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HOMOGENEOUS UNIVERSAL MODELS

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In two recent papers [8], [9], B. Jónsson has established sufficient conditions for the existence and uniqueness up to isomorphism of relational systems of power $\aleph_\alpha$ which are homogeneous and universal relative to a given class $\mathscr{M}$ of relational systems. (Notions mentioned in the introduction are defined precisely in Sections 1–3). Fraïssé [6] had discussed related questions for $\alpha = 0$.

We shall show in Section 3 that the conditions of Jónsson are met by the class $\mathscr{M}$ of models of an arbitrary complete theory, or, rather, by a certain variant of $\mathscr{M}$. It follows that (1) assuming the GCH (Generalized Continuum Hypothesis), a complete theory has up to isomorphism exactly one homogeneous universal model in each power $\aleph_{\alpha+1}$. In a similar way, we also establish (2) a number of related results concerning the existence or uniqueness in arbitrary powers of models satisfying various conditions weaker than homogeneous-universal.

One consequence of (1) is a considerable improvement (Theorem 4.1) of a theorem of [8], which dealt with the problem of the existence of universal systems in powers $> \aleph_0$ for general classes $\mathscr{M}$. Our improvement is subject to the condition that $\mathscr{M}$ be an EC_Δ-class. But this condition is met, for example, by the class of distributive lattices and the class of demigroups, for both of which the problem was left open in [8].

On the other hand, the principal value of (1) and (2) seems to be for the study of models of complete theories itself. It will be seen in 3.6, 3.7 and 3.8 (which is due to H. J. Keisler) that (1) and (2) stand in an interesting relationship to recent results of model theory due to A. Robinson, E. Specker, and R. Lyndon. In Section 5, some results concerning theories categorical in power will be inferred from (1) and (2). Finally, by means of some of the results (2) (one of which is due to W. Craig), we establish in 6.2 what may be called a "Löwenheim-Skolem theorem for two cardinals".

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The applications to model theory have called our attention to various peripheral results obtainable by variants of Jónsson's and Fraïssés arguments. These deal with questions such as (2) above, the situation when there are as many relations as elements, and the problem of avoiding the GCH when possible. For this reason we shall recapitulate much of Jónsson's (and Fraïssés) work in Section 2, paying attention to these various matters. For the most part the improvements obtained are rather obvious from a careful reading of [8], [9] and [6]. However, in some cases, such as Theorem 2.10 or the notion "special" and results concerning it, an essential modification of old arguments is needed.[2]

1. Preliminaries.

A^B is the set of all functions on B into A. $\bar{\bar{A}}$ is the power or cardinal of A. Id_A is the identity function on A. Each ordinal is the set of all its predecessors. The letters m, n (possibly with primes) denote natural numbers, i.e., members of ω. $\alpha, \beta, \gamma, \xi, \eta, \zeta$ denote ordinals, δ *being reserved for* (non-zero) *limit ordinals.* μ and λ denote arbitrary cardinals (initial ordinals), while $\varkappa$ *denotes an infinite cardinal.* $Cf\varkappa$, the character of confinality of $\varkappa$, is the least λ such that $\varkappa$ can be expressed as $\sum(\mu_\xi/\xi<\lambda)$, where each $\mu_\xi<\varkappa$. $\varkappa$ is regular if $cf\varkappa=\varkappa$; otherwise, $\varkappa$ is singular. $\varkappa^+$ is the least $\lambda>\varkappa$. $\varkappa^{\underline{\lambda}}=\sum(\varkappa^\mu/\mu<\lambda)$ and $\varkappa^*=\sum(2^\lambda/\lambda<\varkappa)$. The Beths $\beth_\xi$ are defined by the conditions: $\beth_0=\omega$ and, if $\xi\neq 0$, $\beth_\xi=\sum(2^{\beth_\eta}/\eta<\xi)$.

The following statements are well-known or easily verified (cf., e.g., [1]).

Lemma 1.1. (a) *If the GCH holds then* $\varkappa=\varkappa^*$.

(b) $\varkappa^{*\underline{cf\varkappa}}=\varkappa^*$.

(c) $\varkappa^{\underline{\varkappa}}=\varkappa$ *if and only if* $\varkappa=\varkappa^*$ *and* $\varkappa$ *is regular.*

(d) *If* $\bar{\bar{A}}\leq\varkappa$, *then there are at most* $\varkappa^{\underline{\lambda}}$ *functions* f *into* A *such that the domain of* f *is a subset of* A *of power less than* λ.

(e) $\varkappa^{cf\varkappa}>\varkappa$.

(f) $\varkappa=\varkappa^*$ *if and only if* $\varkappa=\varkappa^{\underline{\varkappa}}$ *or* $\varkappa$ *is of the form* $\beth_\delta$.

(g) $(2^\varkappa)^{\underline{\varkappa^+}}=2^\varkappa$.

Let I be an arbitrary set and $t\in\omega^I$. A system $\mathfrak{A}=\langle A,R_i\rangle_{i\in I}$, formed by a non-empty set A and t_i-ary relations R_i over (i.e., among the elements of) A, is a *relational system*, having *similarity type* $t^{\mathfrak{A}}=t$, *index set* $I_{\mathfrak{A}}=I$, *universe* $|\mathfrak{A}|=A$, *i*-th *relation* $R_i^{\mathfrak{A}}=R_i$ (for $i\in I$), and *power*

[2] Some of the results presented here were announced in [23] by the second author, who then learned that closely related, overlapping results had been found in 1955 by the first author.

$\overline{\overline{\mathfrak{A}}}=\overline{\overline{A}}$. The letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ will always denote relational systems. We write $\mathfrak{A}\cong_f\mathfrak{B}$ to mean that f is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ (to be understood in the obvious way).

