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A VERTEX PROPERTY FOR BANACH ALGEBRAS WITH IDENTITY

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1. Introduction.

Bohnenblust and Karlin [2] proved that any complex Banach algebra with identity has the vertex property: Every unit sphere belonging to a natural norm has the identity as a vertex point (in the complex sense). For real algebras this is not true in general and the main object of this paper is to investigate when a real algebra has the corresponding property (with vertex taken in the real sense). Our first result is

THEOREM 2. *A real Banach algebra has the real vertex property if and only if* $\exp(\alpha x)$ *is an unbounded function of the real variable* α *for every* $x \neq 0$.

Theorem 3 shows that $\exp(\alpha x)$ *is unbounded for every* $x \neq 0$ *in the radical.* From these can be deduced some results involving conditions of a more directly algebraic nature (for definitions of strongly real type and complex type, see p. 28):

THEOREM 4. *A real Banach algebra of strongly real type with identity has the vertex property.*

THEOREM 5. *A real Banach algebra containing a sub-algebra of complex type with identity does not have the vertex property.*

I am very grateful to H. Rådström, who proposed this line of research and continuously guided the work.

2. Preliminary remarks.

This section deals with Banach spaces (and algebras) over the real or complex numbers.

As usual, two norms, N_1 and N_2, are called *equivalent* if there exist two real numbers, c and C, so that for all $x \neq 0$

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$$0 < c \leqq \frac{N_1(x)}{N_2(x)} \leqq C . \tag{1}$$

Two norms define the same topology if and only if they are equivalent. Two complete normed spaces with the same elements and equivalent norms will be regarded as the same Banach space.

A vertex of a convex set, K, is a point, x_0, belonging to the boundary of K with the property that no straight line through x_0 is a tangent of K. The notion of a straight line is of course different when real or complex scalars are regarded (a complex straight line is a real two-dimensional affine subspace). Accordingly we shall speak of a real vertex or a complex vertex as the case may be.

Bohnenblust and Karlin [2] give the following precise definition for the special case of the unit sphere.

DEFINITION 1. A point, u, is said to be a vertex of the unit sphere defined by a norm, N, if $N(u) = 1$ and $\psi(x) = 0$ only if $x = 0$, where

$$\psi(x) = \max_{|\theta|=1} \Phi(\theta x) , \tag{2}$$

where $\Phi(x)$ is the Gateau differential

$$\Phi(x) = \lim_{\alpha \to +0} \frac{N(u+\alpha x) - N(u)}{\alpha} . \tag{3}$$

(This definition is valid both in real and complex spaces where in real spaces $|\theta| = 1$ means $\theta = \pm 1$.)

This paper will deal only with associative algebras with a two-sided identity, mostly denoted e.

By a *natural norm* we mean a norm satisfying

$$||xy|| \leqq ||x|| \cdot ||y|| , \tag{4}$$

$$||e|| = 1 . \tag{5}$$

The notation $||\cdot||$ is reserved for such norms. By use of the left regular representation of the algebra (i.e. as an algebra of linear operators on itself) Gelfand [4] proved that for every Banach algebra there exists at least one natural norm defining the topology. If N is any given norm then

$$||x|| = \sup_{y \neq 0} \frac{N(xy)}{N(y)}$$

is an equivalent natural norm.

For a given natural norm, we will investigate whether the unit element is a vertex of the unit sphere.

A Banach algebra is said to have the *vertex property* if for all natural norms defining the topology the identity is a vertex of the unit sphere. The vertex property is a topological-algebraic property and does not depend on any specific metric.

We recall that in a Banach algebra we can always define

$$\exp x = e + \sum_{n=1}^{\infty} \frac{x^n}{n!}, \tag{7}$$

since this series is absolutely convergent. If x and y commute we have

$$\exp(x+y) = \exp x \cdot \exp y . \tag{8}$$

3. A condition for vertex.

In this section we formulate a necessary and sufficient condition for the identity to be a vertex of the unit sphere. The technique used is to a large extent taken from Bohnenblust and Karlin [2]. A slight improvement is possible because of the following:

Lemma 1. *Let $h(\lambda)$ be a real-valued function defined either on the real axis or on the complex plane.*

