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PLANES OF CURVATURE IN RIEMANNIAN SPACES

FOLKE ERIKSSON

0. Introduction.

We consider n-dimensional Riemannian spaces V_n with positive definite metric. The sectional (or Riemannian) curvature of the space with respect to a 2-dimensional direction at a point P is determined by the formula

$$(00) \qquad \varkappa = \tfrac{1}{4} \sum_{i,j,k,h} R_{ijkh} p^{ij} p^{kh} ,$$

if the 2-direction is given by a simple (cf. p. 6) bivector p^{ij} of norm 1 and R_{ijkh} is the Riemann–Christoffel curvature tensor at P. (See e.g. Cartan [1, p. 195]. The definitions of R_{ijkh} differ in sign with various authors. Thus Cartan's $-R_{ijkh}$ is here denoted by R_{ijkh}.) In the form (00) the formula is valid in any coordinate system. However, we use throughout this paper—when not explicitly stating the contrary—coordinate systems which are locally cartesian (orthonormal) at the point P considered. (Only local properties are studied here.) In such a system covariant and contravariant components coincide, and thus we may use only lower indices in the following. The $N = \binom{n}{2}$ independent components of a bivector $\boldsymbol{p}$ may be numbered by a single index in an arbitrary order chosen once for all. We number the six first components as follows:

$$(01) \quad p_1 = p_{23}, \quad p_2 = p_{31}, \quad p_3 = p_{12}, \quad p_4 = p_{14}, \quad p_5 = p_{24}, \quad p_6 = p_{34} .$$

We thus consider our bivectors $\boldsymbol{p}$ as vectors in an N-dimensional space E_N, where we use the Euclidean norm $|\boldsymbol{p}|^2 = \sum_\nu p_\nu^2$. Correspondingly, the index pairs (ij) and (kh) appearing in R_{ijkh} may be replaced by single indices running from 1 to N. We thus write

$$R_{ijkh} = R_{st} .$$

In order to avoid confusion, we here denote the Ricci tensor $\sum_j R_{ijkj}$ by K_{ik}. Because of the well-known symmetry properties $R_{ijkh} = -R_{jikh} = -R_{ijhk}$ the $N \times N$-matrix $R = (R_{st})$ contains all independent components R_{ijkh}. Further, the identity $R_{ijkh} = R_{khij}$ shows that R is symmetric. The remaining identity

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$$R_{ijkh}+R_{ikhj}+R_{ihjk} = 0$$

is in our notation for $n=4$

$$R_{14}+R_{25}+R_{36} = 0 . \tag{02}$$

For $n>4$ it yields $\binom{n}{4}$ analogous relations.

In our notation the formula (00) for the sectional curvature is written

$$\varkappa = \sum_{s,t} R_{st}p_s p_t ,$$

or with the symbol $(\boldsymbol{u},\boldsymbol{v})$ for the scalar product in E_N

$$\varkappa = (R\boldsymbol{p},\boldsymbol{p}) . \tag{03}$$

If we introduce in E_N a new orthonormal system with eigenvectors of R as basis vectors, the formula is simplified further, R being reduced to diagonal form. Then the variation of the sectional curvature of V_n at P with the 2-direction can be represented very simply in E_N. The eigenvalues of R may be called *principal curvatures* of V_n at P.

However, only *simple* bivectors represent 2-directions. A bivector $\boldsymbol{p}$ is called simple, if it is the alternating product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$: $p_{ik}=\frac{1}{2}(u_i v_k - u_k v_i)$. For $n \geqq 4$ a bivector $\boldsymbol{p}$ is simple, if and only if it satisfies $\binom{n}{4}$ conditions, which in the E_N-notation are very similar to (02). One of them is

$$p_1p_4+p_2p_5+p_3p_6 = 0 \tag{04}$$

(cf. Cartan [1, p. 11]). Now the eigenvectors of R need not satisfy these conditions, and thus need not represent 2-directions in V_n. Then the principal curvatures may lack direct interpretation in V_n. Therefore it is natural to ask, under what conditions the eigenvectors of R represent 2-directions. The eigenvalue problem $R\boldsymbol{p}=\lambda\boldsymbol{p}$ seems not to have been studied from this point of view before. H. S. Ruse calls the eigenvalues "g-roots" [7, p. 11]. He uses the various cases of degeneracy which arise from equalities among the eigenvalues as one of the tools for a classification of curvature tensors. His results are given in terms of the quadratic complex $R_{ijkh}p^{ij}p^{kh}=0$ of lines in S_{n-1} which corresponds to directions $\boldsymbol{p}$ of zero curvature in V_n. In [2] R. V. Churchill splits the matrix R into two parts for the case $n=4$ and considers the eigenvalue problems for each part separately.

In § 2 we deal with the question: Under what conditions does there exist in E_N an orthogonal basis of eigenvectors of R all of which are simple bivectors? Such a basis corresponds in V_n to a set of N 2-directions, which may be called *planes of curvature* of V_n at the point P considered. These planes of curvature are then orthogonal also in V_n, in the sense that each plane contains some vector orthogonal to every vector in the other plane (conditional orthogonality). In fact, it is well-known that this orthogonality in V_n is equivalent to the orthogonality in E_N (Cartan [1, p. 9], cf. also Churchill's theorem 3 [2, p. 132]).

The condition (04) expresses the vanishing of a certain quadratic form and may be written $(B\boldsymbol{p},\boldsymbol{p})=0$ with a symmetric $N\times N$-matrix B. The question dealt with in § 2 is therefore, for $n=4$, of the following general type. Under what conditions on two symmetric $N\times N$-matrices A and B does there exist an orthonormal basis $\{\boldsymbol{v}_\nu\}$ of eigenvectors of A all of which satisfy the condition $(B\boldsymbol{v}_\nu,\boldsymbol{v}_\nu)=0$? In § 1 we prove that a necessary and sufficient condition is that all matrices BA^k, $k=0,1,2,\ldots,N-1$, have trace zero. We consider also the analogous problem needed in § 2 for $n>4$, where we have several conditions of the form $(B_i\boldsymbol{p},\boldsymbol{p})=0$.

