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LIMIT THEOREMS FOR CERTAIN RANDOM WALKS

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1. Introduction.

Let x_n be a stochastic variable taking the values $0, \ldots, n$, and let

$$x_\nu, x_{\nu+1}, \ldots, x_n, \ldots$$

be a Markov chain of such variables. It is completely specified by the distribution of x_ν and the probabilities of the transitions $x_n \to x_{n+1}$. We assume that all transition probabilities vanish except

$$p(n,m) = P(x_{n+1} = m+1 \mid x_n = m) \tag{1}$$

and

$$q(n,m) = P(x_{n+1} = m \mid x_n = m) . \tag{2}$$

As a consequence, $p+q=1$. It is useful to visualize the chain as a random walk in a grid of points (n,m) with $0 \leqq m \leqq n \geqq \nu$ (figure 1). The probabilities of the passages

$$(n,m) \to (n+1, m+1) \qquad \text{and} \qquad (n,m) \to (n+1, m)$$

are then $p(n,m)$ and $q(n,m)$ respectively, and

$$f_n(m) = P(x_n = m)$$

is the probability of passing through the point (n,m). We observe that the recursion formula

$$f_{n+1}(m) = p(n,m-1) f_n(m-1) + q(n,m) f_n(m) \tag{3}$$

is equivalent to (1) and (2). In the special case when p is constant and $\nu = 0$, x_n has a binomial distribution.

It follows from (1) and (2) that the trend of the walk at the point n,m is in the direction of the vector

$$v = v(n,m) = (1,p)$$

provided p does not change too rapidly (see figure 1).

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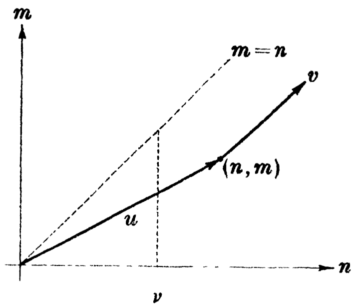

Fig. 1.

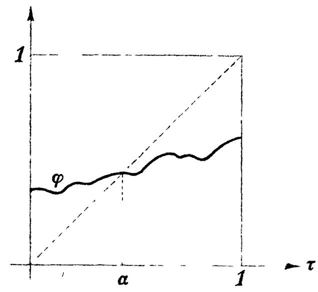

Fig. 2.

It is intuitively clear from the figure that if the walk concentrates around some line $m = n\alpha$ as $n \to \infty$ and v does not oscillate too much, then the vector v ought to have the direction of $(1, \alpha)$ when m is close to $n\alpha$ and n is large, i.e.,

$$p(n, n\alpha) \to \alpha$$

as $n \to \infty$. Conversely, if v drags the position vector $u = (n, m)$ in the direction of $(1, \alpha)$, i.e. if

$$m/n < \alpha \Rightarrow p(n,m) > m/n$$

and

$$m/n > \alpha \Rightarrow p(n,m) < m/n ,$$

we can reasonably expect the walk to concentrate around the line $m = n\alpha$. This is indeed the case and we can give a precise result as follows.

Let $\varphi(\tau)$ be a real function of τ defined when $0 \leqq \tau \leqq 1$ and with values in the same interval. We say that φ is centered at α, where $0 \leqq \alpha \leqq 1$, if

$$\tau < \alpha \Rightarrow \varphi(\tau) > \tau , \qquad \varphi(\alpha) = \alpha , \ \tau > \alpha \Rightarrow \varphi(\tau) < \tau$$

and $|\varphi(\tau) - \tau|$ stays away from zero together with $|\tau - \alpha|$, that is, the infimum of $|\varphi(\tau) - \tau|$ for $|\tau - \alpha| \geqq \varepsilon$ is positive for all positive ε (see figure 2). The simplest example is a linear function

$$\varphi(\tau) = \alpha + a(\tau - \alpha) = a\tau + (1 - a)\alpha , \tag{4}$$

where $0 \leqq a < 1$. We shall also say that $p(n,m)$ is centered at α if there exists a φ centered at α such that

$$m - n\alpha \leqq 0 \Rightarrow p(n,m) \geqq \varphi(m/n) - o(1)$$

and

$$m - n\alpha \geqq 0 \Rightarrow p(n,m) \leqq \varphi(m/n) + o(1)$$

where $o(1)$ refers to $n \to \infty$. We have

THEOREM 1. *If p is centered at α, then*

$$(x_n - n\alpha)/n$$

tends to 0 *in probability. In other words,*

$$P(|x_n - n\alpha| > n\varepsilon) \to 0$$

as $n \to \infty$, for every $\varepsilon > 0$.