If

$$h(\lambda_1+\lambda_2) \leqq h(\lambda_1)+h(\lambda_2), \tag{9}$$

$$h(0) = 0, \tag{10}$$

$$\overline{\lim_{\alpha \to +0}} \frac{h(\alpha \lambda)}{\alpha} \leqq 0, \tag{11}$$

then

$$h(\lambda) \equiv 0 .$$

Proof. Assume that $h(\lambda) > 0$ somewhere, let $h(\lambda_0) > 0$. By repeated use of (9) we find

$$\frac{h(\lambda_0/n)}{1/n} \geqq h(\lambda_0) > 0$$

for every positive integer n. When $n \to \infty$ this contradicts (11). Thus $h(\lambda) \leqq 0$. But from (9) and (10) we have

$$0 = h(0) = h(\lambda-\lambda) \leqq h(\lambda)+h(-\lambda) \leqq 0$$

so equality must hold everywhere and $h(\lambda) \equiv 0$.

Theorem 1. *In a real or complex Banach algebra with a given natural norm, $\|\cdot\|$, the following conditions are equivalent:*

1° *The identity is not a vertex of the unit sphere.*

2° *There exists* $x \neq 0$ *so that for all scalars* λ *the equation* $\|\exp(\lambda x)\| = 1$ *holds.*

PROOF. With

$$\Phi(x) = \lim_{\alpha \to +0} \frac{\|e + \alpha x\| - \|e\|}{\alpha}$$

we have, since $\exp(\alpha x) = e + \alpha x + O(\alpha^2)$, $\alpha \to 0$,

$$\Phi(x) = \lim_{\alpha \to +0} \frac{\|\exp(\alpha x)\| - 1}{\alpha} .$$

We also have $\log(1+t) = t + O(t^2)$, $t \to 0$, so

$$\Phi(x) = \lim_{\alpha \to +0} \frac{\log \|\exp(\alpha x)\|}{\alpha} .$$

For a fixed x we put $h(\lambda) = \log \|\exp(\lambda x)\|$. Because of (4), (5) and (8), $h(\lambda)$ satisfies (9) and (10). The condition $\psi(x) = 0$ in Definition 1 is equivalent to $\Phi(\lambda x) \leqq 0$ for each λ, that is

$$\lim_{\alpha \to +0} \frac{h(\alpha\lambda)}{\alpha} \leqq 0 .$$

Now assume that 1° holds, i.e. there exists an $x \neq 0$ with $\psi(x) = 0$. Lemma 1 shows that

$$\log \|\exp(\lambda x)\| = 0 . \tag{12}$$

Conversely, if an $x \neq 0$ satisfying (12) exists, we get immediately $\Phi(\lambda x) = 0$ and $\psi(x) = 0$, which proves the theorem.

We can now give an alternative proof of Bohnenblust and Karlin's [2] result for complex algebras.

COROLLARY 1. *A complex Banach algebra has the complex vertex property.*

PROOF. Suppose that, for a given x, we have $\|\exp(\lambda x)\| = 1$ for all complex λ. But $\exp(\lambda x)$ is a bounded algebra-valued entire analytic function of the complex variable λ. Then it must be a constant, according to Liouville's theorem. In particular, the coefficient of λ in the power series expansion is zero, so that $x = 0$.

COROLLARY 2. *If in a complex Banach algebra with identity and more than one dimension the topology can be defined by an inner product* (i.e. *the Banach space is a Hilbert space*) *the corresponding norm is not natural.*

PROOF. We prove that no point can be a vertex of the corresponding unit sphere.

Let the norm be $N(x) = (x,x)^{\frac{1}{2}}$ and u an element with $N(u) = 1$. Then

$$N(u+\alpha x) = (u+\alpha x, u+\alpha x)^{\frac{1}{2}} = 1+\alpha \operatorname{Re}(u,x) + O(\alpha^2), \qquad \alpha \to +0,$$

so that (cf. (2) and (3))

$$\Phi(x) = \operatorname{Re}(u,x), \qquad \psi(x) = |(u,x)| .$$

For every x in the hyperplane perpendicular to u we have $\psi(x) = 0$ which shows that u is not a vertex. According to Corollary 1, however, the identity e will be a vertex if N is natural. Hence N is non-natural.