For $n=3$, $N(=3)$ orthogonal planes always intersect in $n(=3)$ orthogonal lines. For $n>3$, however, N planes may be orthogonal to each other without being the coordinate planes of an orthogonal system. In E_4 for example, the basis vectors $\boldsymbol{e}_i$ and the vectors $\boldsymbol{a}=\boldsymbol{e}_1+\boldsymbol{e}_2$, $\boldsymbol{b}=\boldsymbol{e}_1-\boldsymbol{e}_2$ yield by alternating multiplication the six bivectors $[\boldsymbol{e}_1\boldsymbol{e}_2]$, $[\boldsymbol{e}_1\boldsymbol{e}_3]$, $[\boldsymbol{e}_2\boldsymbol{e}_3]$, $[\boldsymbol{e}_3\boldsymbol{e}_4]$, $[\boldsymbol{e}_4\boldsymbol{a}]$ and $[\boldsymbol{e}_4\boldsymbol{b}]$. These represent planes which are obviously pairwise orthogonal, but have *six* different lines of intersection, of which for example $\boldsymbol{e}_1$ and $\boldsymbol{a}$ are not orthogonal. This leads to a question studied in § 3: What are the conditions for a V_n to have, at a point, N planes of curvature, which are the coordinate planes of an orthogonal system? If such a system exists, it is easily proved that its basis vectors determine principal directions in the sense of Ricci [5], i.e. they are eigenvectors of the Ricci tensor K_{ik}. In fact, in a system where the matrix R_{st} has diagonal form, we have $R_{ijkh}=0$ unless $(i,j)=(k,h)$ or $(i,j)=(h,k)$. Then for $i\neq k$, all terms in $K_{ik}=\sum_j R_{ijkj}$ vanish. The solution of § 3 is not complete, and in § 4 an alternative and more complete one for the 4-dimensional case is given. In § 5 we give some examples of spaces satisfying the conditions of § 2–4. Among spaces having, at every point, N planes of curvature, forming coordinate planes of an orthogonal system, we find all conformally flat spaces C_n.

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1. Matrices with eigenvectors lying on given cones.

As stated in the introduction, we study here the possibility of finding in E_N an orthonormal basis $\{\boldsymbol{v}_\nu\}$ so that for two given symmetric $N\times N$-matrices A and B

a) all $\boldsymbol{v}_\nu$ are eigenvectors of A,

b) all $\boldsymbol{v}_\nu$ lie on the cone $(B\boldsymbol{v},\boldsymbol{v})=0$.

A vector $\boldsymbol{v}$ such that $(B\boldsymbol{v},\boldsymbol{v})=0$ will be called *isotropic for* B. As a preparation for the first theorem we give a proof for the following known fact.

LEMMA. *An orthonormal basis* $\{\boldsymbol{v}_\nu\}$, *consisting of vectors isotropic for an* $N\times N$*-matrix* B, *exists in* E_N *if and only if* B *has trace* 0.

PROOF. The condition is necessary since the trace of B is

$$\operatorname{tr} B = \sum_{\nu=1}^{N} b_{\nu\nu} = \sum_{\nu=1}^{N} (B\boldsymbol{v}_\nu,\boldsymbol{v}_\nu) .$$

We prove the sufficiency by induction. If $\operatorname{tr} B=0$, the quadratic form $(B\boldsymbol{v},\boldsymbol{v})$ is not definite. Therefore an isotropic unit vector $\boldsymbol{v}_1$ exists. Introducing an orthonormal basis $\{\boldsymbol{u}_\nu\}$ with $\boldsymbol{u}_1=\boldsymbol{v}_1$, we have

$$0 = \operatorname{tr} B = (B\boldsymbol{v}_1,\boldsymbol{v}_1) + \sum_{\nu=2}^{N} (B\boldsymbol{u}_\nu,\boldsymbol{u}_\nu) = \sum_{\nu=2}^{N} (B\boldsymbol{u}_\nu,\boldsymbol{u}_\nu) .$$

This shows that the restriction of the form $(B\boldsymbol{v},\boldsymbol{v})$ to the subspace E_{N-1} orthogonal to $\boldsymbol{v}_1$ has trace 0. Suppose now that the statement is true for spaces of dimension $N-1$. Then there exists in E_{N-1} a basis of isotropic vectors. Together with $\boldsymbol{v}_1$ this gives a basis of isotropic vectors in E_N. Since the case $N=1$ is trivial, this proves the lemma.

THEOREM 1. *A symmetric* $N\times N$*-matrix* A *has diagonal form in some orthonormal system in* E_N *all the vectors of which are isotropic for another* $N\times N$*-matrix* B *if and only if*

$$\operatorname{tr}(BA^k) = 0 \quad \textit{for} \quad k = 0,1,2,\ldots,N-1 . \tag{10}$$

PROOF. If the basis vector $\boldsymbol{v}_\nu$, $\nu=1,\ldots,N$, is isotropic for B and corresponds to the eigenvalue λ_ν of A, we have for every integer $k\geqq 0$

$$\operatorname{tr}(BA^k) = \sum_\nu (BA^k\boldsymbol{v}_\nu,\boldsymbol{v}_\nu) = \sum_\nu \lambda_\nu{}^k (B\boldsymbol{v}_\nu,\boldsymbol{v}_\nu) = 0 .$$

This shows the necessity of the condition.

To prove the sufficiency we choose a basis $\{\boldsymbol{v}_\nu\}$ of eigenvectors of A with corresponding eigenvalues λ_ν. We denote the distinct λ_ν by $\lambda_{(1)},\ldots,\lambda_{(p)}$ and put

$$y_i = \sum_{\lambda_\nu=\lambda_{(i)}} (B\boldsymbol{v}_\nu,\boldsymbol{v}_\nu) ,$$

where the sum is taken over those ν for which $\lambda_\nu=\lambda_{(i)}$. The p first conditions (10) may then be written

$$\sum_{i=1}^{p} \lambda_{(i)}{}^k y_i = 0, \quad k = 0,1,\ldots,p-1 .$$

Since the determinant of this system of linear equations is the Vandermonde determinant of the $\lambda_{(i)}$ and thus different from zero, it follows that all $y_i = 0$. Now $y_i = 0$ means that the restriction of (Bv, v) to the eigenspace of A belonging to $\lambda_{(i)}$ has trace 0. According to the lemma there exists therefore, in this eigenspace, an orthonormal basis of vectors isotropic for B. Taken together, such bases in the various eigenspaces form a basis of E_N with the required properties.

Remark. As the proof shows, it is sufficient that the condition $\mathrm{tr}(BA^k) = 0$ be satisfied for $k < p$, the number of distinct eigenvalues of A. It follows also that this implies $\mathrm{tr}(BA^k) = 0$ for any integer $k \geqq 0$.

We shall be interested also in the corresponding problem, where the basis vectors are required to be isotropic for several matrices B_i simultaneously. Applying theorem 1 for each B_i separately we get

Theorem 2. *For the existence of an orthonormal basis of eigenvectors of a symmetric $N \times N$-matrix A all of which are isotropic for the $N \times N$-matrices B_i, $i = 1, \ldots, m$, it is necessary that*

(11) $\mathrm{tr}(B_i A^k) = 0 \quad$ *for* $\quad i = 1,\ldots,m;\ k = 0,1,\ldots,N-1 .$

These conditions are sufficient if all eigenvalues of A are distinct.