Theorem 1 seems to indicate that as long as p is centered at α, the behaviour of $x_n - n\alpha$ depends essentially only on the behaviour of p when $m - n\alpha$ is small compared to n. This is indeed true and we have

THEOREM 2. *If p is centered at α and*

$$p(n,m) = \psi(m/n) + O(|mn^{-1} - \alpha|^{1+\delta}) + O(n^{-\frac{1}{2}-\varepsilon}), \tag{5}$$

where ψ is defined by (4), *a is allowed to be negative but $2a < 1$, $0 < \alpha < 1$ and $\delta, \varepsilon > 0$, then*

$$(x_n - n\alpha)/n^{\frac{1}{2}}$$

tends to a normally distributed variable with mean 0 *and variance $\alpha(1-\alpha)(1-2a)^{-1}$. If $2a = 1$, then the same is true of*

$$(x_n - n\alpha)/(n \log n)^{\frac{1}{2}},$$

the variance now being $\alpha(1-\alpha)$.

REMARK. The condition $2a \leqq 1$ is necessary for the result. It is hard to find a convincing heuristic explanation for this. Roughly speaking, the condition says that the direction of the trend vector v is closer to the direction of $(1,\alpha)$ than to the direction of the position vector u. In fact, if m/n is close to α, then by (5) p is close to

$$\alpha + a(mn^{-1} - \alpha) = amn^{-1} + (1-a)\alpha .$$

It is clear that if $p = \varphi(m/n)$ where φ is centered at α and twice differentiable at $\tau = \alpha$ and $2\varphi'(\alpha) \leqq 1$, then the assumptions are fulfilled with $a = \varphi'(\alpha)$. In particular, we can take $p = A(1 + Bmn^{-1})$, where A and B are suitable positive constants. This case occurs in connection with certain duel problems investigated by L. E. Zachrisson and B. Nagel (unpublished). Approximating (3) by a differential equation these authors were lead to conjecture the Theorems 1 and 2 when p is a function of m/n.

Our proofs use recursion formulas for the moments

$$M_n^j = E_n\big((m - n\alpha)^j\big)$$

where E_n denotes the mean with respect to the frequency function f_n of x_n. They are particularly simple when p equals the function (4) in which case it is easy to see that the precise order af magnitude of M_n^j for j even is

$$O(n^{j/2}), \qquad O\big((n \log n)^{j/2}\big), \qquad O(n^{aj}) ,$$

according as $2a < 1$, $2a = 1$ and $2a > 1$. This is the essential step in the proof of Theorem 1, which is based on similar estimates for the moments in the general case. Such estimates combined with inequalities of the Chebyshev type also show that suitable truncations do not influence the order of magnitude of the moments. This last observation is the basic ingredient in the proof of Theorem 2. The proofs of both theorems give some crude error terms, but no effort was made to improve them. It is, however, safe to say that the convergence of Theorem 1 improves if p is made larger for $m - n\alpha \leqq 0$ and smaller for $m - n\alpha \geqq 0$. The convergence of Theorem 2 improves when δ and ε are made larger and a is made smaller, as long as it stays positive, but the convergence is probably never better than in the binomial case. On the other hand, if $\delta \geqq 1, \varepsilon \geqq \frac{1}{2}$ and $a \leqq 0$, it is probably as good.

One might ask what happens when p satisfies the assumptions of Theorem 2 but $2a > 1$. Considering for simplicity the case when p is given by (4), we can assert that then

$$(x_n - n\alpha)n^{-a}$$

converges to a stochastic variable x with positive variance and with a distribution depending on that of x_ν. More precisely, any moment of order j of x is a linear combination of the moments of order $\leqq j$ of x_ν.

The results above indicate that if, e.g., $p(n,m) = \varphi(m/n)$ where the graph of φ is allowed to touch the diagonal or cross it several times, then one has a very complicated situation.

The methods of this paper will probably work also when one wants to give conditions on p that imply the convergence of

$$\big(x_n - g(n)\big)/h(n) ,$$

where g and h are simple functions of n, for instance of the form An^b with positive A and b.

2. Three lemmas.

We shall need a few elementary results about certain recursion formulas. Let A, a, b, c be arbitrary real numbers and let $r > -1$ and $\delta > 0$.

LEMMA 1. *If*

$$L_{n+1} \leqq L_n\big(1 + an^{-1} + O(n^{-1-\delta})\big) + O\big(n^{c-1}(\log n)^r\big)$$

then

$$L_n \leqq O\big(n^a + n^c(\log n)^r\big)$$

provided $a \neq c$. *If* $a = c$, n^c *should be replaced by* $n^c \log n$.

REMARK. In the applications, the error term $O\big(n^{c-1}(\log n)^r\big)$ is usually replaced by a sum of terms of the same form with varying constants c and r. Since one of them is larger than the others, the conclusion of the lemma holds also in this slightly more general situation provided $O\big(n^c(\log n)^r\big)$ is replaced by the corresponding sum. The same remark applies to the next two lemmas of this section.