4. Vertex property in real algebras.

For real algebras it is not generally true that for all natural norms the unit sphere has the identity as a real vertex. For instance the complex number system, as a real algebra, possesses a natural norm with this property ($\|\xi+i\eta\| = |\xi|+|\eta|$) and another one without it $\left(|\xi+i\eta| = \sqrt{\xi^2+\eta^2}\right)$. However, with the help of Theorem 1, it is possible to formulate a condition, suitable for analytic considerations, for an algebra to have the vertex property.

THEOREM 2. *A real Banach algebra has the real vertex property if and only if* $\exp(\alpha x)$ *is an unbounded function of the real variable* α *for every* $x \neq 0$.

PROOF. If for every $x \neq 0$ it its true that $\exp(\alpha x)$ is unbounded it is impossible that, for any norm, $\|\exp(\alpha x)\| = 1$ holds. So the algebra has the vertex property.

Conversely assume that there exists an $x_0 \neq 0$ so that $\exp(\alpha x_0)$ is bounded. For a certain norm (which can be chosen natural according to Ch. 2) we then have a constant K so that

$$\|\exp(\alpha x_0)\| \leqq K \quad \text{for all real } \alpha .$$

We will now re-norm the algebra in two steps. We first put

$$N(x) = \sup_{\alpha} \|\exp(\alpha x_0) \cdot x\|$$

which is a norm. It is also equivalent to $\|x\|$ since

$$\|x\| = \|\exp(0 x_0) \cdot x\| \leqq N(x) \leqq K\|x\| .$$

In general $N(x)$ is non-natural; we therefore repeat the "naturalization" process (6) and form

$$|||x||| = \sup_{y \neq 0} \frac{N(xy)}{N(y)}$$

which is equivalent to $N(x)$ and $\|x\|$. Moreover,

$$|||\exp(\alpha x_0)||| = \sup_{y \neq 0} \frac{\sup_\beta \|\exp(\beta x_0)\cdot \exp(\alpha x_0)\cdot y\|}{\sup_\gamma \|\exp(\gamma x_0)\cdot y\|}$$

$$= \sup_{y \neq 0} \frac{\sup_\beta \|\exp[(\beta+\alpha)x_0]\cdot y\|}{\sup_\gamma \|\exp(\gamma x_0)\cdot y\|} = 1 .$$

Using Theorem 1 we find that the algebra has a natural norm, $|||\cdot|||$, for which the identity is not a vertex of the unit sphere. So the given condition is necessary and sufficient and the theorem is proved.

From Theorem 2 we see that in a real algebra the set of elements x which make the function $\exp(\alpha x)$ bounded is of interest. The intersection of this set with the radical of the algebra is only the origin as is shown in the following theorem.

THEOREM 3. *If $x \neq 0$ belongs to the radical of a real Banach algebra,* $\exp(\alpha x)$ *is an unbounded function of the real variable α.*

PROOF. Assume $\exp(\alpha x)$ bounded. Since x belongs to the radical $\lim_{n\to\infty}(N(x^n))^{1/n} = 0$ for every norm $N(x)$. Let f be any continuous linear functional and put

$$\varphi(\alpha) \equiv f(\exp(\alpha x)) = f(e) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} f(x^n) \equiv \sum_{n=0}^{\infty} a_n \frac{\alpha^n}{n!}.$$

For the coefficients we have

$$|a_n|^{1/n} = |f(x^n)|^{1/n} \leqq (N(f))^{1/n}\,(N(x^n))^{1/n} \to 0, \qquad n \to \infty .$$

Thus $\varphi(\alpha)$ is represented as a power series with infinite radius of convergence; it can then be continued analytically to an entire function in the complex plane. As an entire function it is at most of order 1, minimum type, that is for every $\delta > 0$ we have

$$|\varphi(\alpha)| = O(\exp \delta|\alpha|). \qquad |\alpha| \to \infty .$$

To prove this, we first notice that

$$|\varphi(\alpha)| \leqq \sum_{n=0}^{\infty} |a_n| \frac{|\alpha|^n}{n!} .$$

By comparing coefficients we now find that for every $\delta > 0$ the series of $|\varphi(\alpha)|$ is majorized by that of $\exp \delta|\alpha|$ except possibly for a finite number of terms.

A Phragmén–Lindelöf theorem (Boas [1, p. 84]) shows that a function of this kind, bounded on the real axis, is a constant. This implies that $a_1 = 0$, that is $f(x) = 0$ for every continuous functional f. Hence $x = 0$ against the assumption. Therefore $\exp(\alpha x)$ is unbounded for every $x \neq 0$ in the radical.

