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RECURSIVE ARITHMETIC OF SKOLEM

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Following the fundamental paper of Th. Skolem [5] which introduced recursive arithmetic, we extend the Skolem arithmetic up through the unique resolution theorem with respect to exponent chains of Mycielski numbers.—The author should like to acknowledge his thanks to L. Henkin and V. Vučković for their counsel.

We shall, of course, deal exclusively with the set of natural numbers $0,1,2,\ldots$. Throughout we employ the following notation: ^{non} (a modified Bourbaki notation for negation); $\wedge$ (conjunction); $\vee$ (disjunction); $\Leftrightarrow$ (equivalence); $\bigwedge$ (universal quantifier); $\bigvee$ (existential quantifier); μ (operation of minimalization).

Firstly, we state a list of definitions and properties which we shall need:

$$y \mid x \Leftrightarrow \bigvee z \leqq x\{x = yz \wedge z > 0\}; \tag{1}$$

$$\Pr(x) \Leftrightarrow x \geqq 2 \wedge \bigwedge y \leqq x\{y^{non}| \, x \vee y = 1 \vee y = x\}; \tag{2}$$

$$\begin{aligned} y &= \gcd(x_1, x_2, \ldots, x_n) \\ &\Leftrightarrow \bigwedge r \leqq n\{y \mid x_r \wedge \bigwedge z \leqq \min(x_1, \ldots, x_n)\{z^{non}|\, x_r \vee z \mid y\}\}; \end{aligned} \tag{3}$$

$$\bigvee r \leqq n\{y^{non}|\, zx_r\} \vee y \mid \gcd(zx_1, \ldots, zx_n); \tag{4}$$

$$\gcd(zx_1, \ldots, zx_n) = z\gcd(x_1, \ldots, x_n); \tag{5}$$

$$\begin{cases} \mathsf{E}(x,y;\,1,1) \Leftrightarrow x_1 = y_1\,, \\ \mathsf{E}(x,y;\,1,n) \text{ false for } n > 1\,. \\ \mathsf{E}(x,y;\,m,1) \text{ false for } m > 1\,, \\ \mathsf{E}(x,y;\,m+1,n+1) \Leftrightarrow (x_{m+1} = y_{n+1}) \wedge \mathsf{E}(x,y;\,m,n)\,. \end{cases} \tag{6}$$

Let $[x,y] = x^y$. Using the notation of W. Neumer [4], we define *exponent chains* $[x_n, x_{n-1}, \ldots, x_1]$ for $0 < x_k$, $1 \leqq k \leqq n$, as follows:

$$\begin{cases} [x_1] = x_1\,, \\ [x_{n+1}, x_n, \ldots, x_1] = [x_{n+1}, [x_n, \ldots, x_1]]\,. \end{cases} \tag{7}$$

We now proceed to define a class of natural numbers introduced by Jan Mycielski [3]:

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(8) $$y \mathbf{1} x \Leftrightarrow \bigvee z \leqq x\{x=[y,z] \wedge z>0 \wedge y>1\}$$

($y\mathbf{1}x$ means that x is a power of y);

(9) $$\mathrm{My}(x) \Leftrightarrow x \geqq 2 \wedge \bigwedge y \leqq x\{y \,{}^{non}\mathbf{1} x \vee y=x\}$$

($\mathrm{My}(x)$ means that x is a Mycielski number);

(10) $$\begin{cases} m_1 = 2 \\ m_{n+1} = \mu z \leqq [2,2,n+1]\{z>m_n \wedge \mathrm{My}(z)\} \end{cases}$$

(m_n is the nth Mycielski number).

The following array of lemmata lead to a proof of the unique resolution theorem with respect to exponent chains of Mycielski numbers.

First we simply state the following two obvious lemmata.

(11) LEMMA.

$$^{non}\left\{m = \prod_{1 \leq r \leq \mu} [q_r, \alpha_r] \wedge \bigwedge r \leqq \mu\{\mathrm{Pr}(q_r)\} \wedge \mathrm{My}\,(m)\right\} \vee \gcd(\alpha_1, \ldots, \alpha_\mu) = 1 .$$

(12) LEMMA.

$$^{non}\{[x,y]=[w,z] \wedge y \mid z\} \vee w \mathbf{1} x .$$

We prove the following crucial lemma.

(13) LEMMA.

$$^{non}\{m \mathbf{1}[x,y] \wedge \mathrm{My}\,(m)\} \vee m \mathbf{1} x .$$

PROOF. Let $[x,y]=[m,z]=\prod_{1\leq r\leq\mu}[q_r,\alpha_r]$, $x=\prod_{1\leq r\leq\mu}[q_r,\beta_r]$, $m=\prod_{1\leq r\leq\mu}[q_r,\gamma_r]$ and $\bigwedge r\leqq\mu\{\mathrm{Pr}(q_r)\}$. Clearly

$$\bigwedge r \leqq \mu\{\alpha_r=\beta_r y\} \wedge \bigwedge r \leqq \mu\{\alpha_r=\gamma_r z\} ,$$

so that it is easy to see that $\bigwedge r\leqq\mu\{y \mid \gamma_r z\}$. Hence, by virtue of (4), $y \mid \gcd(\gamma_1 z, \ldots, \gamma_\mu z)$, and furthermore on the basis of (5), (9) and Lemma (11) we have

$$\gcd(\gamma_1 z, \ldots, \gamma_\mu z) = z \gcd(\gamma_1, \ldots, \gamma_\mu) = z ,$$

which means that $y \mid z$. Therefore, on the strength of Lemma (12) it follows that $m\mathbf{1}x$.