PROOF. Let us put

$$L_n = n^a K_n .$$

Then we get the following recursion formula for K_n,

$$K_{n+1} \leqq K_n\big(1 + O(n^{-1-\delta} + n^{-2})\big) + O\big(n^{c-a-1}(\log n)^r\big) \tag{1}$$

and it suffices to show that this implies

$$K_n \leqq O\big(1 + n^{c-a}(\log n)^r\big)$$

when $c \neq a$ and

$$K_n \leqq O\big(1 + (\log n)^{r+1}\big)$$

when $c = a$. To do this, put $M_n = \max(0, K_n)$. Then (1) holds for M_n and hence

$$M_{n+1} - M_n \leqq M_n O(n^{-s}) + O\big(n^{c-a-1}(\log n)^r\big)$$

where $s = \min(1+\delta, 2)$. Summing we get

$$M_n - M_\nu \leqq P_n\Big(\sum_\nu^{n-1} k^{-s}\Big) + \sum_\nu^{n-1} O\big(k^{c-a-1}(\log k)^r\big)$$

where

$$P_n = \max(M_\nu, M_{\nu+1}, \ldots, M_n) .$$

Now choose ν so large that

$$\sum_\nu^\infty k^{-s} < \tfrac{1}{2} .$$

Then

$$M_n \leqq \tfrac{1}{2}P_n + M_\nu + \sum_\nu^{n-1} O\big(k^{c-a-1}(\log k)^r\big) .$$

Since the right side is not decreasing, it majorizes all M_k with $\nu \leqq k \leqq n$ and hence also P_n. Consequently

$$P_n \leqq 2M_\nu + 2\sum_\nu^{n-1} O\big(k^{c-a-1}(\log k)^r\big) .$$

Since the right side is $O\big(1 + n^{c-a}(\log n)^r\big)$ and $O\big(1 + (\log n)^{r+1}\big)$ according as $c \neq a$ and $c = a$, this is the desired result.

LEMMA 2. *If* $a \neq b+1$ *then*

$$L_{n+1} = L_n\big(1+an^{-1}+O(n^{-1-\delta})\big) + An^b + O\big(n^{c-1}(\log n)^r\big)$$

implies

$$L_n = A(b+1-a)^{-1}n^{b+1} + O\big(n^a+n^b+n^{b+1-\delta}+n^c(\log n)^r\big)$$

unless a *is equal to* b, $b+1-\delta$ *or* c. *If this happens, the corresponding powers of* n *acquire a factor* $\log n$.

PROOF. Writing

$$L_n = K_n + A(b+1-a)^{-1}n^{b+1}$$

we get the recursion formula

$$K_{n+1} = K_n\big(1+an^{-1}+O(n^{-1-\delta})\big) + O\big(n^{b-1}+n^{b-\delta}+n^{c-1}(\log n)^r\big).$$

Replacing K_n by $|K_n|$ we get the same formula with $=$ replaced by $\leqq$ and hence Lemma 1 gives the desired result.

LEMMA 3. *If* $a \neq 0$ *then*

$$L_{n+1} = L_n\big(1+an^{-1}+O(n^{-1-\delta})\big) + A(n\log n)^{a-1} + O\big(n^{c-1}(\log n)^r\big)$$

implies

$$L_n = Aa^{-1}(n\log n)^a + O\big(n^a+n^c(\log n)^r\big)$$

provided $a \neq c$. *If* $a=c$, n^c *acquires the factor* $\log n$.

PROOF. Inserting

$$L_n = K_n + Aa^{-1}(n\log n)^a$$

into the recursion formula for L_n and collecting terms we get

$$K_{n+1} = K_n\big(1+an^{-1}+O(n^{-1-\delta})\big) + \\ + O\big(n^{a-2}(\log n)^a+n^{a-1-\delta}(\log n)^a+n^{c-1}(\log n)^r\big).$$

Applying Lemma 1 to the sequence $|K_n|$ finishes the proof.

3. Moments and absolute moments.

Let us denote by

$$E_n(g) = E_n\big(g(m)\big) = \sum g(m)f_n(m)$$

the mean value of the function $g(m)$ relative to the frequency function f_n of x_n. Let

$$M_n^j = E_n\big((m-n\alpha)^j\big)$$

and

$$\bar{M}_n^j = E_n(|m-n\alpha|^j)$$

be the moments and absolute moments of $x_n - n\alpha$. They are equal when j is even. We notice the inequalities

$$\bar{M}_n^{2k-1} \leqq n^{\varrho(2k-1)} + n^{-\varrho}M_n^{2k}, \tag{1}$$

$$\bar{M}_n^{2k-1} \leqq (n \log n)^{\varrho(2k-1)} + (n \log n)^{-\varrho} M_n^{2k}, \tag{2}$$

which hold when $\varrho \geqq 0$. The first one is obtained by majorizing $|m-n\alpha|^{2k-1}$ by $n^{\varrho(2k-1)}$ or $|m-n\alpha|^{2k}n^{-\varrho}$ according as $|m-\alpha n| \leqq n^\varrho$ or $|m-\alpha n| \geqq n^\varrho$, respectively. The proof of (2) is similar.