Corollary. *If an identity is adjoined to a radical Banach algebra, the resulting algebra (with the usual topology) has the vertex property.*

Proof. Let the radical algebra be R and the new algebra A. Then R is the radical of A. Every $x \in A$ can be written

$$x = \gamma e + x',$$

where $x' \in R$. A norm for A is

$$N(x) = N(\gamma e + x') = |\gamma| + \|x'\|,$$

where $\|x'\|$ is the norm for x' in R. Moreover

$$\begin{aligned} \exp(\alpha x) &= \exp(\alpha\gamma e + \alpha x') \\ &= \exp(\alpha\gamma)\exp(\alpha x') \\ &= \exp(\alpha\gamma)\cdot[e + r(\alpha)], \qquad r(\alpha) \in R, \end{aligned}$$

and

$$N(\exp(\alpha x)) = \exp(\alpha\gamma)(1 + \|r(\alpha)\|).$$

If $\gamma \neq 0$ this function is obviously unbounded and if $\gamma = 0$ but $x' \neq 0$ it is unbounded according to the theorem. So $\exp(\alpha x)$ bounded implies $x = 0$, and the algebra has the vertex property.

5. Some criteria in real algebras.

We begin with some definitions for real algebras, applying also to algebras without identity.

Definitions.

2. The *realization* of a complex algebra is the real algebra with the same elements obtained when only real scalars are admitted.
3. A real algebra is of *complex type* if it is the realization of a complex algebra; it is of *real type* if it is not of complex type.
4. A real algebra is of *strongly real type* if, for every x, the element $-x^2$ is quasi-regular.

In order to justify the notation we prove

Lemma 2. *A non-radical Banach algebra of strongly real type is of real type.*

PROOF. Let A be strongly real and P a primitive ideal. Then A/P is a primitive, strongly real Banach algebra. An argument by Kaplansky [5] (proof of Theorem 4.8, p. 405) shows that A/P is isomorphic with the reals. If A is at the same time of complex type then to every x belongs y_x, satisfying

$$x^2 = -y_x^2 .$$

Since this relation is preserved under homomorphisms we find that every x is mapped on 0 in A/P, i.e. $x \in P$ for every primitive P. Then every x belongs to the radical, against the assumption.

For algebras with identity e the property of *complex type* is equivalent to the requirement:

There exists an element j in the center of the algebra with $j^2+e=0$.

The same statement is true of Banach algebras, i.e. a real Banach algebra with identity can be regarded as the realization of a complex Banach algebra if and only if such a j exists. To prove this it only remains to show that a real-normed space is complex-normable. A complex norm can easily be constructed (see for instance Kaplansky [5])

$$N(x) = \sup_{\varphi} \|(e \cos\varphi + j \sin\varphi)x\|$$

where $\|\cdot\|$ is a natural real norm; then $N(x)$ is a complex-homogeneous norm equivalent to $\|x\|$. Strongly real for algebras with identity means that $e+x^2$ is regular for every x; it was used in this form by Gelfand [4].

It is now possible to formulate some criteria making it possible to decide in some cases whether a given real Banach algebra with identity has the vertex property or not.

THEOREM 4. *A real Banach algebra of strongly real type with identity has the vertex property.*

PROOF. Let the algebra be B and its radical R. $B' = B/R$ is semi-simple and strongly real; according to a theorem by Kaplansky [5] it is then necessarily commutative. Belonging to every maximal ideal in B' there is a continuous (canonical) homomorphism $x' \to \Gamma(x')$, mapping B' on the real numbers (see for instance Gelfand [4]). Then we have

$$\Gamma(\exp(\alpha x')) = \exp(\alpha\Gamma(x')) .$$

Now assume that we have an $x \in B$ making $\exp(\alpha x)$ bounded. Then in B' $(\exp(\alpha x))' = \exp(\alpha x')$ is bounded and therefore $\exp(\alpha\Gamma(x'))$ is bounded in the field of real numbers. Then $\Gamma(x')=0$ for every canonical homomorphism, giving $x'=0$ since B' is semi-simple. Thus $x \in R$, and according to Theorem 3, $x=0$.

As mentioned above not all real algebras have the vertex property and the next theorem will characterize a class of algebras lacking this property. This class will contain every algebra of complex type but also several of real type. For convenience we first prove a lemma.

