lambda_n) = F(\lambda_n), \qquad n = 1,2,\ldots . \tag{3.1}$$

Let

$$X_{(n)}^* = \max_{1\leqq k\leqq n} X_k^*, \qquad n = 1,2,\ldots , \tag{3.2}$$

and

$$E_{(n)}^* = \{X_{(n)}^* \leqq \lambda_n\}, \qquad n = 1,2,\ldots . \tag{3.3}$$

The range space of the sequences $\{X_n\}$ and $\{X_n^*\}$ is the product space $\prod_{n=1}^{\infty} R_n$, where for each n, R_n is a real line with points x_n. Let P_0 and P_0^* denote the probability measures induced on the Borel-field $\mathscr{B}$ of $\prod_{n=1}^{\infty} R_n$ by $\{X_n\}$ and $\{X_n^*\}$, respectively, i.e. for each $B \in \mathscr{B}$

$$P_0(B) = P\big(\{(X_1, X_2, \ldots, X_n, \ldots) \in B\}\big) \tag{3.4}$$

and

$$P_0^*(B) = P^*\big(\{(X_1^*, X_2^*, \ldots, X_n^*, \ldots) \in B\}\big) . \tag{3.5}$$

Because of (3.1) we have $P_0^*(B) = P_0(B)$ for all B of the form

$$B = \{x_i \leqq \lambda_k\} \tag{3.6}$$

and hence, since P_0 and P_0^* are product measures (for a proof hereof see e.g. [4, p. 230]) they must coincide on the minimal σ-field containing all such sets. Thus, in particular, we may conclude

$$P\big(\{E_n \text{ i.o.}\}\big) = P^*(\{E_n^* \text{ i.o.}\}) . \tag{3.7}$$

Lemma 3 is an immediate consequence of this relation and the obvious equality

$$\sum_{n=3}^{\infty}(F^*(\lambda_n))^n \frac{\log\log n}{n} = \sum_{n=3}^{\infty}(F(\lambda_n))^n \frac{\log\log n}{n}. \tag{3.8}$$

Applying this result we can prove

LEMMA 4. *Without loss of generality it may be assumed that*

$$\alpha_n \leqq F(\lambda_n) \leqq \beta_n \quad \textit{for} \quad n>2, \tag{3.9}$$

where

$$\alpha_n = \exp\left(-2\frac{\log\log n}{n}\right) \tag{3.10}$$

and

$$\beta_n = \exp\left(-\frac{1}{2}\frac{\log\log n}{n}\right). \tag{3.11}$$

REMARK. The present lemma (and its proof) is quite similar to the lemma (and its proof) in Section 2 of [2].

PROOF. Suppose (in accordance with Lemma 3) that F is continuous and that theorem 1 has been proved for sequences $\{\lambda_n\}$ satisfying the additional condition (3.9).

To any nondecreasing sequence $\{\lambda_n\}$ for which $\{(F(\lambda_n))^n\}$ is nonincreasing, let us define a sequence $\{\lambda_n'\}$ by

$$\lambda_n' = \begin{cases} \sup\{\lambda;\, F(\lambda)\leqq\alpha_n\} & \text{if} \quad F(\lambda_n)<\alpha_n\,, \\ \lambda_n & \text{if} \quad \alpha_n\leqq F(\lambda_n)\leqq\beta_n\,, \\ \inf\{\lambda;\, F(\lambda)\geqq\beta_n\} & \text{if} \quad F(\lambda_n)>\beta_n\,. \end{cases} \tag{3.12}$$

Then $\{\lambda_n'\}$ is nondecreasing and on account of the assumed continuity of F, $\{(F(\lambda_n'))^n\}$ is nonincreasing and

$$\alpha_n \leqq F(\lambda_n) \leqq \beta_n \tag{3.13}$$

for every n. Hence, setting

$$E_n' = \{X_{(n)}\leqq\lambda_n'\} \tag{3.14}$$

we find

$$P(\{E_n' \text{ i.o.}\}) = \begin{cases} 0 \\ 1 \end{cases} \text{if} \quad \sum_{n=3}^{\infty}(F(\lambda_n'))^n \frac{\log\log n}{n} \begin{cases} <\infty \\ =\infty. \end{cases} \tag{3.15}$$