The next two lemmata are easy to prove:

(14) LEMMA.

$$^{non}\{m \mathbf{1}[q_\mu, \ldots, q_1] \wedge \mathrm{My}(m)\} \vee m \mathbf{1} q_\mu .$$

(15) LEMMA.

$$^{non}\{m \mathbf{1}[q_\mu, \ldots, q_1] \wedge \bigwedge r \leqq \mu\{\mathrm{My}\,(q_r)\} \wedge \mathrm{My}\,(m)\} \vee q_\mu = m .$$

Finally, we prove our unique resolution theorem:

(16) THEOREM.

$$[q_\mu, q_{\mu-1}, \ldots, q_1]^{non} = [q_\nu{}', q_{\nu-1}{}', \ldots, q_1{}']$$
$$\vee \mathsf{V} r \leqq \mu\{{}^{non}\mathrm{My}(q_r)\} \vee \mathsf{V} s \leqq \nu\{{}^{non}\mathrm{My}(q_s{}')\} \vee \mathsf{E}(q,q';\mu,\nu).$$

PROOF. The proof is by induction. First we prove the theorem for $\mu = 1$. From

$$q_1 = [q_\nu{}', \ldots, q_1{}'] \wedge \mathrm{My}(q_1) \wedge \mathsf{\Lambda} s \leqq \nu\{\mathrm{My}(q_s{}')\},$$

on the grounds of Lemma (15) we obtain $q_1 = q_\nu{}'$. Since $q_1 = q_\nu{}'$ and $q_1 = [q_\nu{}', [q_{\nu-1}{}', \ldots, q_1{}']]$, it follows that $[q_{\nu-1}{}', \ldots, q_1{}'] = 1$, and so it is impossible for $\nu > 1$.

Let us assume that the theorem is true for some μ. Then from the assumption of the theorem we have

$$[q_{\mu+1}, \ldots, q_1] = [q_\nu{}', \ldots, q_1{}'] \wedge \mathsf{\Lambda} r \leqq \mu+1\{\mathrm{My}(q_r)\} \wedge \mathsf{\Lambda} s \leqq \nu\{\mathrm{My}(q_s{}')\},$$

and consequently

$$q_{\mu+1} \mathbin{1} [q_\nu{}', \ldots, q_1{}'] \wedge \mathsf{\Lambda} s \leqq \nu\{\mathrm{My}(q_s{}')\} \wedge \mathrm{My}(q_{\mu+1}),$$

from which it follows that $q_{\mu+1} = q_\nu{}'$ by virtue of Lemma (15). Furthermore, since $[q_\nu{}', q_{\nu-1}{}', \ldots, q_1{}'] = [q_\nu{}', [q_{\nu-1}{}', \ldots, q_1{}']]$, we have $[q_\mu, \ldots, q_1] = [q_{\nu-1}{}', \ldots, q_1{}']$, from which, by applying the inductive hypothesis, we obtain

$$(q_{\mu+1} = q_\nu{}') \wedge \mathsf{E}(q,q';\mu,\nu-1) \Leftrightarrow \mathsf{E}(q,q';\mu+1,\nu).$$

In conclusion, we should like to remark that it is easy to see that the class of consecutive prime numbers is in fact a subclass of the class of consecutive Mycielski numbers. It is equally evident that the class of Mycielski numbers can be successively extended in the following way.

Modifying somewhat the so-called Hilbert–Ackermann class of primitive recursive functions [2,1], we introduce the class of primitive recursive functions $\xi_1(x,y) = [x,y]$, $\xi_2(x,y) = [_y x]$, where $[_0 x] = 1$ and $[_{y+1}x] = [x,[_y x]]$ and such that each successive primitive recursive function is defined by the following primitive recursive scheme:

$$\begin{cases} \xi_{k+1}(x,0) = 1, \\ \xi_{k+1}(x,y+1) = \xi_k(x,\xi_{k+1}(x,y)). \end{cases} \tag{17}$$

Using the above class of Hilbert–Ackermann functions, we can easily define the following class of relations ($k = 1, 2, \ldots$):

$$\mathsf{R}_k(x,y) \Leftrightarrow \mathsf{V} z \leqq x\{x = \xi_k(y,z) \wedge z > 0 \wedge y > 1\}, \tag{18}$$

(19) $$\mathsf{M}_k(x) \Leftrightarrow x \geqq 2 \wedge \bigwedge y \leqq x\{{}^{non}\mathsf{R}_k(x,y) \vee x=y\}\,,$$

where $\mathsf{R}_1(x,y)$ is the relation $y\,\mathbf{1}\,x$ and $\mathsf{M}_1(x)$ is the relation $\mathrm{My}(x)$, and so on. Finally, we define the recursively definable class of natural numbers we mentioned above ($k=1,2,\ldots$):

(20) $$\begin{cases} m_1^{(k)} = 2\,, \\ m_{n+1}^{(k)} = \mu z \leqq [2,2,n+1]\{z > m_n^{(k)} \wedge \mathsf{M}_k(z)\}\,, \end{cases}$$

where $m_n^{(1)}$ is the nth Mycielski number and so on.

So, evidently, Skolem's recursive arithmetic can be even more considerably extended in the way we have outlined in this paper following the methods of Th. Skolem.

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