Let $1>\gamma>0$ and define a function $\theta=\theta(\gamma,n,m-\alpha n)$ by putting

$$\theta = 1 \quad \text{when} \quad |m-\alpha n| \leqq n^\gamma$$

and

$$\theta = 0 \quad \text{when} \quad |m-\alpha n| > n^\gamma .$$

This function will be used to define a truncated mean value

$$E_n^\theta(g) = E_n\big(\theta(\gamma,n,m-n\alpha)g(m)\big)$$

and truncated moments and absolute moments

$$L_n^j = E_n^\theta\big((m-n\alpha)^j\big)$$

and

$$\bar{L}_n^j = E_n^\theta(|m-n\alpha|^j) .$$

We observe that, with an obvious definition of $E_n^{1-\theta}$,

$$|M_n^j-L_n^j| \leqq \bar{M}_n^j-\bar{L}_n^j = E_n^{1-\theta}(|m-n\alpha|^j) . \tag{3}$$

In the sections that follow we shall find it convenient to use the notation R_n^l for any function of n which has the property that

$$R_n^l = O(1+\bar{M}_n^1+\ldots+\bar{M}_n^l) = O(1+\bar{M}_n^l) , \tag{4}$$

the second equality being obvious.

4. Recursion formulas.

The recursion formula (1.3) shows that

$$E_{n+1}\big(g(m)\big) = E_n\big(g(m)\big) + E_n\Big(p(n,m)\big(g(m+1)-g(m)\big)\Big) .$$

Applying this to the moments we get

$$M_{n+1}^j = E_n\Big(\big(m-(n+1)\alpha\big)^j\Big) + E_n(p(n,m) \cdot \\ \cdot \Big(\big((m+1)-(n+1)\alpha\big)^j-\big(m-(n+1)\alpha\big)^j\Big) .$$

After a few rearrangements this can be written as

$$\begin{aligned} M_{n+1}^j = M_n^j &+ jE_n\Big(\big(p(n,m)-\alpha\big)(m-n\alpha)^{j-1}\Big) + \\ &+ \binom{j}{2}(1-2\alpha)E_n\Big(\big(p(n,m)-\alpha\big)(m-n\alpha)^{j-2}\Big) + \\ &+ \binom{j}{2}\alpha(1-\alpha)M_n^{j-2} + R_n^{j-3} , \end{aligned} \tag{1}$$

where we have used the notation (3.4). Notice the special case

$$(2) \qquad M_{n+1}^{j} = M_{n}^{j} + jE_n\big((p(n,m)-\alpha)(m-n\alpha)^{j-1}\big) + R_n^{j-2}\,.$$

The formulas (1) and (2) will be our main tools in the sequel.

5. Order of magnitude of the moments.

It is obvious that

$$\bar{M}_n^j \leqq n^j\,.$$

We shall see that under suitable assumptions, the rate of growth of $\bar{M}_n^j$ is considerably less.

LEMMA 1. *Let ψ be defined by (1.4), let $1>\beta\geqq 0$ and $\varepsilon>0$ and assume that*

$$(1) \qquad p(n,m) \geqq \psi(m/n)-O(n^{\beta-\varepsilon-1}) \quad \textit{for} \quad m\leqq n\alpha\,,$$

$$(2) \qquad p(n,m) \leqq \psi(m/n)+O(n^{\beta-\varepsilon-1}) \quad \textit{for} \quad m\geqq n\alpha\,.$$

Then

$$(3) \qquad \bar{M}_n^j = (\log n)^{j/2}\,O(n^{j/2}+n^{\alpha j}+n^{\beta j})$$

and, if $\gamma>\max(\frac{1}{2},\alpha,\beta)$, then

$$(4) \qquad E_n^{1-\theta}(|m-n\alpha|^j) = O(n^{-\varrho})$$

for all $\varrho>0$.

REMARK 1. If $a\neq\frac{1}{2}$ and $a\neq\beta$, the logarithmic factor in (3) is not necessary.

REMARK 2. Let $F_n(t)=P(x_n\leqq t)$ be the distribution function of x_n. Chebyshev's inequality,

$$F_n\big(n(\alpha-\delta)\big)+1-F_n\big(n(\alpha+\delta)\big) \leqq (\delta n)^{-2}M_n^2 \quad (\delta>0)\,,$$

combined with (3) shows that $F_n(n)$ tends to 0 or 1 according as $\tau<\alpha$ or $\tau>\alpha$.