LEMMA 3. *A real algebra contains a sub-algebra of complex type with identity if and only if there exists an element k in the algebra satisfying $k^3 = -k$.*

PROOF. If such an element exists, the elements $\alpha k^2 + \beta k$ form a sub-algebra of complex type with identity $-k^2$. If, on the other hand, such a sub-algebra exists with identity e', there exists by definition an element k with $k^2 = -e'$; so $k^3 = -k$.

THEOREM 5. *A real Banach algebra containing a sub-algebra of complex type with identity does not have the vertex property.*

PROOF. According to Lemma 3 there exists an element k with $k^3 = -k$. Then

$$\exp(\alpha k) = e + k \sin\alpha + k^2(1 - \cos\alpha)$$

is a bounded function of α. So according to Theorem 2 the algebra does not have the vertex property.

6. Concluding remarks and examples.

In this paper one necessary and sufficient condition for a real Banach algebra to have the vertex property has been given (Theorem 2). Further one necessary and one sufficient condition have been formulated in more directly algebraic terms. Neither of these conditions are both necessary and sufficient and it does not seem possible to change them in any obvious way to give such conditions. This will be illustrated by Examples 1 and 2. Finally, in Example 3 it is shown that for two-dimensional real Banach algebras with identity the vertex-free natural unit spheres are all ellipses.

In Examples 1 and 2 will be considered closed sub-algebras of *real* Banach algebras, $C(\Omega)$; $C(\Omega)$ consists of all complex-valued continuous functions on a compact Hausdorff space Ω, the topology defined by the maximum norm.

EXAMPLE 1. $\Omega =$ the closed unit disc of a complex z-plane. The sub-algebra consists of all functions analytic for $|z| < 1$ and taking only real values at the point $z = 1$.

This algebra is not of strongly real type (for instance $z^2 + 1$ has a zero in the disc and so has no inverse) but it does have the vertex property.

For if $\exp(\alpha f(z))$ is bounded, then $\mathrm{Re} f(z) = 0$. Since $f(z)$ is analytic it is a constant with real part zero. This constant is also $= f(1) =$ real, leading to $f(z) \equiv 0$. This algebra contains a non-zero sub-algebra of complex type but without identity, namely the set of functions satisfying $f(1) = 0$.

EXAMPLE 2. $\Omega =$ closed interval $[0,1]$ of the real line. The sub-algebra consists of those functions taking only real values at 1. This algebra does not have the vertex property (take $f(x) = i(1-x)$, $0 \leqq x \leqq 1$; then $\exp(\alpha f(x))$ is bounded) but still does not contain any non-zero sub-algebra of complex type with identity. An identity e' satisfies $e'^2 = e'$ and since it is a continuous function either $e'(x) \equiv 0$ or $e'(x) \equiv 1$. In the first case the sub-algebra is zero and in the second case it cannot be of complex type, since then ie' would take an imaginary value at $x = 1$, against the assumption.

EXAMPLE 3. It is well known that any two-dimensional real algebra with identity is isomorphic to one of the three following (see for instance Dickson [3]):

(a) The direct sum of the real line with itself. This algebra is of strongly real type, it has thus the vertex property (Theorem 4).

(b) The algebra of number pairs with the multiplication

$$(a,b)\cdot(c,d) = (ac, ad+bc) .$$

This algebra is obtained when an identity is adjoined to the radical algebra of real numbers with trivial multiplication. The Corollary of Theorem 3 now shows that this algebra has the vertex property.

(c) The complex numbers. Let $\|\cdot\|$ be a natural norm and $x_0 = \xi_0 + i\eta_0 \neq 0$ an element with

$$\|\exp(\alpha x_0)\| = \exp(\alpha \xi_0)\cdot\|\cos\alpha\eta_0 + i \sin\alpha\eta_0\| = 1 .$$

This implies $\xi_0 = 0$ and, since $\eta_0 \neq 0$, $\|\cos\varphi + i\sin\varphi\| = 1$, for every φ, that is $\|\xi + i\eta\| = \sqrt{\xi^2 + \eta^2}$. Hence the usual absolute value is the only natural norm for the complex numbers for which the unit sphere does not have a vertex at the identity.

Considering all three cases we have the conclusion: A natural unit sphere for a two-dimensional real Banach algebra with identity is either an ellipse or has a vertex at the identity.

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