Next we note that the series

$$\sum_{n=3}^{\infty}(F(\lambda_n))^n \frac{\log\log n}{n} \tag{3.16}$$

diverges if $\lambda_n > \lambda_n'$ for infinitely many n, $n_1, n_2, \ldots, n_k, \ldots$ say, since in that case (for $n_k > 81$)

$$(3.17) \qquad \sum_{n=81}^{n_k} (F(\lambda_n))^n \frac{\log\log n}{n} \geqq (F(\lambda_{n_k}))^{n_k} \sum_{n=81}^{n_k} \frac{1}{n}$$
$$\geqq (\log n_k)^{-\frac{1}{2}}\big(\log(n_k+1) - \log 81\big) \to \infty \quad \text{for} \quad k \to \infty .$$

Also, let us observe that

$$(3.18) \qquad \sum_{n=3}^{\infty} (\alpha_n)^n \frac{\log\log n}{n} < \infty .$$

Consequently the series (3.16) and

$$(3.19) \qquad \sum_{n=3}^{\infty} \big(F(\lambda_n')\big)^n \frac{\log\log n}{n},$$

converge and diverge simultaneously. In the case of convergence we have $E_n \subset E_n'$ for all except finitely many n and hence

$$(3.20) \qquad P(\{E_n \text{ i.o.}\}) \leqq P(\{E_n' \text{ i.o.}\}) = 0;$$

in the case of divergence, in view of (3.19) and Lemma 1*, we find

$$(3.21) \quad P\big(\{E_n \text{ i.o.}\}\big) \geqq P\big(\{E_n \text{ i.o.}\} \cap \{E_n' \text{ i.o.}\}\big) = P\big(\{E_n' \text{ i.o.}\}\big) = 1 .$$

as claimed.

Proof of Theorem 1. According to Lemma 4, we may assume that $F(\lambda_n)$ (for $n > 2$) is of the form

$$(3.22) \qquad F(\lambda_n) = \exp\left(-\gamma_n \frac{\log\log n}{n}\right) \quad \text{with} \quad \tfrac{1}{2} \leqq \gamma_n \leqq 2 .$$

Under this assumption, convergence of (3.17) entails convergence of

$$(3.23) \qquad \sum_{n=1}^{\infty} P(E_n \cap E_{n+1}^c) = \sum_{n=1}^{\infty} \big(F(\lambda_n)\big)^n \big(1 - F(\lambda_{n+1})\big)$$

and this in conjunction with Lemma 1* shows the validity of the convergence part of Theorem 1.

Let

$$(3.24) \qquad m_n = \left[e^{\frac{n}{\log n}}\right] \quad \text{for} \quad n = 2, 3, \ldots .$$

We conclude the proof of Theorem 1 by showing that divergence of (3.16) entails

$$(3.25) \qquad P\big(\{E_{m_n} \text{ i.o.}\}\big) = 1 .$$

The device we shall apply is similar to one used by Erdös in his proof of the general form of the law of the iterated logarithm (cf. Section 3 in [1]).

For any sequence $i_1, i_2, \ldots i_n, \ldots$ of positive integers such that $i_n = o(\log n)$ we have

$$\frac{m_{n+i_n} - m_n}{m_{n+i_n}} \sim \frac{i_n}{\log(n+i_n)} \quad \text{as} \quad n \to \infty . \tag{3.26}$$

This relation will be used several times in the sequel.

If (3.16) diverges, then so does

$$\sum_{n=2}^{\infty} P(E_{m_n}) = \sum_{n=2}^{\infty} (F(\lambda_{m_n}))^{m_n} \tag{3.27}$$

since

$$\begin{aligned} \sum_{n=m_2}^{\infty} (F(\lambda_n))^n \frac{\log\log n}{n} &= \sum_{n=2}^{\infty} \sum_{r=m_n}^{m_{n+1}-1} (F(\lambda_r))^r \frac{\log\log r}{r} \\ &= \sum_{n=2}^{\infty} (F(\lambda_{m_n}))^{m_n} \frac{m_{n+1} - m_n}{m_n} \log\log m_{n+1} \end{aligned} \tag{3.28}$$

and

$$\frac{m_{n+1} - m_n}{m_n} \log\log m_{n+1} \to 1 \quad \text{as} \quad n \to \infty . \tag{3.29}$$