Proof. The necessity is clear. On the other hand, if the conditions (11) are satisfied, there exists according to theorem 1, for each i, an orthonormal basis of eigenvectors of A which are isotropic for B_i. If the eigenvalues of A are distinct, there is (up to changes of sign) only one orthonormal basis of eigenvectors of A, and this must satisfy the requirements.

If A has multiple eigenvalues, the conditions (11) are in general not sufficient. For example, an orthonormal basis $\{v_\nu\}$ of eigenvectors of the 3×3-matrix $A = \mathrm{diag}(1,1,2)$ may contain any pair of orthogonal unit vectors v_1 and v_2 in the "xy-plane". Thus there exists such a basis on any circular cone with three ortogonal generators one of which is the "z-axis". But no such basis lies simultaneously on two general such cones.

2. Conditions for the existence of planes of curvature.

We now try to answer the question: Under what conditions does R have diagonal form in some orthonormal basis in E_N consisting of simple bivectors? For $n=4$ we obtain the complete answer by applying theorem 1 with $A=R$ and B twice the matrix of the quadratic form in (04), i.e. (in our coordinate system) the 6×6-matrix

$$B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{20}$$

where I is the unit 3×3-matrix. The first two conditions

$$\operatorname{tr} B = 0, \qquad \operatorname{tr}(BR) = 2(R_{14}+R_{25}+R_{36}) = 0$$

are here automatically satisfied—the latter is (02). We can thus formulate the result as

THEOREM 3. *The curvature matrix R of V_4 has diagonal form in some orthonormal basis in E_6 consisting of simple bivectors if and only if*

$$\operatorname{tr}(BR^k) = 0 \qquad \textit{for} \qquad k = 2,3,4,5\,,$$

where $(B\boldsymbol{p},\boldsymbol{p})=0$ is the condition (04) *for a bivector $\boldsymbol{p}$ to be simple.*

For $n>4$, we can apply theorem 2 in the same way and get the following partial answer:

THEOREM 4. *If the curvature matrix R of V_n has diagonal form in some orthonormal basis in E_N consisting of simple bivectors, then*

$$\operatorname{tr}(B_iR^k) = 0 \qquad \textit{for} \quad i = 1,2,\ldots,\tbinom{n}{4};\ k = 0,1,2,\ldots,N-1\,, \tag{21}$$

where $(B_i\boldsymbol{p},\boldsymbol{p})=0$ are the conditions for a bivector $\boldsymbol{p}$ to be simple. If all eigenvalues of R are distinct and (21) *holds, then the eigenvectors of R are simple bivectors. (The conditions* (21) *for $k=0$ and 1 are automatically satisfied.)*

The question whether the conditions (21) imply the existence of N orthogonal planes of curvature also if $n>4$ and multiple eigenvalues occur, remains open. We give an affirmative answer for eigenvalues of multiplicity less than or equal to 2.

We use here the usual notation p_{ij} with two indices for the components of a bivector $\boldsymbol{p}$, and write the conditions for a simple $\boldsymbol{p}$

$$\tfrac{1}{2}B_{ijhl}(\boldsymbol{p}) \equiv p_{ij}p_{hl}+p_{ih}p_{lj}+p_{il}p_{jh} = 0, \qquad i,j,h,l \text{ distinct}\,.$$

They are not independent but satisfy $n\binom{n-1}{4}$ linear identities:

$$\theta_{ijhlm}(\boldsymbol{p}) \equiv p_{ij}B_{ihlm}(\boldsymbol{p})+p_{ih}B_{jilm}(\boldsymbol{p})+p_{il}B_{jhim}(\boldsymbol{p})+p_{im}B_{jhli}(\boldsymbol{p}) \equiv 0\,.$$

We denote by B_{ijhl} also the matrix of the form $B_{ijhl}(\boldsymbol{p})$.

Consider now a double eigenvalue λ of the $N \times N$-matrix R which satisfies the conditions (21), that is, $\operatorname{tr}(B_{ijhl}R^k)=0$. Take an orthonormal basis of two bivectors p and q in the corresponding eigenspace. For each set of 4 distinct indices i, j, h, l we get

$$B_{ijhl}(p)+B_{ijhl}(q) \;=\; 0\,, \tag{22}$$

as we obtained $y_i=0$ in the proof of theorem 1. Looking for a simple bivector of the form

$$p' \;=\; p\cos\alpha+q\sin\alpha\,,$$

we have to find an α satisfying the equations

$$B_{ijhl}(p') \;\equiv\; B_{ijhl}(p)\cos^2\alpha+2(B_{ijhl}p,q)\sin\alpha\,\cos\alpha+B_{ijhl}(q)\sin^2\alpha \;=\; 0\,,$$

or, because of (22),

$$B_{ijhl}(\alpha) \;\equiv\; B_{ijhl}(p)\cos 2\alpha+(B_{ijhl}p,q)\sin 2\alpha \;=\; 0\,. \tag{23}$$

A common solution of these equations exists if all the determinants

$$D^{ijhl}_{mkgf} = \begin{vmatrix} B_{ijhl}(p) & (B_{ijhl}p,q) \\ B_{mkgf}(p) & (B_{mkgf}p,q) \end{vmatrix}$$

vanish. First we prove this for those determinants, where three of the indices in one row are equal to three indices in the other. We have, in fact

$$\begin{aligned} &B_{ijhl}(p)(B_{ijhm}p,q) \\ &\quad\equiv\; B_{ijhl}(p)(p_{ij}q_{hm}+p_{hm}q_{ij}+p_{ih}q_{mj}+p_{mj}q_{ih}+p_{im}q_{jh}+p_{jh}q_{im})\,. \end{aligned}$$

Expressing $p_{hm}B_{ijhl}(p)$, $p_{jm}B_{ijhl}(p)$ and $p_{im}B_{ijhl}(p)$ by means of the identities $\theta_{hijml}(p)\equiv 0$, $\theta_{jimhl}(p)\equiv 0$ and $\theta_{imjhl}(p)\equiv 0$, respectively, and treating $q_{hm}B_{ijhl}(p)=-q_{hm}B_{ijhl}(q)$, $q_{mj}B_{ijhl}(p)$ and $q_{im}B_{ijhl}(p)$ in the same way, we get

$$\begin{aligned} &B_{ijhl}(p)(B_{ijhm}p,q) \\ &\quad\equiv\; -q_{ij}[p_{hi}B_{hjml}(p)+p_{hj}B_{ihml}(p)+p_{hl}B_{ijmh}(p)] \\ &\qquad +q_{ih}[p_{ji}B_{jmhl}(p)+p_{jh}B_{imjl}(p)+p_{jl}B_{imhj}(p)] \\ &\qquad -q_{jh}[p_{ij}B_{mihl}(p)+p_{ih}B_{mjil}(p)+p_{il}B_{mjhi}(p)] \\ &\qquad +p_{ij}[q_{hi}B_{hjml}(q)+q_{hj}B_{ihml}(q)+q_{hl}B_{ijmh}(q)] \\ &\qquad -p_{ih}[q_{ji}B_{jmhl}(q)+q_{jh}B_{imjl}(q)+q_{jl}B_{imhj}(q)] \\ &\qquad +p_{jh}[q_{ij}B_{mihl}(q)+q_{ih}B_{mjil}(q)+q_{il}B_{mjhi}(q)]\,. \end{aligned}$$