If $0\neq X\subseteq|\mathfrak{A}|$, we denote by $\mathfrak{A}|X$ the subsystem $\mathfrak{B}$ of $\mathfrak{A}$ with universe X, i.e., the system $\langle X,S_i\rangle_{i\in I_{\mathfrak{A}}}$ such that $S_ix_0\ldots x_{n-1}$ if and only if $R_i^{\mathfrak{A}}x_0\ldots x_{n-1}$, whenever $i\in I_{\mathfrak{A}}$, $n=t_i^{\mathfrak{A}}$, and $x_0,\ldots,x_{n-1}\in X$. Under the same conditions we write $\mathfrak{B}\subseteq\mathfrak{A}$. If, moreover, the set (=singulary relation) X coincides with $R_i^{\mathfrak{A}}$, for some $i\in I_{\mathfrak{A}}$, then we call $\mathfrak{B}$ a relativization of $\mathfrak{A}$.

$\langle\mathfrak{A}_\xi/\xi<\delta\rangle$ is a *chain* if $\mathfrak{A}_\xi\subseteq\mathfrak{A}_\eta$ whenever $\xi<\eta<\delta$. The *union* $\bigcup(\mathfrak{A}_\xi/\xi<\delta)$ is the system

$$\langle\bigcup(|\mathfrak{A}_\xi|/\xi<\delta),\ \bigcup(R_i^{\mathfrak{A}_\xi}/\xi<\delta)\rangle_{i\in I_{\mathfrak{A}_0}}.$$

The *J-reduct* $\mathfrak{A}\rceil J$ of $\mathfrak{A}$ is the system $\langle\mathfrak{A},R_i^{\mathfrak{A}}\rangle_{i\in I_{\mathfrak{A}}\cap J}$. On the other hand, if, for $k\in K$, S_k is a finitary relation over $|\mathfrak{A}|$, then we denote by $(\mathfrak{A},S_k)_{k\in K}$ the system $\langle|\mathfrak{A}|,R_i\rangle_{i\in I_{\mathfrak{A}}\cup K}$, where $R_i=R_i^{\mathfrak{A}}$, if $i\in I_{\mathfrak{A}}$, and $R_i=S_i$, if $i\in K$. It is understood that K is to be replaced by another index set (in some standard way), if necessary, so that $I_{\mathfrak{A}}\cap K=0$.

Let $\mathscr{M}$ be a class of relational systems. $\mathscr{I}(\mathscr{M})$ is the class of all isomorphs of members of $\mathscr{M}$. $\mathscr{M}\rceil J=\{\mathfrak{A}\rceil J/\mathfrak{A}\in\mathscr{M}\}$, $\mathscr{M}_\varkappa=\{\mathfrak{A}\in\mathscr{M}/\overline{\overline{\mathfrak{A}}}<\varkappa\}$, $\mathscr{S}(\mathfrak{A})=\{\mathfrak{B}/\mathfrak{B}\subseteq\mathfrak{A}\}$ and $\mathscr{S}(\mathscr{M})=\bigcup\{\mathscr{S}(\mathfrak{A})/\mathfrak{A}\in\mathscr{M}\}$. We put $\mathscr{S}_\varkappa^{\mathscr{M}}(\mathfrak{A})=\{\mathfrak{B}\subseteq\mathfrak{A}/\mathfrak{B}\in\mathscr{M}$ and $\mathfrak{B}<\varkappa\}$; $\mathscr{S}^{\mathscr{M}}(\mathfrak{A})$ and $\mathscr{S}_\varkappa(\mathfrak{A})$ are to be understood analogously. For a set X we also write $\mathscr{S}(X)=\{Y/Y\subseteq X\}$, etc.

In some auxiliary constructions we shall want to deal with systems of the more general form $\langle A,R_i,d_k\rangle_{i\in I,k\in K}$, where $\mathfrak{A}=\langle A,R_i\rangle_{i\in I}$ and each $d_k\in A$. We denote such a system by $\mathfrak{D}=(\mathfrak{A},d_k)_{k\in K}$; its similarity type is $\langle t^{\mathfrak{A}},K\rangle$; and $d_k^{\mathfrak{D}}=d_k$ for $k\in K$. All the notions above extend in an obvious way to such systems. ($\mathfrak{D}\subseteq\mathfrak{D}'$ requires $d^{\mathfrak{D}}=d^{\mathfrak{D}'}$).

We turn now to metamathematical preliminaries (which are not needed until Section 3). Suppose $t\in\omega^J$ and $t'=\langle t,K\rangle$ ($=t$ if $K=0$). Corresponding to t' is a first order language $L_{t'}$, whose (distinct) symbols are $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, $\approx$, the (individual) variables $v_0,\ldots,v_n,\ldots$, the t_i-placed relation symbols P_{i,t_i} for $i\in I$, and the individual constants c_k, for $k\in K$. A (wellformed) formula φ of $L_{t'}$ is *open* if no quantifiers occur in it; φ is *universal* if it is of the form $\forall v_m\ldots\forall v_n\theta$, where θ is an open formula; φ is a *sentence* if it has no free variables. We write $\vdash_{\mathfrak{D}}\varphi[x_0,\ldots,x_{n-1}]$, to mean that ϕ has at most the free variables $v_0,\ldots,v_{n-1}$, $t^{\mathfrak{D}}=t'$, and $x_0,\ldots x_{n-1}$ are members of $|\mathfrak{D}|$ which (in the usual sense) satisfy φ in $\mathfrak{D}$, when x_m is assigned to v_m for each $m<n$. $\mathfrak{D}$ is a *model* of a sentence σ (or a set Σ of sentences) if σ (or each member of Σ) is satisfied in $\mathfrak{D}$. The *theory* of $\mathfrak{D}$, Th$\mathfrak{D}$, is the set of all sentences

of which $\mathfrak{D}$ is a model. The *diagram* $D(\mathfrak{D})$ is the set of all open sentences of which the system $(\mathfrak{D},a)_{a\in|\mathfrak{D}|}$ is a model.