Hence, in view of (3.22), to any δ, $0 < \delta < 1$, and any positive integer n_0. there exists positive integers n_1 and n_2 such that $n_0 \leqq n_1 < n_2$ and

$$\delta < \sum_{n=n_1+1}^{n_2} P(E_{m_n}) < 2\delta . \tag{3.30}$$

An immediate application of Kolmogorov's zero-one law (see e.g. [4, p. 229]) shows that $P(\{E_{m_n} \text{ i.o.}\})$ equals either 0 or 1. Hence, to complete the proof it clearly suffices to prove the existence of a constant $c > 0$, such that for all sufficiently large n_0

$$P\left(\bigcup_{n=n_1+1}^{n_2} E_{m_n} \right) \geqq c . \tag{3.31}$$

where n_1 and n_2 correspond to a value of δ to be determined later.

Now we have

$$\begin{aligned} P\left(\bigcup_{n=n_1+1}^{n_2} E_{m_n} \right) &= \sum_{n=n_1+1}^{n_2} P\left(E_{m_n} \cap \bigcap_{r=n+1}^{n_2} E_r^{c} \right) \\ &= \sum_{n=n_1+1}^{n_2} \left(P(E_{m_n}) - P\left(E_{m_n} \cap \bigcup_{r=n+1}^{n_2} E_r \right) \right), \end{aligned} \tag{3.32}$$

and therefore it suffices to prove the existence of a constant δ_0, $0<\delta_0<1$, and a constant θ, $0<\theta<1$, such that for $\delta=\delta_0$, $n_1<n\leqq n_2$, and all sufficiently large n_0

$$P\left(E_{m_n}\cap\bigcup_{r=n+1}^{n_2}E_r\right)\leqq\theta P(E_{m_n}). \tag{3.33}$$

Indeed, (3.30), (3.32) and (3.33) imply

$$P\left(\bigcup_{n=n_1+1}^{n_2}E_{m_n}\right)\geqq(1-\theta)\delta_0. \tag{3.34}$$

Let us put

$$a_n = 5[\log\log n] \tag{3.35}$$

and

$$b_n = 2[\log n\,\log\log n]. \tag{3.36}$$

Then

$$P\left(E_{m_n}\cap\bigcup_{r=n+1}^{n_2}E_{m_r}\right) = S_1+S_2+S_3+S_4, \tag{3.37}$$

where

$$S_1 = P(E_{m_n}\cap E_{m_{n+1}}), \tag{3.38}$$

$$S_2 = P\left(E_{m_n}\cap E_{m_{n+1}}{}^{c}\cap\bigcup_{r=n+2}^{n+a_n}E_{m_r}\right), \tag{3.39}$$

$$S_3 = P\left(E_{m_n}\cap\bigcup_{r=n+a_n+1}^{n+b_n}E_{m_r}\right), \tag{3.40}$$

$$S_4 = P\left(E_{m_n}\cap\bigcup_{r=n+b_n+1}^{n_2}E_{m_r}\right). \tag{3.41}$$

Setting for convenience

$$\varphi_n = F(\lambda_n) \tag{3.42}$$

we obtain in the first place

$$S_1 = P(E_{m_n})\,\varphi_{m_{n+1}}^{m_{n+1}-m_n}\leqq P(E_{m_n})\exp\left(-\frac{1}{2}\frac{m_{n+1}-m_n}{m_{n+1}}\log\log m_{n+1}\right)\leqq P(E_{m_n})e^{-\frac{1}{4}} \tag{3.43}$$

for all sufficiently large n_0, since from (3.26) we have

$$\frac{m_{n+1}-m_n}{m_{n+1}}\log\log m_{n+1}\to 1\qquad\text{as}\qquad n\to\infty. \tag{3.44}$$