Because of (22) and the obvious anti-symmetry of $B_{ijhl}(p)$ with respect to its indices, this reduces to

$$B_{ijhm}(\boldsymbol{p})(q_{ij}p_{hl}+q_{ih}p_{lj}+p_{il}q_{jh}+p_{ij}q_{hl}+p_{ih}q_{lj}+q_{il}p_{jh}) \\ \equiv B_{ijhm}(\boldsymbol{p})(B_{ijhl}\boldsymbol{p},\boldsymbol{q}) .$$

For $n=5$ this applies to all the determinants, and then the equations (23) have a common solution α_0, and obviously another one, $\alpha_0+\frac{1}{2}\pi$. The corresponding simple bivectors $\boldsymbol{p}'(\alpha_0)$ and $\boldsymbol{p}'(\alpha_0+\frac{1}{2}\pi)$ constitute the required basis.

For $n \geqq 6$ it remains to exclude the possibility that two of the equations (23) with at least two indices different could lack common solutions. Suppose for example

$$(24) \qquad \begin{cases} B_{ijhl}(\alpha_1) = 0, & B_{ijhl}(\alpha_2) \neq 0 , \\ B_{ijmk}(\alpha_1) \neq 0, & B_{ijmk}(\alpha_2) = 0 . \end{cases}$$

Since, as proved above, $D^{ijhl}_{ijhm}=D^{ijhm}_{ijmk}=0$, this would imply $B_{ijhm}(\alpha_1)=0$, $B_{ijhm}(\alpha_2)=0$ and thus

$$B_{ijhm}(\alpha) \equiv 0 .$$

Analogously

$$B_{ijlm}(\alpha) \equiv B_{ijhk}(\alpha) \equiv B_{ijlk}(\alpha) \equiv 0 .$$

We prove now that also, for example, $B_{ihlm}(\alpha)\equiv 0$. For suppose $B_{ihlm}(\alpha)\not\equiv 0$. Then as above $B_{ihmk}(\alpha)\equiv 0$, and because of (24) the identity $\theta_{ijhmk}(\boldsymbol{p}')\equiv 0$ would yield $p'_{ih}(\alpha)\equiv 0$. Analogously $p'_{il}(\alpha)\equiv 0$. Then

$$\begin{aligned} B_{ijhl}(\alpha) &\equiv 2p'_{ij}(\alpha)p'_{hl}(\alpha) &\Rightarrow\quad p'_{ij}(\alpha_1) &= 0 \text{ or } p'_{hl}(\alpha_1) = 0 , \\ B_{ihlm}(\alpha) &\equiv 2p'_{im}(\alpha)p'_{hl}(\alpha) &\Rightarrow\quad p'_{im}(\alpha_1) &= 0 \text{ or } p'_{hl}(\alpha_1) = 0 , \\ B_{ihlk}(\alpha) &\equiv 2p'_{ik}(\alpha)p'_{hl}(\alpha) &\Rightarrow\quad p'_{ik}(\alpha_1) &= 0 \text{ or } p'_{hl}(\alpha_1) = 0 . \end{aligned}$$

Thus $p'_{hl}(\alpha_1)=0$, because $B_{ijmk}(\alpha_1)\neq 0$. Analogously $p'_{hl}(\alpha_1+\frac{1}{2}\pi)=0$, because

$$B_{ijhl}(\alpha_1+\tfrac{1}{2}\pi) = -B_{ijhl}(\alpha_1) = 0 .$$

Since $p'_{hl}(\alpha)\equiv p_{hl}\cos\alpha+q_{hl}\sin\alpha$, this is possible only if $p'_{hl}(\alpha)\equiv 0$. We would thus have $p'_{hl}(\alpha)\equiv p'_{ih}(\alpha)\equiv p'_{il}(\alpha)\equiv 0$, which implies $B_{ihlm}(\alpha)\equiv 0$.

The identity $\theta_{imjhl}(\boldsymbol{p}')\equiv 0$ would now reduce to

$$p'_{im}(\alpha)B_{ijhl}(\alpha) \equiv 0 ,$$

and imply $p'_{im}(\alpha)\equiv 0$ because of (24). Analogously we would get $p'_{ik}(\alpha)\equiv 0$. Using the equations

$$B_{ijmk}(\alpha_2) = B_{ihmk}(\alpha_2) = B_{ilmk}(\alpha_2) = 0$$

we would then get $B_{ijmk}(\alpha)\equiv 0$ just as we obtained $B_{ihlm}(\alpha)\equiv 0$. Since this contradicts (24), the statement is proved for $n=6$.

For $n \geqq 7$ we encounter a new possibility of the type

$$(25)\qquad \begin{cases} B_{ijhl}(\alpha_1) = 0, & B_{ijhl}(\alpha_2) \neq 0\,, \\ B_{imkg}(\alpha_1) \neq 0, & B_{imkg}(\alpha_2) = 0\,. \end{cases}$$

Since (24) is impossible, this would imply

$$B_{ijhm}(\alpha) \equiv B_{ijlm}(\alpha) \equiv B_{ihlm}(\alpha) \equiv 0\,.$$

Then the identity $\theta_{ijhlm}(\boldsymbol{p}') \equiv 0$ would yield $p'_{im}(\alpha) \equiv 0$. Together with the analogous results $p'_{ik}(\alpha) \equiv 0$, $p'_{ig}(\alpha) \equiv 0$, this would imply $B_{imkg}(\alpha) \equiv 0$, contradicting (25).