$\mathfrak{D}$ and $\mathfrak{D}'$ are elementarily equivalent, written $\mathfrak{D}\equiv\mathfrak{D}'$, if $\mathrm{Th}\,\mathfrak{D}=\mathrm{Th}\,\mathfrak{D}'$. $\mathfrak{D}$ is an elementary subsystem of $\mathfrak{D}'$ (written $\mathfrak{D}\prec\mathfrak{D}'$) if $\mathfrak{D}\subseteq\mathfrak{D}'$ and $(\mathfrak{D},a)_{a\in|\mathfrak{D}|}\equiv(\mathfrak{D}',a)_{a\in|\mathfrak{D}|}$. If for some similarity type t and some set Σ of sentences (or universal sentences) of L_t, $\mathscr{M}$ is the class of all models of Σ, then we write $\mathscr{M}\in \boldsymbol{EC}_\Delta$ (or $\mathscr{M}\in\boldsymbol{UC}_\Delta$). We say that $\mathscr{M}$ is an elementary type ($\mathscr{M}\in\boldsymbol{ET}$) if, for some *infinite* $\mathfrak{A}$, $\mathscr{M}=\{\mathfrak{B}/\mathfrak{B}\equiv\mathfrak{A}\}$. Clearly $\mathscr{M}\in\boldsymbol{ET}$ implies $\mathscr{M}\in\boldsymbol{EC}_\Delta$ and $\mathscr{M}\in\boldsymbol{EC}_\Delta$ implies $\mathscr{I}(\mathscr{M})=\mathscr{M}$.

LEMMA 1.2. (a) *If $\mathfrak{D}\equiv\mathfrak{D}'$, then for some $\mathfrak{D}''\equiv\mathfrak{D}$ we have $\mathfrak{D},\mathfrak{D}'\in\mathscr{I}\mathscr{S}(\mathfrak{D}'')$.*

(b) *Suppose $\mathfrak{D}\equiv\mathfrak{D}'$; $x_k\in|\mathfrak{D}|$, for each $k\in K$; and S_k is a finitary relation over $|\mathfrak{D}'|$, for each $k\in K'$. Then there is a system $\mathfrak{D}''\equiv\mathfrak{D}$ such that, for some $y_k(k\in K)$ and $T_k(k\in K')$, $(\mathfrak{D},x_k)_{k\in K}\equiv(\mathfrak{D}'',y_k)_{k\in K}$ and $(\mathfrak{D}',S_k)_{k\in K'}\equiv(\mathfrak{D}'',T_k)_{k\in K'}$.*[3]

PROOF. By considering $D(\mathfrak{D})$ and $D(\mathfrak{D}')$, (a) becomes a special case of (b). In view of the completeness theorem, it suffices for (b) to show that any finite subset of $\mathrm{Th}(\mathfrak{D},x_k)_{k\in K}\cup\mathrm{Th}(\mathfrak{D}',S_k)_{k\in K'}$ is consistent. (We can assume $K\cap K'=0$.) This easily follows from the observation that if $x_{k_0},\ldots,x_{k_{n-1}}$ satisfy φ in $\mathfrak{D}$, then $\mathfrak{D}$ and hence also $\mathfrak{D}'$ are models of $\exists v_0\ldots\exists v_{n-1}\varphi$.

An elementary type $\mathscr{M}$ is called *$\varkappa$-categorical* if $\mathfrak{A},\mathfrak{B}\in\mathscr{M}$ and $\overline{\overline{\mathfrak{A}}}=\overline{\overline{\mathfrak{B}}}=\varkappa$ implies $\mathfrak{A}\cong\mathfrak{B}$; *model-complete* if $\mathfrak{A},\mathfrak{B}\in\mathscr{M}$ and $\mathfrak{A}\subseteq\mathfrak{B}$ implies $\mathfrak{A}\prec\mathfrak{B}$. (Cf. [13], [15].)

2. Classes fulfilling Jónsson's conditions.

Henceforth *we denote by $\mathscr{M}$ a class of similar relational systems containing members of arbitrarily large powers and such that $\mathscr{I}(\mathscr{M})=\mathscr{M}$* (cf. conditions (I) and (II) of [9]). $I_{\mathscr{M}}$ is the index set for members of $\mathscr{M}$. We allow $\overline{\overline{I}}_{\mathscr{M}}$ to be arbitrary.[4] (In most applications one has $\overline{\overline{I}}_{\mathscr{M}}\leq\omega$; however, that assumption would yield very little simplification.) Consider now the following conditions on $\mathscr{M}$:

(III) *If $\mathfrak{A},\mathfrak{B}\in\mathscr{M}$, then there exists $\mathfrak{C}\in\mathscr{M}$ with $\mathfrak{A},\mathfrak{B}\in\mathscr{I}\mathscr{S}(\mathfrak{C})$.*

[3] 1.2 (a) is due to A. Robinson [15, Theorem 4.2.2) and R. Fraïssé [6]; 1.2 (b) to A. Robinson [16].

[4] In [6] it is assumed that $\overline{\overline{I}}_{\mathscr{M}}<\omega$ and in [8], [9], [23] that $\overline{\overline{I}}_{\mathscr{M}}\leq\omega$. C. C. Chang found that by a device involving ultraproducts he could infer from 4.1, below, for $\overline{\overline{I}}_{\mathscr{M}}\leq\omega$ its own extension to arbitrary $\overline{\overline{I}}_{\mathscr{M}}$. Later, the authors saw that the proofs in [8], [9], [23] can all be extended to arbitrary $I_{\mathscr{M}}$; for elementary types (cf. Section 3) Keisler had noted the same possibility.

(IV) *If* $\mathfrak{A}, \mathfrak{B} \in \mathscr{M}$, $\mathfrak{A} \cong_f \mathfrak{A}'$ *and* $\mathfrak{A}, \mathfrak{A}' \subseteq \mathfrak{B}$, *then there exist* $\mathfrak{C} \in \mathscr{M}$ *such that* $\mathfrak{C} \supseteq \mathfrak{B}$ *and there is an isomorphism* $g \supseteq f$ *of* $\mathfrak{B}$ *into* $\mathfrak{C}$.

(V) *The union of any chain of members of* $\mathscr{M}$ *belongs to* $\mathscr{M}$.

($\mathrm{VI}_\varkappa$) *If* $\mathfrak{A} \in \mathscr{M}$ *and* $X \in \mathscr{S}_\varkappa(|\mathfrak{A}|)$, *then* $X \subseteq |\mathfrak{B}|$ *for some* $\mathfrak{B} \in \mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A})$.