PROOF. Since $M_n^0=1$ for all n, (3) is true when $j=0$. We proceed by induction. By (4.2),

$$(5) \qquad M_{n+1}^{2k} = M_n^{2k} + 2kE_n\big((p(n,m)-\alpha)(m-n\alpha)^{2k-1}\big) + R_n^{2k-2}$$

and hence (1) and (2) show that

$$M_{n+1}^{2k} \leqq M_n^{2k}(1+2akn^{-1}) + \bar{M}^{2k-1}\,O(n^{\beta-\varepsilon-1}) + R_n^{2k-2}\,.$$

Now by (3.1)

$$(6) \qquad \bar{M}_n^{2k-1} \leqq n^{\beta(2k-1)} + n^{-\beta}M_n^{2k}$$

so that we get

$$M_{n+1}^{2k} \leqq M_n^{2k}\big(1+2akn^{-1}+O(n^{-1-\varepsilon})\big) + O(n^{2\beta k-1}) + R_n^{2k-2} .$$

Now if (3) holds for $j=2k-2$, we have

$$R_n^{2k-2} = (\log n)^{k-1}\, O(n^{k-1}+n^{2a(k-1)}+n^{2\beta(k-1)})$$

so that we can apply Lemma 2.1. Since $M_n^{2k} \geqq 0$, the result is

$$M_n^{2k} = (\log n)^k\, O(n^k+n^{2ak}+n^{2ak+1-2a}+n^{2\beta k}+n^{2\beta k+1-\beta}) .$$

Now, if $2a \leqq 1$,

$$k \geqq 2ak+1-2a \geqq 2ak$$

and if $2a \geqq 1$, we have the same inequalities reversed. The same applies to β. Hence we can cancel the third and fifth terms on the right so that (3) follows by induction when j is even.

Next, put $b=\max(\frac{1}{2},a,\beta)$. By (3.2)

$$\bar{M}_n^{2k-1} \leqq (n \log n)^{(2k-1)b} + (n \log n)^{-b} M_n^{2k} .$$

Hence, since (3) holds with $j=2k$, it holds also with $j=2k-1$.

Finally, since $|m-n\alpha| > n^\gamma$ in the sum on the left, we have

$$E_n^{1-\theta}(|m-n\alpha|^j) < n^{\gamma(j-k)} \bar{M}_n^k$$

for all $k \geqq j$. Taking k large, (4) follows.

Next, we shall see that the true orders of the moments for a permitted p depend essentially only on the behaviour of $p(n,m)$ when m/n is close to α.

Lemma 2. *Let $\delta>0$. If $p(n,m)$ is centered at α and* (1) *and* (2) *are required to hold only if $-n\delta \leqq m-n\alpha \leqq 0$ and $0 \leqq m-n\alpha \leqq n\delta$ respectively, then the conclusions of Lemma* 1 *and the two remarks following it are still true.*

Proof. The assumption that p is centered at α means that $p(n,m)$ stays strictly above m/n for $m/n \leqq \alpha-\delta$ and below m/n for $m/n \geqq \alpha+\delta$ when n is large enough. Hence (1) and (2) hold with ψ replaced by some $\psi'=\alpha+b(\tau-\alpha)$ with $a \leqq b<1$. Hence, if we take $1>\gamma>\max(\frac{1}{2},b,\beta)$, Lemma 1 and (3.3) show that, e.g.,

$$|M_n^j-L_n^j| \leqq \bar{M}_n^j-\bar{L}_n^j = E_n^{1-\theta}(|m-n\alpha|^j) = O(1)$$

for all j. Hence it follows from (5) that

$$L_{n+1}^{2k} = L_n^{2k} + 2kE_n^\theta\big((p(n,m)-\alpha)(m-n\alpha)^{2k-1}\big) + R_n^{2k-2} .$$

Since $\theta = 0$ when $|m - n\alpha| \geqq \delta n$ and n is large enough, we can use (1) and (2) in the middle term. The result is

$$L_{n+1}^{2k} \leqq L_n^{2k}(1+2akn^{-1}) + O(n^{\beta-\varepsilon-1})\bar{L}_n^{2k-1} + R_n^{2k-2} .$$

Since (6) holds with $o(1)$ added to the right and M_n^{2k} replaced by L_n^{2k}, we deduce from this that

$$L_{n+1}^{2k} \leqq L_n^{2k}\big(1+2akn^{-1}+O(n^{-1-\varepsilon})\big) + O(n^{2\beta k-1-\varepsilon}) + R_n^{2k-2} .$$

Hence, by induction, we get (3) for L_n^{2k} and hence also for M_n^{2k}. The rest of the proof now runs as before.

6. Proof of Theorem 1.

Together with $x_\nu, x_{\nu+1}, \ldots$, consider another Markov chain $x'_\nu, x'_{\nu+1}, \ldots$ with transition probabilities $p'(n,m)$ and $q'(n,m)$ corresponding to (1.1) and (1.2). Let $F_n(t) = P(x_n \leqq t)$ and $F'_n(t) = P(x'_n \leqq t)$ be the distribution functions of x_n and x'_n respectively. We shall use the following principle of domination:

$$(1) \qquad F'_\nu = F_\nu,\ p' \leqq p \text{ for } n \geqq \nu \ \Rightarrow\ F'_n \geqq F_n \text{ for } n \geqq \nu .$$

To prove this we observe that a summation of (1.3) gives

$$F_{n+1}(m) = p(n,m)F_n(m-1) + q(n,m)F_n(m) .$$

Since the right side is a non-decreasing function of F_n and a non-increasing function of $p(n,m)$, (1) follows.