Secondly, we find

$$S_2 \leqq P(E_{m_n}) \sum_{r=n+2}^{n+a_n} P\left(\left\{\lambda_{m_{n+1}} < \max_{m_n < \nu \leqq m_{n+1}} X_\nu \leqq \lambda_{m_r}\right\}\right) \tag{3.45}$$

$$\leqq P(E_{m_n})\, a_n\, (\varphi_{m_{n+a_n}}^{m_{n+1}-m_n} - \varphi_{m_{n+1}}^{m_{n+1}-m_n})$$

$$\leqq P(E_{m_n})\, a_n\, (m_{n+1}-m_n)\, (\varphi_{m_{n+a_n}} - \varphi_{m_{n+1}})\,,$$

where

$$\varphi_{m_{n+a_n}} - \varphi_{m_{n+1}} \leqq \gamma_{m_{n+1}} \frac{\log\log m_{n+1}}{m_{n+1}} - \gamma_{m_{n+a_n}} \frac{\log\log m_{n+a_n}}{m_{n+a_n}} \tag{3.46}$$

$$\leqq 2 \log\log m_{n+1} \left(\frac{1}{m_{n+1}} - \frac{1}{m_{n+a_n}}\right)$$

since

$$\varphi_{m_{n+1}}^{m_{n+1}} \geqq \varphi_{m_{n+a_n}}^{m_{n+a_n}}$$

implies

$$\gamma_{m_{n+a_n}} \log\log m_{n+a_n} \geqq \gamma_{m_{n+1}} \log\log m_{n+1}\,. \tag{3.47}$$

It follows from (3.26) that

$$a_n \frac{m_{n+a_n} - m_{n+1}}{m_{n+a_n}} \to 0 \qquad \text{as} \quad n \to \infty\,. \tag{3.48}$$

Thus, for all sufficiently large n_0 we have

$$S_2 \leqq P(E_{m_n})\, 2\, a_n \left(1 - \frac{m_{n+1}}{m_{n+a}}\right) \frac{m_{n+1} - m_n}{m_{n+1}} \log\log m_{n+1} \tag{3.49}$$

$$\leqq P(E_{m_n}) \frac{1-e^{-\frac{1}{4}}}{4}\,.$$

Thirdly, we find

$$S_3 \leqq P(E_{m_n}) \sum_{r=n+a_n+1}^{n+b_n} \varphi_{m_r}^{m_r-m_n} \leqq P(E_{m_n})\, b_n\, \varphi_{m_{n+a_n}}^{m_{n+a_n}-m_n}\,, \tag{3.50}$$

where

$$\varphi_{m_{n+a_n}}^{m_{n+a_n}-m_n} \leqq \exp\left(-\frac{1}{2} \frac{m_{n+a_n} - m_n}{m_{n+a_n}} \log\log m_{n+a_n}\right). \tag{3.51}$$

From (3.27) we see that

$$\frac{m_{n+a_n} - m_n}{m_{n+a_n}} \log\log m_{n+a_n} \sim a_n \sim 5 \log\log n\,. \tag{3.52}$$

Hence, for all sufficiently large n_0

$$S_3 \leqq P(E_{m_n}) b_n e^{-2\log\log n} \leqq P(E_{m_n}) \frac{1-e^{-\frac{1}{4}}}{4}\,. \tag{3.53}$$

Finally, we obtain

$$S_4 \leqq P(E_{m_n}) \sum_{r=n+b_n+1}^{n_2} \varphi_{m_r}^{m_r-m_n} \leqq P(E_{m_n})\, \varphi_{m_{n+b_n}}^{-m_n}\, 2\,\delta\,, \tag{3.54}$$

where

$$\varphi_{m_{n+b_n}}^{-m_n} \leqq \exp\left(2\frac{m_n}{m_{n+b_n}} \log\log m_{n+b_n}\right). \tag{3.55}$$

It is easy to see that

$$\frac{m_n}{m_{n+b_n}} \log\log m_{n+b_n} \sim \exp\left(-\frac{b_n}{\log(n+b_n)}\right) \log n \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3.56}$$

The numbers $\varphi_{m_{n+b_n}}^{-m_n}$ are consequently bounded from above by some constant c_0 and

$$S_4 \leqq P(E_{m_n})\, 2\, c_0\, \delta\,. \tag{3.57}$$

To summarize, from (3.37), (3.43), (3.49), (3.53) and (3.57) we have

$$P\left(E_{m_n} \cap \bigcup_{r=n+1}^{n_2} E_{m_r}\right) \leqq \left(\tfrac{1}{2}e^{-\frac{1}{2}} + \tfrac{1}{2} + 2c_0\delta\right) P(E_{m_n})\,. \tag{3.58}$$

Hence, if we choose

$$\delta = \delta_0 = \frac{1}{2c_0}\,\frac{1-e^{-\frac{1}{2}}}{4} \tag{3.59}$$

and

$$\theta = \frac{3+e^{-\frac{1}{2}}}{4}\,, \tag{3.60}$$

then (3.34) will be satisfied for all sufficiently large n_0.