These conditions (and their numbering) are from [9], except that IV is slightly weaker than either IV or IV′ in [9]. However, the conjunction of our III and IV is obviously equivalent to the conjunction of III and IV′ of [9].

Various familiar classes were shown in [8] to have these properties. For example, the class of all groups was shown to have III, IV, V, and $\mathrm{VI}_\varkappa$ for $\varkappa > \omega$. We mention here one other example, namely, the class of all metric spaces. From each metric space $\langle X, d\rangle$ we can obtain a relational system $\langle X, R_r\rangle_{r \in Rt}$, where Rt is the set of rationals and $R_r xy$ if and only if $d(x,y) < r$. If $\mathscr{M}$ is the class of all systems thus obtained, then $\mathscr{M}$ obviously has V and VI_ω. That $\mathscr{M}$ has III and IV is in essence shown in Sierpínski [17]. By a construction related to the proof of (4) below, Sierpinski established from the GCH the existence of universal metric spaces in each power $> \omega$. From 2.8 below we can conclude that in fact there is up to isometry exactly one special (cf. 2.2) metric space in each power $> \omega$, assuming the GCH. This example differs from any of those discussed in [8] in the fact that $\mathscr{M} \notin \boldsymbol{EC}_\Delta$.

The following Lemmas 2.1 (a), (b) are proved in [8, Lemmas 2.6, 2.5].

LEMMA 2.1. (a) *If* $\mathscr{M}$ *has* V *and* $\mathrm{VI}_\varkappa$, *then* $\mathscr{M}$ *has* VI_λ *for every* $\lambda > \varkappa$.

(b) *If* $\mathscr{M}$ *has* $\mathrm{VI}_\varkappa$, $\mathfrak{A} \in \mathscr{M}$, *and* $\overline{\overline{\mathfrak{A}}} = \varkappa$, *then* $\mathfrak{A}$ *is the union of some chain of members of* $\mathscr{M}_\varkappa$.

2.1. (a) will be used henceforth without explicit reference.

DEFINITION 2.2. *Let* $\overline{\overline{\mathfrak{A}}} = \varkappa$. (a) $\mathfrak{A}$ *is* $\mathscr{M}$*-homogeneous* (*of degree* λ) *if* $\mathfrak{A} \in \mathscr{M}$ *and any isomorphism of a member of* $\mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A})$ (*of* $\mathscr{S}_\lambda{}^{\mathscr{M}}(\mathfrak{A})$) *into* $\mathfrak{A}$ *can be extended to an automorphism of* $\mathfrak{A}$.

(b) $\mathfrak{A}$ *is* $\mathscr{M}$*-universal if* $\mathfrak{A} \in \mathscr{M}$ *and* $\mathscr{M}_{\varkappa+} \subseteq \mathscr{I}\mathscr{S}(\mathfrak{A})$.

(c) $\mathfrak{A}$ *is* $\mathscr{M}, \mathscr{Q}$*-homogeneous if* $\mathfrak{A} \in \mathscr{M}$, $\mathscr{Q} \subseteq \mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A})$, *and the following conditions hold*:

(i) $\mathscr{S}(\mathscr{Q}) \cap \mathscr{M} \subseteq \mathscr{Q}$.

(ii) $\mathscr{Q}$ *is* $cf\varkappa$-directed, *i.e.*, *whenever* $\mathscr{Q}' \subseteq \mathscr{Q}$ *and* $\overline{\overline{\mathscr{Q}'}} < cf\varkappa$, *there exist* $\mathfrak{B} \in \mathscr{Q}$ *with* $\mathscr{Q}' \subseteq \mathscr{S}(\mathfrak{B})$.

(iii) $\mathfrak{A}$ *is the union of some chain of members of* $\mathscr{Q}$; *and*

(iv) *whenever* $\mathfrak{B}_1 \subseteq \mathfrak{C}_1 \in \mathscr{Q}$, $\mathfrak{B}_2 \in \mathscr{Q}$, *and* $\mathfrak{B}_1 \cong_f \mathfrak{B}_2$, *there exist* $f' \supseteq f$ *and* $\mathfrak{C}_2 \in \mathscr{Q}$ *such that* $\mathfrak{C}_1 \cong_{f'} \mathfrak{C}_2$.

(d) $\mathfrak{A}$ *is* $\mathscr{M}$*-special if, for some* $\mathscr{Q}$, $\mathfrak{A}$ *is* $\mathscr{M}, \mathscr{Q}$*-homogeneous and* $\mathscr{M}_\varkappa \subseteq \mathscr{I}(\mathscr{Q})$

If $\mathfrak{A}$ is $\mathscr{M}$-homogeneous and $\mathscr{M}$-universal, we will say simply that $\mathfrak{A}$ is $\mathscr{M}$-homogeneous-universal. The following remarks clarify the relationship between the new notions (c) and (d) and the notions, $\mathscr{M}$-homogeneous and $\mathscr{M}$-homogeneous-universal, respectively (which are from [9]). Suppose $\mathscr{M}$ has V and $\mathrm{VI}_\varkappa$, $\mathfrak{A} \in \mathscr{M}$, and $\overline{\overline{\mathfrak{A}}} = \varkappa$, and put $\mathscr{Q} = \mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A})$. Conditions (i) and (ii) obviously hold, and (iii) holds by 2.1 (b). If $\mathfrak{A}$ is $\mathscr{M}$-homogeneous, then clearly also (iv) holds; if, moreover, $\mathfrak{A}$ is $\mathscr{M}$-universal, then obviously $\mathfrak{A}$ is $\mathscr{M}$-special. From 2.4 (a), (d), proved below, we see that two converse statements are also valid. Thus the notions (c) and (d) are direct generalizations of (a) and (b). Moreover, if $\mathfrak{A}$ is $\mathscr{M},\mathscr{Q}'$-homogeneous, then, by (i) and (iii), $\mathscr{S}_{cf\varkappa}{}^{\mathscr{M}}(\mathfrak{A}) \subseteq \mathscr{Q}'$; and hence, if $\varkappa$ is regular, $\mathscr{Q}' = \mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A})$. Thus the new notions differ from the old only when $\varkappa$ is singular.