The last possibility, represented by

$$(26)\qquad \begin{cases} B_{ijhl}(\alpha_1) = 0, & B_{ijhl}(\alpha_2) \neq 0\,, \\ B_{mkgf}(\alpha_1) \neq 0, & B_{mkgf}(\alpha_2) = 0\,, \end{cases}$$

appears for $n \geqq 8$. Since (25) and (24) are impossible, this would imply for example

$$B_{ijhm}(\alpha) \equiv B_{ijlm}(\alpha) \equiv B_{ihlm}(\alpha) \equiv B_{ijmk}(\alpha) \equiv 0\,.$$

As above, we would get $p'_{im}(\alpha) \equiv 0$ by means of the identity $\theta_{ijhlm}(\boldsymbol{p}') \equiv 0$, and analogously for example $p'_{ik}(\alpha) \equiv 0$. Then

$$\begin{aligned} 0 \equiv B_{ijmk}(\alpha) &\equiv 2p'_{ij}(\alpha)p'_{mk}(\alpha) \Rightarrow p'_{ij}(\alpha) \equiv 0 \text{ or } p'_{mk}(\alpha) \equiv 0\,, \\ 0 \equiv B_{ihmk}(\alpha) &\equiv 2p'_{ih}(\alpha)p'_{mk}(\alpha) \Rightarrow p'_{ih}(\alpha) \equiv 0 \text{ or } p'_{mk}(\alpha) \equiv 0\,, \\ 0 \equiv B_{ilmk}(\alpha) &\equiv 2p'_{il}(\alpha)p'_{mk}(\alpha) \Rightarrow p'_{il}(\alpha) \equiv 0 \text{ or } p'_{mk}(\alpha) \equiv 0\,. \end{aligned}$$

Because $B_{ijhl}(\alpha) \not\equiv 0$ by (26), it would follow that $p'_{mk}(\alpha) \equiv 0$. Analogously $p'_{mg}(\alpha) \equiv p'_{mf}(\alpha) \equiv 0$, and thus $B_{mkgf}(\alpha) \equiv 0$, which contradicts (26). The proof is thus completed.

Remark 1. The conditions (21) for $k > 0$ are $\binom{n}{4}(N-1)$ in number. Already for $n = 6$ this number exceeds the number of independent elements of a symmetric $N \times N$-matrix, which is $\binom{N+1}{2}$. Thus, in general, our conditions cannot be independent.

Remark 2. It is not difficult to write the conditions (21) in theorem 4 in invariant form. With the usual notation of R with four indices, and distinguishing between lower (covariant) and upper (contravariant) indices, we have

$$(R^2)_{st} = \sum_{\tau} R_{s\tau}R_{\tau t} = \tfrac{1}{2}\sum_{\mu,\nu} R_{ij\mu\nu}R^{\mu\nu}{}_{kh}\,,$$

$$(R^3)_{st} = \sum_{\tau,\alpha} R_{s\tau}R_{\tau\alpha}R_{\alpha t} = \tfrac{1}{4}\sum_{\mu,\nu,\varrho,\sigma} R_{ij\mu\nu}R^{\mu\nu}{}_{\varrho\sigma}R^{\varrho\sigma}{}_{kh}\,,$$

etc. The conditions $\operatorname{tr}(B_iR) = 0$ are the well-known identities

$$R_{ijkh} + R_{ikhj} + R_{ihjk} = 0\,,$$

which with a notation of Schouten [9, p. 14] may be written

$$R_{i[jkh]} = 0 .$$

The corresponding conditions on $R^2, R^3, \ldots$ take the forms

$$3 \sum_{\mu,\nu} R_{i[j|\mu\nu} R^{\mu\nu}{}_{|kh]} \equiv \sum_{\mu,\nu} (R_{ij\mu\nu} R^{\mu\nu}{}_{kh} + R_{ik\mu\nu} R^{\mu\nu}{}_{hj} + R_{ih\mu\nu} R^{\mu\nu}{}_{jk}) = 0 .$$

$$\sum_{\mu,\nu,\varrho,\sigma} R_{i[j|\mu\nu} R^{\mu\nu}{}_{\varrho\sigma} R^{\varrho\sigma}{}_{|kh]} = 0 ,$$

etc.

3. A condition for the diagonalizability of the curvature matrix by an orthogonal transformation in the tangent space of V_n.

As we saw in the introduction, the coordinate axes of an orthonormal system S in the tangent space T_n at a point of V_n, in which R has diagonal form, must determine Ricci directions. Now, the linear transformation of the vectors in T_n, which is effected by the $n \times n$-matrix K with the components K_{ik} of the Ricci tensor as elements,

$$\boldsymbol{v} \to K\boldsymbol{v} ,$$

induces in the space of bivectors the transformation

$$[\boldsymbol{u}\boldsymbol{v}] \to [K\boldsymbol{u}K\boldsymbol{v}] .$$

The components of this transformed bivector are

$$\begin{aligned} [K\boldsymbol{u}K\boldsymbol{v}]_{ij} &= (K\boldsymbol{u})_i (K\boldsymbol{v})_j - (K\boldsymbol{u})_j (K\boldsymbol{v})_i \\ &= \sum_{\nu} K_{i\nu} u_\nu \cdot \sum_{\mu} K_{j\mu} v_\mu - \sum_{\nu} K_{j\nu} u_\nu \cdot \sum_{\mu} K_{i\mu} v_\mu \\ &= \sum_{\nu,\mu} (K_{i\nu} K_{j\mu} - K_{i\mu} K_{j\nu}) u_\nu v_\mu = \sum_{\nu<\mu} (K_{i\nu} K_{j\mu} - K_{i\mu} K_{j\nu})(u_\nu v_\mu - u_\mu v_\nu) . \end{aligned}$$

If we here, as in (01), replace the index pairs (ij) and $(\nu\mu)$ by single indices s and τ, respectively, and put

$$Q_{s\tau} = K_{i\nu} K_{j\mu} - K_{i\mu} K_{j\nu} , \tag{30}$$

we get the bivector transformation expressed by this $N \times N$-matrix Q:

$$p_s' = \sum_{\tau} Q_{s\tau} p_\tau .$$

It is obvious that the coordinate planes of S correspond to eigenvectors of Q, with the products $\varrho_i \varrho_k$ of the Ricci curvatures as eigenvalues. If now these coordinate planes are also planes of curvature, i.e. correspond to eigenvectors of R, the matrices R and Q must commute. In fact, it is well-known (cf. e.g. [10, p. 189]) that two symmetric matrices commute if, and only if, there exists an orthonormal system in which both have diagonal form. We have thus proved the first part of

THEOREM 5. *If the curvature matrix R can be reduced to diagonal form by an orthogonal transformation in T_n, then*

$$RQ = QR\ ,$$

where Q is the matrix defined by (30).

If $RQ=QR$ and all the products $\varrho_i\varrho_k$, $i<k$, of the Ricci curvatures ϱ_i are distinct, then R can be reduced to diagonal form by an orthogonal transformation in T_n.

PROOF OF THE SECOND PART. Because $RQ=QR$, there exists in E_N an orthonormal system in which both R and Q have diagonal form. Because of the other assumption, Q has diagonal form in only one system (up to changes of sign), namely that system which is determined by the Ricci principal directions of V_n.

One cannot dispense with the assumptions that all products $\varrho_i\varrho_k$ are distinct. If the Ricci directions are unique, but the eigenvalues of Q degenerate because of some equality $\varrho_i\varrho_k=\varrho_l\varrho_m$, then Q has also eigenvectors which do not correspond to a plane determined by two Ricci directions, and it may happen, that R and Q can be simultaneously diagonalized only by using such an eigenvector in the basis. We give an example in 4 dimensions.