4. Comments on Theorem 1.

In this section we give an example which shows, that divergence of (3.16) does not necessarily imply

$$P(\{E_n \text{ i.o.}\}) = 1\,. \tag{4.1}$$

Thus the condition in Theorem 1 that $(F(\lambda_n))^n$ be nonincreasing is not extraneous to the conclusion of the theorem. On the other hand it is not difficult to see that convergence of (3.16) always implies

$$P(\{E_n \text{ i.o.}\}) = 0\,. \tag{4.2}$$

Indeed, in view of (3.18) and Lemma 3 we may assume $F(\lambda_n) \geqq \alpha_n$, $\forall n$, and under this assumption convergence of (3.16) entails

$$\sum_{n=1}^{\infty} P(E_n \cap E_{n+1}{}^{c}) < \infty \tag{4.3}$$

and hence (4.2) (by Lemma 1*).

Let the random variables X_n be uniformly distributed on the interval $[0,1]$, let

$$n_k = 2^{2^k} \quad \text{for} \quad k = 1,2,\ldots \tag{4.4}$$

and define the sequence $\{\lambda_n\}$ by

$$\lambda_n = \exp\left(-2\frac{\log k}{n_k}\right) \quad \text{for} \quad n_k \leqq n < n_{k+1} . \tag{4.5}$$

Then $F(\lambda_n)=\lambda_n$, $\forall n$, and

$$P(\{E_n \text{ i.o.}\}) = P(\{E_{n_k} \text{ i.o.}\}) = 0 \tag{4.6}$$

since

$$\sum_{k=1}^{\infty} P(E_{n_k}) = \sum_{k=1}^{\infty} \lambda_{n_k}^{\ n_k} < \infty . \tag{4.7}$$

Nevertheless, the series

$$\sum_{n=3}^{\infty} \lambda_n^{\ n} \frac{\log\log n}{n} \tag{4.8}$$

diverges. To prove this we first note that

$$\sum_{n=n_k}^{n_{k+1}-1} \lambda_n^{\ n} \frac{\log\log n}{n} \geqq \log\log n_k \int_{n_k}^{n_{k+1}} \frac{\lambda_{n_k}^{\ t}}{t}\, dt > \log 2\cdot k \int_{n_k}^{n_{k+1}} \frac{\lambda_{n_k}^{\ t}}{t}\, dt . \tag{4.9}$$

We find by partial integration

$$\begin{aligned} -\log\lambda_{n_k} \int_{n_k}^{n_{k+1}} \frac{\lambda_{n_k}^{\ t}}{t}\, dt &= \frac{\lambda_{n_k}^{\ n_k}}{n_k} - \frac{\lambda_{n_k}^{\ n_{k+1}}}{n_{k+1}} - \int_{n_k}^{n_{k+1}} \frac{\lambda_{n_k}^{\ t}}{t^2}\, dt \\ &\geqq \frac{\lambda_{n_k}^{\ n_k}}{n_k}\left(1-\frac{n_k}{n_{k+1}}\right) - \frac{1}{n_k}\int_{n_k}^{n_{k+1}} \frac{\lambda_{n_k}^{\ t}}{t}\, dt \end{aligned} \tag{4.10}$$

or

$$\int_{n_k}^{n_{k+1}} \frac{\lambda_{n_k}^{\ t}}{t}\, dt \geqq \tfrac{3}{4}\lambda_{n_k}^{\ n_k}(1-n_k\log\lambda_{n_k})^{-1} . \tag{4.11}$$

It follows that

$$\sum_{n=n_k}^{n_{k+1}-1} \lambda_n^{\ n} \frac{\log\log n}{n} \geqq \frac{3\log 2}{4}\, \frac{1}{k(1+2\log k)}, \tag{4.12}$$

and hence (4.8) diverges.

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