2.3 (a), (b) (and also 2.5 (a), (b)) below are generalizations of Cantor's two famous arguments concerning the order of the rational numbers.

THEOREM 2.3. *Suppose* $\mathscr{M}$ *has* V *and* $\mathrm{VI}_\varkappa$, $\mathfrak{A}'$ *is* $\mathscr{M},\mathscr{Q}'$*-homogeneous and* $\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{A}'}} = \varkappa$.

(a) *If* $\mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A}) \subseteq \mathscr{I}(\mathscr{Q}')$ *then* $\mathfrak{A} \in \mathscr{I}\mathscr{S}(\mathfrak{A}')$.

(b) *If* $\mathfrak{A}$ *is* $\mathscr{M},\mathscr{Q}$*-homogeneous,* $\mathscr{I}(\mathscr{Q}) = \mathscr{I}(\mathscr{Q}')$, $\mathfrak{A}_0 \in \mathscr{Q}$, $\mathfrak{A}_0' \in \mathscr{Q}'$, *and* $\mathfrak{A}_0 \cong_{f_0} \mathfrak{A}_0'$, *then, for some* $f \supseteq f_0$, $\mathfrak{A} \cong_f \mathfrak{A}'$.

PROOF. (a) By 2.1 (b) we can obviously express $\mathfrak{A}$ as the union of a chain $\langle \mathfrak{A}_\xi / \xi < cf\varkappa \rangle$ of members of $\mathscr{S}_\varkappa(A)$. By hypothesis there is an isomorphism f_0 of $\mathfrak{A}_0$ onto a member of $\mathscr{Q}'$. By (transfinite) recursion we shall define isomorphisms f_ξ of $\mathfrak{A}_\xi$ onto members of $\mathscr{Q}'$ in such a way that $f_\xi \subseteq f_\eta$, for $\xi < \eta \leq cf\varkappa$. By 2.2 (ii) (iii), we can put $f_\delta = \bigcup(f_\eta / \eta < \delta)$. Let $f_{\xi+1}$ be the "first" (say, in a fixed well-ordering of $\mathscr{S}\mathscr{S}\mathscr{S}(A \cup A') \cup \mathscr{Q} \cup \mathscr{Q}'$) isomorphism $g \supseteq f_\xi$ of $\mathfrak{A}_{\xi+1}$ onto a member of $\mathscr{Q}'$. To find such a g we can obtain, by hypothesis an isomorphism h of $\mathfrak{A}_{\xi+1}$ onto a member $\mathfrak{C}$ of $\mathscr{Q}'$. Then we apply 2.2 (iv), with $f_\xi \circ h^{-1}$ for "f" and $\mathfrak{C}$ for "$\mathfrak{C}_1$", to obtain f', and take $g = f' \circ h$. Clearly $f_{cf\varkappa}$ is the desired isomorphism of $\mathfrak{A}$ into $\mathfrak{A}'$. (Note that for (a) we needed neither 2.2 (iii) nor the assumption $\overline{\overline{\mathfrak{A}'}} = \varkappa$.)

(b) By 2.2 (iii) we can express $\mathfrak{A}$ and $\mathfrak{A}'$ as unions of chains $\langle \mathfrak{B}_\xi / \xi < cf\varkappa \rangle$ and $\langle \mathfrak{B}_\xi' / \xi < cf\varkappa \rangle$ of members of $\mathscr{Q}$ and $\mathscr{Q}'$, respectively. By recursion we shall define $\mathfrak{A}_\xi, \mathfrak{A}_\xi', f_\xi$ for $1 \leq \xi < cf\varkappa$ in such a way that $\mathfrak{A}_\xi \in \mathscr{Q}$, $\mathfrak{A}_\xi' \in \mathscr{Q}'$ and $\mathfrak{A}_\xi \cong_{f_\xi} \mathfrak{A}_\xi'$. Put $f_\delta = \bigcup(f_\eta / \eta < \delta)$ and similarly for $\mathfrak{A}_\delta$ and $\mathfrak{A}_\delta'$. If ξ is even, then by 2.2 (ii) we can let $\mathfrak{A}_{\xi+1}$ be the "first" member of $\mathscr{Q}$ such that $\mathfrak{A}_\xi, \mathfrak{B}_\xi \subseteq \mathfrak{A}_{\xi+1}$. Exactly as in the proof of (a) we can take for $f_{\xi+1}$ the "first" isomorphism of $\mathfrak{A}_{\xi+1}$ onto a member of $\mathscr{Q}$ extending f_ξ; and for $\mathfrak{A}_{\xi+1}'$ we take the image of $f_{\xi+1}$. When ξ is odd we work instead from $\mathfrak{A}'$ toward $\mathfrak{A}$, ensuring that $\mathfrak{B}_\xi' \subseteq \mathfrak{A}_{\xi+1}'$. Then clearly $f = \bigcup(f_\eta / \eta < cf\varkappa)$ is as desired.

From 2.3 (and the remarks after 2.2) follows at once:

COROLLARY 2.4. *Suppose $\mathscr{M}$ has* V *and* $\mathrm{VI}_\varkappa$, $\mathfrak{A} \in \mathscr{M}$, *and* $\overline{\overline{\mathfrak{A}}} = \varkappa$.