EXAMPLE 1.

$$R = \begin{pmatrix} 11 & 0 & 0 & 12 & 0 & 0 \\ 0 & 7 & 0 & 0 & -12 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 12 & 0 & 0 & -21 & 0 & 0 \\ 0 & -12 & 0 & 0 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}, \quad K = \begin{pmatrix} -7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & -25 \end{pmatrix},$$

$$Q = \operatorname{diag}(175, -175, -49, 175, -175, -625)\ .$$

Here the only basis in E_6, which diagonalizes both R and Q, is

$$\{10^{-\frac{1}{2}}(3\boldsymbol{e}_1+\boldsymbol{e}_4),\ 5^{-\frac{1}{2}}(2\boldsymbol{e}_2-\boldsymbol{e}_5),\ \boldsymbol{e}_3,\ 10^{-\frac{1}{2}}(\boldsymbol{e}_1-3\boldsymbol{e}_4),\ 5^{-\frac{1}{2}}(\boldsymbol{e}_2+2\boldsymbol{e}_5),\ \boldsymbol{e}_6\}$$

The corresponding principal curvatures are 15, 13, 7, -25, -17, 7, respectively. Only the third and sixth of these are sectional curvatures for planes of curvature in V_4, of the six new basis vectors only $\boldsymbol{e}_3$ and $\boldsymbol{e}_6$ being simple bivectors.

Example 1 thus also shows that the condition $RQ=QR$ does not imply the existence of N planes of curvature forming a basis in E_N. Now one might ask whether the existence of such a basis could make the assumption $\varrho_i\varrho_k \neq \varrho_l\varrho_m$ dispensable in theorem 5. This is not true in general. We give a counter-example in 5 dimensions.

EXAMPLE 2. The correspondence (01) between index pairs (ij) and single indices s is here and in example 3 extended to $j=5$ in the following way: $(i5) \leftrightarrow i+6$ for $i=1, \ldots, 4$. Let the 10 diagonal elements of R be in order

$$(3, -3, 3, -3, 3, 0, -7, 1, -20, 20) ,$$

and let all other elements vanish except

$$R_{14} = R_{41} = -R_{25} = -R_{52} = 4 .$$

We then get

$$K = \mathrm{diag}(-10, 10, -20, 20, -6) ,$$

$$Q = \mathrm{diag}(-200, 200, -100, -200, 200, -400, 60, -60, 120, -120) .$$

If we replace the basis vectors e_1, e_2, e_4 and e_5 by $e_1{}'=5^{-\frac{1}{2}}(e_1-2e_4)$, $e_2{}'=5^{-\frac{1}{2}}(2e_2+e_5)$, $e_4{}'=5^{-\frac{1}{2}}(2e_1+e_4)$ and $e_5{}'=5^{-\frac{1}{2}}(e_2-2e_5)$, we get a system in which both R and Q have diagonal form. Thus $RQ=QR$. The basis vectors $e_1{}'$ and $e_2{}'$ correspond to the same principal curvature -5, to which thus also the planes of curvature defined by the simple bivectors $e_1{}'+e_2{}'$ and $e_1{}'-e_2{}'$ correspond. Likewise $e_4{}'$ and $e_5{}'$ correspond to the same principal curvature 5 to which also $e_4{}'+e_5{}'$ and $e_4{}'-e_5{}'$ correspond, these yielding planes of curvature. The other principal curvatures $(3, 0, -7, 1, -20, 20)$ are all different and correspond to uniquely determined planes of curvature given by e_3, e_6, e_7, e_8, e_9 and e_{10}, respectively. The Ricci directions are unique and determine the original coordinate system S. As R has not diagonal form in S, it cannot be diagonalized by any orthogonal transformation in T_5.

The situation is similar if the Ricci curvatures ϱ_i are not distinct. In spite of the freedom of choice for the Ricci directions, it is not always possible to choose them so that they determine N planes of curvature. We show this by an example in 5 dimensions.

EXAMPLE 3. As diagonal of R we take $(3, 2, 1, 4, 5, 6, 7, 5, 8, 9)$. If then all other elements vanish except

$$R_{45} = R_{54} = -R_{78} = -R_{87} = 1 ,$$

the Ricci tensor has the diagonal form

$$\mathrm{diag}(14, 14, 19, 24, 29) ;$$

$\varrho_1 = \varrho_2 = 14$ implies $Q_{44} = Q_{55}$ and $Q_{77} = Q_{88}$, and thus $RQ = QR$. In order to diagonalize R by an orthogonal transformation in T_5, we must change the basis vectors in the plane which corresponds to the double Ricci curvature 14. Any such change yields, however, an element $R_{12} \neq 0$.

It may be of some interest to express the condition $RQ = QR$ of theorem 5 more explicitly in terms of the Riemann and Ricci tensors only. For the transposed matrix of RQ we have $(RQ)' = Q'R' = QR$ because Q and R are symmetric. Thus the condition is that RQ shall be symmetric. Using the definition (30) of Q and the usual notation of R with four indices, we get

$$\begin{aligned}(RQ)_{st} &= \sum_{\tau} R_{s\tau} Q_{\tau t} = \tfrac{1}{2} \sum_{\nu,\mu} R_{ij\nu\mu}(K_{\nu k} K_{\mu h} - K_{\nu h} K_{\mu k}) \\ &= \tfrac{1}{2} \sum_{\nu,\mu} R_{ij\nu\mu} K_{\nu k} K_{\mu h} - \tfrac{1}{2} \sum_{\mu,\nu} R_{ij\mu\nu} K_{\mu h} K_{\nu k} = \sum_{\nu,\mu} R_{ij\nu\mu} K_{\nu k} K_{\mu h} ,\end{aligned}$$

Then the condition $RQ = QR$ is that the last expression shall not change, if the index pairs (ij) and (kh) are interchanged:

$$\sum_{\nu,\mu} R_{ij\nu\mu} K_{\nu k} K_{\mu h} = \sum_{\nu,\mu} R_{kh\nu\mu} K_{\nu i} K_{\mu j} .$$

If we again introduce co- and contravariant indices, we can write the condition so that it applies in any coordinate system:

$$\sum_{\nu,\mu} R_{ij}{}^{\nu\mu} K_{\nu k} K_{\mu h} = \sum_{\nu,\mu} R_{kh}{}^{\nu\mu} K_{\nu i} K_{\mu j} .$$

4. Conditions for the diagonizability of the curvature matrix by an orthogonal transformation in the tangent space of V_4.

In 4 dimensions we can use the notion of *duality* as developed by Churchill in [2, pp. 129ff.]. We formulate the definitions in terms of the 6×6-matrix B given by (20).