Now, let p be centered at $\alpha > 0$ and let $\alpha' < \alpha$. A look at the figure 2 makes it clear that we can choose $1 > a > 0$ in

$$\psi'(\tau) = \alpha' + a(\tau - \alpha')$$

in such a way that

$$m \leqq n\alpha' \Rightarrow p(n,m) \geqq \psi'(m/n)$$

when n is large enough, say $n \geqq \nu'$.

Next, put $p' = p$ when $n < \nu'$ and

$$p'(m,n) = \min\big(p(n,m),\ \psi'(m/n)\big)$$

when $n \geqq \nu'$, and let $x'_\nu, \ldots$ be the corresponding Markov chain determined so that x'_ν and x_ν have the same distribution. Then (1) applies so that

$$F_n \leqq F'_n .$$

It is also clear that p' is centered at α' and that we have $p' = \psi'(m/n)$ when m/n is close to α'. Hence Lemma 5.2 can be applied so that, in particular,

$$F_n(n\alpha'') \leqq F'_n(n\alpha'') \to 0$$

as $n \to \infty$, provided $\alpha'' < \alpha'$. Since α'' is arbitrarily close to α', and α' arbitrarily close to α, this means that $F_n(n(\alpha-\varepsilon)) \to 0$ for every $\varepsilon > 0$.

In the same way, chosing $\alpha' > \alpha$ and $0 < a < 1$ such that

$$m \geqq n\alpha' \Rightarrow p(n,m) \leqq \psi'(m/n)$$

when n is large enough, and putting

$$p'(n,m) = \max\big(p(n,m),\ \psi'(m/n)\big)\,,$$

we conclude that $F_n((\alpha+\varepsilon)n) \to 1$ for every $\varepsilon > 0$. The proof is finished.

7. Proof of Theorem 2.

We shall prove more precise results.

LEMMA 1. *Under the assumptions of Theorem 2 and if* $2a < 1$, *one has*[1]

(1) $$M_n^{2k} = (2k-1)!!\,\sigma^k n^k + n^k O(n^{2a-1}) + (n \log n)^k O(n^{-\delta'} + n^{\gamma-1} + n^{-\varepsilon})\,,$$

(2) $$M_n^{2k-1} = n^{k-\frac12} O(n^{a-\frac12}) + (\log n)^k\, n^{k-\frac12}\, O(n^{-\varepsilon'} + n^{-\varepsilon} + n^{-\frac12})\,,$$

where γ *is any number such that* $1 > \gamma > \frac12$, δ' *and* ε' *are defined by*

$$\delta' = \min\big(\varepsilon, \delta(1-\gamma)\big), \qquad (1+\delta)(\gamma-1) = -\varepsilon' - \tfrac12$$

and

$$\sigma = \alpha(1-\alpha)(1-2a)^{-1}\,.$$

REMARK. In most cases we do not need $\log n$ in the error term; chosing γ so close to $\frac12$ that $\varepsilon' > 0$ we see that the moments of $y_n = (x_n - \alpha n)/n^{\frac12}$ tend to the moments of a normally distributed variable y with mean 0 and variance σ. In other words, if $K_n(t) = P(y_n \leqq t)$ and $K(t) = P(y \leqq t)$ are the distribution functions of y_n and y respectively, then

$$\int t^j\, dK_n(t) \to \int t^j\, dK(t)$$

for $j = 0, 1, 2, \ldots$. It is well known that this implies that $K_n(t)$ converges to $K(t)$ for all t, or, more generally, that

$$\int h(t)\, dK_n(t) \to \int h(t)\, dK(t)$$

for every h in the space H of all piece-wise continuous functions of at most polynomial growth. For completeness we give the proof. In fact, by Helly's selection theorem, we can find a subsequence L_n of K_n and a non-decreasing L such that

[1] Editor's note: $(2k-1)!! = 1 \cdot 3 \cdot 5 \ldots (2k-1)$.

$$\int h(t)\,dL_n(t) \to \int h(t)\,dL(t)$$

when $h \in H$ is continuous and has compact support. But since

$$\int\limits_{|t|\geqq A} t^{2j}\,dL_n(t) \leqq A^{-2}\int t^{2j+2}\,dL_n(t)$$

tends to zero with $1/A$, uniformly in n, we see immediately that $L(t)$ has finite moments of all orders and that we have the same convergence for every continuous $h \in H$. In particular, L and K have the same moments, so that, by the uniqueness part of the theory of the moment problem, $L=K$. Hence every convergent subsequence of K_n converges to K on the continuous elements of H so that the same holds for the sequence K_n itself. But then, K having a continuous derivative, we have convergence on all of H.