(a) *If $\mathfrak{A}$ is $\mathscr{M}$-special, then $\mathfrak{A}$ is $\mathscr{M}$-universal.*

(b) *Any two $\mathscr{M}$-special (or, a fortiori, $\mathscr{M}$-homogenous-universal) systems of the same power are isomorphic.*

(c) *If $\mathfrak{A}$ is $\mathscr{M}, \mathscr{Q}$-homogeneous, then $\mathfrak{A}$ is $\mathscr{M}$-homogeneous of degree $cf\varkappa$.*

(d) *A sufficient (and necessary) condition for $\mathfrak{A}$ to be $\mathscr{M}$-homogeneous is that: whenever $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathscr{S}_\varkappa^{\mathscr{M}}(\mathfrak{A})$ and $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, then any isomorphism of $\mathfrak{B}_1$ into $\mathfrak{A}$ can be extended to one of $\mathfrak{B}_2$ into $\mathfrak{A}$.*

(e) *A sufficient (and necessary) condition for $\mathfrak{A}$ to be $\mathscr{M}$-homogeneous-universal is that: whenever $\mathfrak{B} \in \mathscr{M}_\varkappa$, $X \subseteq |\mathfrak{B}|$, and either $X = 0$ or $\mathfrak{B} \mid X \in \mathscr{S}^{\mathscr{M}}(\mathfrak{A})$, then there exists an isomorphism $f \supseteq Id_X$ of $\mathfrak{B}$ into $\mathfrak{A}$.*[5]

The next theorem, 2.5, asserts that when $\mathscr{S}(\mathscr{M}) \subseteq \mathscr{M}$ and the two systems have the same cardinality one can replace certain subsystems by single elements in 2.4 (d), (e). (However, this does not seem to apply to the notions involving $\mathscr{Q}$.) That 2.5 holds for $\varkappa > \omega$ was brought to our attention by Keisler.

THEOREM 2.5. *Suppose $\mathscr{S}(\mathscr{M}) \subseteq \mathscr{M}$, $\mathfrak{A} \in \mathscr{M}$ and $\overline{\overline{\mathfrak{A}}} = \varkappa$.*

(a) *$\mathfrak{A}$ is $\mathscr{M}$-homogeneous if (and only if), whenever $0 \neq X \in \mathscr{S}_\varkappa(|\mathfrak{A}|)$ and $a \in |\mathfrak{A}|$, any isomorphism of $\mathfrak{A} \mid X$ into $\mathfrak{A}$ can be extended to one of $\mathfrak{A} \mid (X \cup \{a\})$ into $\mathfrak{A}$.*

(b) *$\mathfrak{A}$ is $\mathscr{M}$-homogeneous-universal if (and only if) whenever $X \in \mathscr{S}(|\mathfrak{A}|)$, $X \cup \{c\} = |\mathfrak{C}|$, $\mathfrak{C} \in \mathscr{M}$, and either $X = 0$ or $\mathfrak{A} \mid X \subseteq \mathfrak{C}$, then there exists an isomorphism $f \supseteq Id_X$ of $\mathfrak{C}$ into $\mathfrak{A}$.*

The proofs are similar to those above, but proved "an element at a time".

THEOREM 2.6. *Suppose $\mathscr{M}$ has* IV, V, *and* $\mathrm{VI}_\varkappa$, $\mathfrak{A}_0 \in \mathscr{M}$, *and* $\overline{\overline{\mathfrak{A}}}_0 = \varkappa = \varkappa^*$.

(a) *If $\mathscr{Q}_0 \subseteq \mathscr{S}_\varkappa^{\mathscr{M}}(\mathfrak{A}_0)$ and $\overline{\overline{\mathscr{Q}}}_0 \leq \varkappa$, then there exist $\mathfrak{A} \supseteq \mathfrak{A}_0$ and $\mathscr{Q} \supseteq \mathscr{Q}_0$ such that $\mathfrak{A}$ is $\mathscr{M}, \mathscr{Q}$-homogeneous and $\overline{\overline{\mathfrak{A}}} = \varkappa$.*

(b) *If $\varkappa$ is regular, then there is an $\mathscr{M}$-homogeneous system $\mathfrak{A} \supseteq \mathfrak{A}_0$ such that $\overline{\overline{\mathfrak{A}}} = \varkappa$.*

PROOF. (b) follows from (a), by 2.4 (c). Alternatively a short direct

[5] 2.4 (e) and 2.4 (b) for $\mathscr{M}$-homogeneous-universal are due to Jónsson [8], [9]. For the case when $\mathscr{S}(\mathscr{M}) \subseteq \mathscr{M}$, $\bar{I}_{\mathfrak{A}} < \omega = \varkappa$, $\mathscr{Q} = \mathscr{S}_\omega(\mathfrak{A})$, and $\mathscr{Q}' = \mathscr{S}_\omega(\mathfrak{A}')$, 2.3, 2.4, and 2.5 (a), below, are in Fraïssé [6]. The special case of 2.3 (b) stated later in 6.1 (b) and a related special case of 2.3 (a) are due to Craig [3].

proof of (b) is obtained by simplifying in an obvious way the proof of (a) which we now give.

We first show that there exists a system $\mathfrak{A}_1$, such that

$$(3)\quad \begin{cases} \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \in \mathscr{M}_{\varkappa^+} \textit{ and, whenever } \mathfrak{B} \subseteq \mathfrak{C} \in \mathscr{Q}_0,\ \mathfrak{B}' \in \mathscr{Q}_0, \textit{ and } \mathfrak{B} \cong_f \mathfrak{B}'. \\ \textit{there exist } g \supseteq f \textit{ and } \mathfrak{C}' \subseteq A_1 \textit{ such that } \mathfrak{C} \cong_g \mathfrak{C}'. \end{cases}$$

By $\mathrm{VI}_\varkappa$, we can clearly assume $\mathscr{Q}_0 \neq 0$. From 1.1 (c), (d) we easily infer that there is a list $\langle f_\xi \rangle_{\xi<\varkappa}$ (possibly with repetitions) of all isomorphisms between members of $\mathscr{Q}_0$. Proceeding by recursion, we put $\mathfrak{A}_0' = \mathfrak{A}_0$ and $\mathfrak{A}_\delta' = \bigcup(\mathfrak{A}_\eta'/\eta < \delta)$, if $\delta \leqq \varkappa$. If $\xi < \varkappa$, then, by IV and $\mathrm{VI}_{\varkappa^+}$, we can take for $\mathfrak{A}_{\xi+1}'$ the "first" (in a fixed well-ordering of $\{\mathfrak{A} \in \mathscr{M} / |\mathfrak{A}| \subseteq |\mathfrak{A}_0| \cup \mathscr{S}(\varkappa)\}$) system with $\mathfrak{A}_\xi' \subseteq \mathfrak{A}_{\xi+1}' \in \mathscr{M}_{\varkappa^+}$ and such that there exists an isomorphism $g \supseteq f_\xi$ of $\mathfrak{A}_0$ into $\mathfrak{A}_{\xi+1}'$. Then clearly $\mathfrak{A}_1 = \mathfrak{A}_\varkappa'$ satisfies (3).