DEFINITIONS. The matrix $\bar{A} = BAB$ is called the *dual* matrix of A. The bivector $\bar{\boldsymbol{p}} = B\boldsymbol{p} = (p_4, p_5, p_6, p_1, p_2, p_3)$ is called the *dual* of the bivector $\boldsymbol{p}$. A matrix A (bivector $\boldsymbol{p}$) is called *selfdual* if $\bar{A} = A$ ($\bar{\boldsymbol{p}} = \boldsymbol{p}$), and *antiselfdual* if $\bar{A} = -A$ ($\bar{\boldsymbol{p}} = -\boldsymbol{p}$).

The condition (04) for a bivector $\boldsymbol{p}$ to be simple may now be written

$$(B\boldsymbol{p}, \boldsymbol{p}) = (\boldsymbol{p}, \bar{\boldsymbol{p}}) = 0 . \tag{40}$$

If $\boldsymbol{p}$ is simple, $\bar{\boldsymbol{p}}$ determines in T_4 the plane which contains all vectors orthogonal to the plane of $\boldsymbol{p}$ [2, p. 129]. It is obvious that $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}) = (\boldsymbol{p}, \boldsymbol{q})$.

We shall obtain the condition for the possibility of diagonalizing R by an orthogonal transformation in T_4 as a corollary of the following more general theorem.

THEOREM 6. *Let A and B be symmetric $N \times N$-matrices, N even, and suppose that B is involutory ($B^2 = I, B \neq I$). Then there exists in E_N an orthonormal basis of the form*

$$\{v_1, \ldots, v_{N/2}, Bv_1, \ldots, Bv_{N/2}\}$$

consisting of eigenvectors of A if and only if

$$\operatorname{tr}(BA^k) = 0 \qquad \text{for} \quad k \geqq 0$$

and $A\bar{A} = \bar{A}A$, where $\bar{A} = BAB$.

PROOF. 1° "if". Let $\lambda_1, \ldots, \lambda_p$ be the different eigenvalues of A, and then also of $\bar{A} = BAB^{-1}$. When $A\bar{A} = \bar{A}A$, there exists an orthonormal basis $\{v_\nu\}$, referred to which both A and $\bar{A}$ have diagonal form (cf. above p. 15). Then each v_ν is an eigenvector of A corresponding to a certain eigenvalue λ_i and simultaneously an eigenvector of $\bar{A}$ corresponding to a certain eigenvalue λ_j. For every pair of eigenvalues λ_i, λ_j we put

$$y_{ij} = \sum (Bv_\nu, v_\nu) ,$$

where the sum is taken over those ν for which

$$Av_\nu = \lambda_i v_\nu \qquad \text{and} \qquad \bar{A}v_\nu = \lambda_j v_\nu . \tag{41}$$

For any non-negative integers k and l we then have

$$\begin{aligned} \sum_{i,j=1}^{p} \lambda_i^k \lambda_j^l y_{ij} &= \sum_{\nu=1}^{N} (BA^k\bar{A}^l v_\nu, v_\nu) = \operatorname{tr}(BA^k\bar{A}^l) \\ &= \operatorname{tr}(BA^kBA^lB) = \operatorname{tr}(A^kBA^l) = \operatorname{tr}(BA^{l+k}) = 0 , \end{aligned} \tag{42}$$

because of the assumptions and the general relation $\operatorname{tr}(CD) = \operatorname{tr}(DC)$. The equations (42) for $k < p$, $l < p$ form a system of p^2 linear equations with the p^2 unknowns y_{ij} and the determinant

$$\prod_{i<j} (\lambda_i - \lambda_j)^{2p} \neq 0 .$$

Hence, all y_{ij} vanish. Now, $y_{ij} = 0$ means that the quadratic form (Bv, v) has trace 0 in the subspace E_{ij} spanned by those v_ν which satisfy (41). According to the lemma in § 1, it is then possible to choose in each E_{ij} an orthonormal basis of vectors isotropic for B. From such a basis $\{u_1, \ldots, u_m\}$ in E_{ij} we get an analogous basis

$$\{Bu_1, \ldots, Bu_m\} \tag{43}$$

in E_{ji}. For, if u is isotropic for B, so is Bu. Further,

$$ABu_h = B\bar{A}u_h = \lambda_j Bu_h, \qquad \bar{A}Bu_h = BAu_h = \lambda_i Bu_h$$

and, B being orthogonal,

$$(Bu_h, Bu_g) = (u_h, u_g) = \delta_{hg} .$$

In particular, there exist in a non-empty E_{ii} two vectors $\boldsymbol{u}_1$ and $B\boldsymbol{u}_1$ isotropic for B. If they do not span the whole E_{ii}, the form $(B\boldsymbol{v},\boldsymbol{v})$ has again trace 0 in the subspace of E_{ii} orthogonal to $\boldsymbol{u}_1$ and $B\boldsymbol{u}_1$. Then we get another pair of isotropic vectors $\boldsymbol{u}_2$ and $B\boldsymbol{u}_2$ by the lemma in § 1 and so on until E_{ii} is exhausted. Applying this procedure to each E_{ii}, choosing bases of vectors isotropic for B in all non-empty E_{ij} with $i<j$, and completing with the corresponding bases (43) in E_{ji}, we get a basis in E_N which, numbered suitably, has the required properties.

2° "only if". Suppose the vectors $\boldsymbol{v}_\mu$ and $B\boldsymbol{v}_\mu=\boldsymbol{v}_\nu$, where $\mu=1,2,\ldots,N/2$ and $\nu=\mu+N/2$, are eigenvectors of A and form an orthonormal basis. Then

$$(B\boldsymbol{v}_\mu,\boldsymbol{v}_\mu) = (\boldsymbol{v}_\nu,B\boldsymbol{v}_\nu) = (\boldsymbol{v}_\nu,\boldsymbol{v}_\mu) = 0\,,$$

that is, they are isotropic for B. This implies $\operatorname{tr}(BA^k)=0$ for $k\geqq 0$ according to theorem 1 and the remark following its proof. Further, because of

$$\bar{A}\boldsymbol{v}_\mu = BAB\boldsymbol{v}_\mu = BA\boldsymbol{v}_\nu = \lambda_i B\boldsymbol{v}_\nu = \lambda_i\boldsymbol{v}_\mu\,,$$
$$\bar{A}\boldsymbol{v}_\nu = \bar{A}B\boldsymbol{v}_\mu = BA\boldsymbol{v}_\mu = \lambda_j B\boldsymbol{v}_\mu = \lambda_j\boldsymbol{v}_\nu$$

for certain i and j, all of the basis vectors are also eigenvectors of $\bar{A}$. This implies $A\bar{A}=\bar{A}A$.

Corollary. *The curvature matrix R has diagonal form for some orthonormal basis in T_4, if and only if $R\bar{R}=\bar{R}R$ and $\operatorname{tr}(BR^k)=0$ for $k=2,3,4,5$, where B is the matrix* (20).