Proof. Some reflection shows that the assumptions of Lemma 5.2 are true with a replaced by $b>a$ but arbitrarily close to a, and with $\beta=\frac{1}{2}$. In particular, for any $1>\gamma>\frac{1}{2}$ and every $\varrho>0$

$$(3) \qquad |M_n^j - L_n^j| \leqq \bar{M}_n^j - \bar{L}_n^j = E_n^{1-\theta}(|m-n\alpha|^j) = O(n^{-\varrho})\,.$$

Let us now use the recursion formula (4.1) replacing the mean value E_n by the truncated mean value E_n^θ and using (3) with $\varrho=2$. The result is

$$(4) \qquad \begin{aligned} L_{n+1}^{2k} = L_n^{2k} &+ 2kE_n^\theta\big((p(n,m)-\alpha)(m-n\alpha)^{2k-1}\big) + \\ &+ \binom{2k}{2}(1-2\alpha)\,E_n^\theta\big((p(n,m)-\alpha)(m-n\alpha)^{2k-2}\big) + \\ &+ \binom{2k}{2}\alpha(1-\alpha)L_n^{2k-2} + R_n^{2k-3} + O(n^{-2})\,. \end{aligned}$$

Now (1.5) implies that if $\theta \neq 0$ then

$$p(n,m)-\alpha = (m-n\alpha)n^{-1}\big(a+O(n^{\delta(\gamma-1)})\big) + O(n^{-\frac{1}{2}-\varepsilon})$$

and consequently also

$$p(n,m)-\alpha = O(n^{\gamma-1})\,.$$

Inserting these estimates into the two mean values on the right in (4) we get

$$\begin{aligned} L_{n+1}^{2k} = L_n^{2k}&\big(1+2akn^{-1}+O(n^{-1-\delta(1-\gamma)})\big) + \\ &+ O(n^{-\frac{1}{2}-\varepsilon})\bar{M}_n^{2k-1} + k(2k-1)\alpha(1-\alpha)L_n^{2k-2} + \\ &+ O(n^{\gamma-1})M_n^{2k-2} + R_n^{2k-3} + O(n^{-2})\,. \end{aligned}$$

Now let us use (3.1) with $\varrho=\frac{1}{2}$ to eliminate the odd moments that occur on the right,

$$\bar{M}_n^{2k-1} \leqq n^{k-\frac{1}{2}} + n^{-\frac{1}{2}} M_n^{2k} ,$$
$$\bar{M}_n^{2k-3} \leqq n^{k-\frac{3}{2}} + n^{-\frac{1}{2}} M_n^{2k-2} .$$

Since we can replace M by L on the right, the result is

$$\text{(5)} \qquad \begin{aligned} L_{n+1}^{2k} = & L_n^{2k}\big(1+2akn^{-1}+O(n^{-1-\delta'})\big) + \\ & + \big(k(2k-1)\alpha(1-\alpha) + O(n^{\gamma-1})\big)L_n^{2k-2} + \\ & + n^{k-1}O(n^{-\varepsilon}+n^{-\frac{1}{2}}) . \end{aligned}$$

Here we have taken into account that $\gamma > \frac{1}{2}$ and used the definition of δ' given in the lemma. Now if we interpret $-1!!$ as 1, (1) holds for $k=0$ since $M_n^0 = 1$. We proceed by induction. Assuming (1) for $k-1$ and inserting into (5) we get, after some collecting of terms,

$$\begin{aligned} L_{n+1}^{2k} = & L_n^{2k}\big(1+2akn^{-1}+O(n^{-1-\delta'})\big) + \\ & + k(2k-1)!!\,\alpha(1-\alpha)\sigma^{k-1}n^{k-1} + \\ & + n^{k-1}O(n^{2a-1}) + (n\log n)^{k-1}O(n^{-\delta'}+n^{-\varepsilon}+n^{\gamma-1}) . \end{aligned}$$

Hence, by Lemma 2.2, we get

$$L_n^{2k} = (2k-1)!!\,n^k\sigma^k + n^kO(n^{2a-1}) + (n\log n)^kO(n^{-\delta'}+n^{-\varepsilon}+n^{\gamma-1}) + O(n^{2ak}) .$$

Here $2ak \leqq k+2a-1$, so that the last term is redundant and, by virtue of (3), (1) follows. As a consequence, we also have the estimate

$$\bar{M}_n^{2k-1} \leqq n^{k-\frac{1}{2}} + n^{-\frac{1}{2}}M^{2k} = O(n^{k-\frac{1}{2}}) .$$

Next, let us verify (2) when $k=1$. Using (4.1) and (3) with a suitable ϱ we have

$$L_{n+1}^1 = L_n^1 + E_n^\theta\big(p(n,m)-\alpha\big) + O(n^{-\frac{1}{2}-\varepsilon}) .$$