Now we define $\mathfrak{A}_\xi$ and $\mathscr{Q}_\xi$ by recursion in such a way that, for $\xi < \eta \leqq cf\varkappa$, $\mathfrak{A}_\xi \in \mathscr{M}_{\varkappa^+}$, $\mathscr{Q}_\xi \subseteq \mathscr{S}_\varkappa^{\mathscr{M}}(\mathfrak{A}_\xi)$, $\overline{\overline{\mathscr{Q}}}_\xi \leqq \varkappa$, $\mathfrak{A}_\xi \subseteq \mathfrak{A}_\eta$, and $\mathscr{Q}_\xi \subseteq \mathscr{Q}_\eta$. We put $\mathfrak{A}_\delta = \bigcup(\mathfrak{A}_\eta/\eta < \delta)$ and $\mathscr{Q}_\delta = \bigcup(\mathscr{Q}_\eta/\eta < \delta)$. Let $\mathfrak{A}_{\xi+1}$ be the "first" system related to $\mathfrak{A}_\xi$ as $\mathfrak{A}_1$ is to $\mathfrak{A}_0$ in (3). Let $\mathscr{Q}_{\xi+1}$ consist of all systems $\mathfrak{A}'$ for which one of the following four conditions holds: (i) $\mathfrak{A}' \in \mathscr{S}(\mathscr{Q}_\xi) \cap \mathscr{M}$. (ii) For some $\mathscr{Q}' \subseteq \mathscr{Q}_\xi$, $\overline{\overline{\mathscr{Q}'}} < cf\varkappa$ and $\mathfrak{A}'$ is the "first" member of $\mathscr{S}_\varkappa^{\mathscr{M}}(\mathfrak{A}_\xi)$ with $\mathscr{Q}' \subseteq \mathscr{S}(\mathfrak{A}')$ (such an $\mathfrak{A}'$ exists by $\mathrm{VI}_\varkappa$). (iii) $\mathfrak{A}'$ is one of the entries in the "first" $cf\varkappa$-chain of members of $\mathscr{S}_\varkappa^{\mathscr{M}}(A_\xi)$ whose union is $\mathfrak{A}_\xi$ (such a chain exists by 2.1 (b)). (iv) For some $\mathfrak{B}$, $\mathfrak{B}'$, f, $\mathfrak{C}$ as in (3) with $\mathscr{Q}_\xi$ for "$\mathscr{Q}_0$", $\mathfrak{A}'$ is the "first" system $\mathfrak{C}'$ as in (3) with $\mathfrak{A}_{\xi+1}$ for "$\mathfrak{A}_1$". By 1.1 (b), (d) and our assumption that $\varkappa = \varkappa^*$, it is clear that $\overline{\overline{\mathscr{Q}}}_{\xi+1} \leqq \varkappa$.

Taking $\mathfrak{A} = \mathfrak{A}_{cf\varkappa}$ and $\mathscr{Q} = \mathscr{Q}_{cf\varkappa}$, all conditions of (a) are now obvious except 2.2 (iii). But by our construction there are chains $\langle \mathfrak{C}_{\eta\xi} \rangle_{\xi < cf\varkappa}$ of members of $\mathscr{Q}$ with $\mathfrak{A}_\eta = \bigcup(\mathfrak{C}_{\eta\xi}/\xi < cf\varkappa)$ for each $\eta < cf\varkappa$. By 2.2 (ii) we can define $\mathfrak{B}_\xi$ recursively as the "first" member of $\mathscr{Q}$ such that, for each $\eta < \xi$, $\mathfrak{B}_\eta$, $\mathfrak{C}_{\eta\xi} \subseteq \mathfrak{B}_\xi$. Then clearly $\mathfrak{A} = \bigcup(\mathfrak{B}_\xi/\xi < cf\varkappa)$ snd (a) is proved.

LEMMA 2.7. (a) *If $\bar{I}_{\mathscr{M}} < \varkappa$ then the set of isomorphism types of $\mathscr{M}_\varkappa$ has power $\leqq \varkappa^*$.*

(b) *If $\varkappa$ is regular, $\overline{\overline{\mathfrak{A}}} \leqq \varkappa^*$, and $\mathscr{M}_\varkappa \subseteq \mathscr{I}\mathscr{S}(\mathfrak{A})$, then the set of isomorphism types of $\mathscr{M}_\varkappa$ has power $\leqq \varkappa^*$.*

(c) *If $\mathscr{M}$ has* III, V, *and* $\mathrm{VI}_{\varkappa^+}$ *and if the set of isomorphism types of $\mathscr{M}_\varkappa$ has power $\leqq \varkappa^*$ then for some $\mathfrak{A} \in \mathscr{M}$, $\overline{\overline{\mathfrak{A}}} = \varkappa^*$ and $\mathscr{M}_\varkappa \subseteq \mathscr{I}\mathscr{S}(\mathfrak{A})$.*

PROOF. (a) is easily checked. (b) follows from 1.1 (b), (d). To prove (c) we first note that, by $\mathrm{VI}_{\varkappa^+}$ and our assumption that $\mathscr{M}$ has arbitrarily large members, $\mathscr{M}$ has a member $\mathfrak{B}_0$ of power $\varkappa^*$. Form a list $\langle \mathfrak{A}_\xi \rangle_{\xi<\varkappa^*}$ of members of $\mathscr{M}_\varkappa$ containing at least one of each isomorphism type. Put $\mathfrak{B}_\delta = \bigcup(\mathfrak{B}_\eta/\eta < \delta)$. By III and $\mathrm{VI}_{\varkappa^+}$ we can let $\mathfrak{B}_{\xi+1}$ be the "first" (in a fixed well-ordering of $\{\mathfrak{B} \in \mathscr{M} / |\mathfrak{B}| \leqq |\mathfrak{B}_0| \cup \mathscr{S}(\varkappa^*)\}$) member of $\mathscr{M}$ such that $\mathfrak{A}_\xi \in \mathscr{I}\mathscr{S}(\mathfrak{B}_{\xi+1})$, $\mathfrak{B}_\xi \subseteqq \mathfrak{B}_{\xi+1}$, and $\overline{\overline{\mathfrak{B}}}_{\xi+1} = \varkappa^*$. Then clearly $\mathfrak{A} = \mathfrak{B}_{\varkappa^*}$ is as desired.