Proof. The conditions imply, by the remark following theorem 1 and theorem 6, the existence in E_6 of an orthonormal basis $\{\boldsymbol{v}_\nu\}$ of eigenvectors of R which are dual in pairs. According to Churchill's theorem 4 [2, p. 133], such a basis corresponds to an orthonormal basis in T_4. Conversely, the bivectors $\boldsymbol{v}_\nu=[\boldsymbol{e}_i\boldsymbol{e}_k]$ determined by a four-dimensional basis $\{\boldsymbol{e}_i\}$ are dual in pairs (cf. [2, p. 129]). If these $\boldsymbol{v}_\nu$ are eigenvectors of R, the "only if" in theorem 6 yields $R\bar{R}=\bar{R}R$ and $\operatorname{tr}(BR^k)=0$.

Remark 1. The single condition $R\bar{R}=\bar{R}R$ is not sufficient. We show this by an example, where selfdual and anti-selfdual eigenvectors appear. If we take $(3,-3,7,3,-3,1)$ for diagonal elements of R and let all other elements vanish except $R_{14}=R_{41}=-R_{25}=-R_{52}=1$, we get the (uniquely determined) eigenvectors $\boldsymbol{e}_1\pm\boldsymbol{e}_4$, $\boldsymbol{e}_2\pm\boldsymbol{e}_5$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_6$ which are not dual in pairs. It is easily verified that $R\bar{R}=\bar{R}R$.

Remark 2. In order to formulate the condition $R\bar{R}=\bar{R}R$ in an arbitrary coordinate system we have to introduce a tensor corresponding to the dual matrix $\bar{R}$ by putting

$$\bar{R}^{ij}{}_{kh} = R_{i'j'}{}^{k'h'}\,,$$

where $(iji'j')$ and $(khk'h')$ are even permutations of the numbers (1234). With this "dual curvature tensor" $\bar{R}$ our condition can be written

$$\sum_{\mu,\nu} R^{ij}{}_{\mu\nu}\bar{R}^{\mu\nu}{}_{kh} = \sum_{\mu,\nu} \bar{R}^{ij}{}_{\mu\nu}R^{\mu\nu}{}_{kh} .$$

5. Some examples.

We mention here some classes of Riemannian spaces V_n at each point of which the curvature matrix R can be reduced to diagonal form by an orthogonal transformation in the tangent space. At every point of such a space there exist thus N planes of curvature which are the coordinate planes of an orthonormal system.

a) The spaces of constant curvature, S_n, are obvious examples since the curvature matrix is proportional to the unit matrix. Every plane is a plane of curvature.

b) For spaces V_4 with anti-selfdual curvature matrix ($\bar{R} = -R$) the reduction to diagonal form has been carried out by Churchill [2, pp. 149 ff.] using a result of Einstein [3]. These spaces are known to be conformally flat [6, p. 71].

The first condition, $R\bar{R} = \bar{R}R$, of the corollary in the preceding section is here trivially satisfied. Also the second condition is easily verified. For any symmetric and anti-selfdual matrix A, any symmetric matrix C, and the symmetric involutory matrix B we have

$$\begin{aligned} \operatorname{tr}(BAC) &= \operatorname{tr}(BABBC) = -\operatorname{tr}(ABC) = -\operatorname{tr}(CAB) \\ &= -\operatorname{tr}(B'A'C') = -\operatorname{tr}(BAC) = 0 , \end{aligned}$$

and hence, in particular, $\operatorname{tr}(BR^k) = \operatorname{tr}(BRR^{k-1}) = 0$ for $k = 2,3,\ldots$.

c) In every conformally flat space C_n the curvature matrix R can be reduced to diagonal form by an orthogonal transformation in the tangent space.

This follows for example from the well-known condition

$$\begin{aligned} C_{ijkh} = R_{ijkh} - (n-2)^{-1}\{\delta_{ik}K_{jh} - \delta_{ih}K_{jk} + g_{jh}K_{ik} - g_{jk}K_{ih} + \\ + k(n-1)^{-1}(\delta_{ih}g_{jk} - \delta_{ik}g_{jh})\} = 0 \end{aligned} \tag{50}$$

for a V_n to be conformally flat (cf. e.g. [4, pp. 517 ff.]). Here δ_{ik} is the Kronecker symbol, g_{ik} the fundamental tensor, and k the scalar curvature $\operatorname{tr}K$. In the principal coordinate system (determined by the Ricci directions) at a point, K_{ik} has diagonal form and $g_{ik} = \delta_{ik}$. Consequently. (50) reduces to

$$\begin{aligned} R_{ijij} &= (n-2)^{-1}\{K_{jj} + K_{ii} - k(n-1)^{-1}\} = -R_{ijji} , \qquad i \neq j , \\ R_{ijkh} &= 0 \quad \text{for} \quad (k,h) \neq (i,j) \quad \text{and} \neq (j,i) . \end{aligned} \tag{51}$$

These equations show that the curvature matrix too has diagonal form in the principal system.

As mentioned above, a V_4 with anti-selfdual curvature matrix is a C_4. The converse is not true in general, but it follows easily from (51) that in a C_4 the matrix $R-cI$, where I denotes the unit 6×6-matrix, is anti-selfdual for $c=k/12$.

d) If the Riemannian spaces V_r and V_{n-r}, $1<r<n-1$, have the property in question, then also their product $V_r \times V_{n-r}$ (cf. [9, p. 285]) has it. This is an obvious consequence of the definition. Examples of such products are the recurrent spaces with positive definite metric, these being decomposable into a V_2 and a Euclidean space (cf. [8, p. 173]).

REFERENCES

1. E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, 2e éd., Paris, 1951.
2. R. V. Churchill, *On the geometry of the Riemann tensor*, Trans. Amer. Math. Soc. 34 (1932), 126–152.
3. A. Einstein, *Über die formale Beziehung des Riemannschen Krümmungstensors zu den Feldgleichungen der Gravitation*, Math. Ann. 97 (1926), 99–103.
4. J. Favard, *Cours de géométrie différentielle locale*, Paris, 1957.
5. G. Ricci, *Direzioni e invarianti principali di una varietà qualunque*, Atti R. Istituto Veneto (8) 6 (=63) (1904), 1233–1239.
6. H. S. Ruse, *On the line-geometry of the Riemann tensor*, Proc. Roy. Soc. Edinburgh Sect. A 62 (1944), 64–73.
7. H. S. Ruse, *The five-dimensional geometry of the curvature tensor in a Riemannian V_4*, Quart. J. Math. Oxford 17 (1946), 1–15.
8. H. S. Ruse, A. G. Walker and T. J. Willmore, *Harmonic spaces*, Rome, 1961.
9. J. A. Schouten, *Ricci-calculus*, Berlin · Göttingen · Heidelberg, 1954.
10. R. Thrall and L. Tornheim, *Vector spaces and matrices*, New York · London, 1957.

CHALMERS' INSTITUTE OF TECHNOLOGY, GOTHENBURG, SWEDEN