Here let us use (1.5) in the form

$$\text{(6)} \qquad p(n,m)-\alpha = a(m-n\alpha)n^{-1} + O(n^{(1+\delta)(\gamma-1)}) + O(n^{-\frac{1}{2}-\varepsilon})$$

when $\theta \neq 0$. Since $(1+\delta)(\gamma-1) = -\frac{1}{2}-\varepsilon'$, we get

$$L_{n+1}^1 = L_n^1(1+an^{-1}) + O(n^{-\frac{1}{2}-\varepsilon'}+n^{-\frac{1}{2}-\varepsilon}) .$$

Hence, by Lemma 2.1,

$$L_n^1 = n^{\frac{1}{2}}O(n^{a-\frac{1}{2}}) + n^{\frac{1}{2}}\log n\, O(n^{-\varepsilon'}+n^{-\varepsilon})$$

so that, by virtue of (3), (2) holds for $k=1$.

We proceed by induction. Using (4.1) and (3) we get

$$\begin{aligned} L_{n+1}^{2k+1} = & L_n^{2k+1} + (2k+1)E_n^\theta\big(\big(p(n,m)-\alpha\big)(m-n\alpha)^{2k}\big) + \\ & + k(2k+1)(1-2\alpha)E_n^\theta\big(\big(p(n,m)-\alpha\big)(m-n\alpha)^{2k-1}\big) + \\ & + k(2k+1)\alpha(1-\alpha)L_n^{2k-1} + R_n^{2k-2} . \end{aligned}$$

Here we use (6) in the first mean value and

$$p(n,m)-\alpha = (m-n\alpha)O(n^{-1}) + O(n^{-\frac{1}{2}})$$

in the second. The result is

$$\begin{aligned} L_{n+1}^{2k+1} = {} & L_n^{2k+1}\big(1+(2k+1)an^{-1}\big) + \\ & + O(n^{-\frac{1}{2}-\varepsilon'}+n^{-\frac{1}{2}-\varepsilon}+n^{-1})M_n^{2k} + O(1)L_n^{2k-1} + \\ & + O(n^{-\frac{1}{2}})\bar{M}_n^{2k-1}+R_n^{2k-2} . \end{aligned}$$

Inserting the known estimates for M_n^{2k} and $\bar{M}_n^{2k-1}$ and estimating L_n^{2k-1} by (2) we get, after some collecting of terms,

$$\begin{aligned} L_{n+1}^{2k+1} = {} & L_n^{2k+1}\big(1+(2k+1)an^{-1}\big) + \\ & + n^{k-\frac{1}{2}}O(n^{a-\frac{1}{2}}) + (\log n)^k n^{k-\frac{1}{2}}O(n^{-\varepsilon'}+n^{-\varepsilon}+n^{-\frac{1}{2}}) . \end{aligned}$$

Hence, taking absolute values and using Lemma 2.1, we get

$$L_n^{2k+1} = n^{k+\frac{1}{2}}O(n^{a-\frac{1}{2}}) + (\log n)^{k+1}n^{k+\frac{1}{2}}O(n^{-\varepsilon'}+n^{-\varepsilon}+n^{-\frac{1}{2}}) + O(n^{2ak+a}) .$$

Here, $2ak+a<k+a$, so that the last term is redundant. Hence, using (3), we obtain (2) with k replaced by $k+1$ and the proof is finished.

Lemma 2. *Under the assumptions of Theorem 2 and if $a=\frac{1}{2}$, one has*

$$M_n^{2k} = (2k-1)!!\,\big(\alpha(1-\alpha)\big)^k\,(n\log n)^k + O\big(n^k(\log n)^{k-1}\big), \tag{7}$$

$$M_n^{2k-1} = O\big(n^{k-\frac{1}{2}}(\log n)^{k-1}\big) . \tag{8}$$

Remark. These estimates and the remark of Lemma 1 show that $(x_n-n\alpha)/(n\log n)^{\frac{1}{2}}$ tends to a normally distributed variable with mean 0 and variance $\alpha(1-\alpha)$.

Proof. The estimate (3) for the truncated moments is still true and (5) still holds. As before, we observe that (7) holds for $k=0$ and proceed by induction. Inserting the estimate for L_n^{2k-2} into (5) and majorizing the error terms suitably, we get

$$\begin{aligned} L_{n+1}^{2k} = {} & L_n^{2k}\big(1+kn^{-1}+O(n^{-1-\delta'})\big) + \\ & + k(2k-1)!!\,\big(\alpha(1-\alpha)\big)^k\,(n\log n)^{k-1} + O\big(n^{k-1}(\log n)^{k-2}\big) \end{aligned}$$

so that, by Lemma 2.3, (7) holds for L_n^{2k} and hence also for M_n^{2k}. The proof of (8) runs as before and is left to the reader.

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