THEOREM 2.8. *Suppose $\mathscr{M}$ has* III, IV, V, *and* $\mathrm{VI}_\varkappa$ *and* $\varkappa = \varkappa^*$. *Suppose that the set of isomorphism types of $\mathscr{M}_\varkappa$ has power $\leqq \varkappa$ (or, a fortiori, that $\bar{\bar{I}}_{\mathscr{M}} < \varkappa$). Then there is up to isomorphism exactly one $\mathscr{M}$-special system $\mathfrak{A}$ of power $\varkappa$. $\mathfrak{A}$ is $\mathscr{M}$-universal and is $\mathscr{M}$-homogeneous of degree $cf\varkappa$.*[6]

PROOF. By 2.7 (c) there exists $\mathfrak{A}_0 \in \mathscr{M}$ and $\mathscr{Q}_0 \subseteqq \mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A}_0)$ such that $\overline{\overline{\mathfrak{A}}}_0 = \varkappa$, $\overline{\overline{\mathscr{Q}}}_0 \leqq \varkappa$, and $\mathscr{M}_\varkappa \subseteqq \mathscr{I}(\mathscr{Q}_0)$. The proof is now completed by applying 2.6 (a) and 2.4 (b), (a), (c).

For $\varkappa$ regular it is clear that 2.8 can be proved without any reference to notions involving $\mathscr{Q}$. The same can be said for $\varkappa$ singular, if we assume that the GCH holds and, for some $\varkappa' < \varkappa$, $\mathscr{M}$ has $\mathrm{VI}_{\varkappa'}$ and $\bar{\bar{I}}_{\mathscr{M}} < \varkappa'$. Indeed, there then exists a system $\mathfrak{A}$ of power $\varkappa$ which is the union of a chain $\langle \mathfrak{A}_\lambda / \varkappa' \leqq \lambda < \varkappa \rangle$, where each $\mathfrak{A}_\lambda$ is $\mathscr{M}$-homogeneous-universal and of power λ^+; moreover (as one easily sees by applying the proofs of 2.3 (a), (b) specialized to $\mathscr{Q} = \bigcup(\mathscr{S}_\varkappa{}^{\mathscr{M}}(\mathfrak{A}_\lambda) / \varkappa' \leqq \lambda < \varkappa)$), such an $\mathfrak{A}$ is $\mathscr{M}$-universal, $\mathscr{M}$-homogeneous of degree $cf\varkappa$, and unique up to isomorphism.[7] However, by using the $\mathscr{Q}$-notions, we have obtained in 2.8 a result which avoids altogether the GCH when $\varkappa = \beth_\delta$ (cf. 1.1 (f)).

As regards the existence of "universal" systems, the following strengthening of 2.8 can be made:

(4) $\begin{cases} \textit{If } \varkappa = \varkappa^* \textit{ is omitted from the hypotheses of 2.8, one can still conclude} \\ \textit{that there is a system } \mathfrak{A} \in \mathscr{M} \textit{ of power } \leqq \varkappa^* \textit{ with } \mathscr{M}_{\varkappa^+} \subseteqq \mathscr{I}\mathscr{S}(\mathfrak{A}). \end{cases}$[8]

To see this, apply 2.7 (c) and 2.6 (a) after modifying the latter as follows: assume that $\overline{\overline{\mathfrak{A}}}_0$, $\overline{\overline{\mathscr{Q}}}_0 \leqq \varkappa^*$; omit 2.2 (iii) from the conclusion; and omit the condition (iii) from the construction in the proof. Finally, note the parenthetical remark at the end of the proof of 2.3 (a).

The next theorem, 2.8′, is an addendum to 2.8 and 2.6 which will be needed when we apply the latter to elementary types in Section 3.

THEOREM 2.8′. *Suppose $\mathscr{M}'$ has* V *and* $\mathrm{VI}_{\varkappa^+}$. *Suppose that $\mathscr{M}' \subseteqq \mathscr{M} \subseteqq \mathscr{S}(\mathscr{M}')$; or, more generally, that* (i) $\mathscr{M}' \upharpoonright I_{\mathscr{M}} \subseteqq \mathscr{M} \subseteqq \mathscr{S}(\mathscr{M}' \upharpoonright I_{\mathscr{M}})$ *and*

[6] For α regular and $\bar{\bar{I}}_{\mathscr{M}} < \varkappa$, 2.8 is due to Jónsson [9]. A special instance of the case $\bar{\bar{I}}_{\mathscr{M}} = \varkappa = \omega$ was in [24].

[7] In [8, p. 201] a closely related argument is used to obtain under these hypotheses the existence of an $\mathscr{M}$-universal system of power $\varkappa$. The question is raised there whether one can replace the hypothesis $\mathrm{VI}_{\varkappa'}$ by $\mathrm{VI}_\varkappa$.

[8] Assuming $\mathscr{M} \in \boldsymbol{EC}_\Delta$ and $\bar{\bar{I}}_{\mathscr{M}} \leqq \omega$, this result was obtained in 1955 by the first author. A similar strengthening of 4.1, below, can also be made.

(ii) *whenever* $\mathfrak{B}' \in \mathcal{M}'$ *and* $\mathfrak{B}' \rceil I_{\mathcal{M}} \subseteq \mathfrak{B} \in \mathcal{M}$ *there exists* $\mathfrak{C}' \in \mathcal{M}'$ *such that* $\mathfrak{B}' \subseteq \mathfrak{C}'$ *and* $\mathfrak{B} \subseteq \mathfrak{C}' \rceil I_{\mathcal{M}}$. *Then in* 2.6 (a), (b) *and* 2.8 *we can find* $\mathfrak{A} \in \mathcal{M}' \rceil I_{\mathcal{M}}$.

Proof. It clearly suffices to deal with 2.6 (a). By (i) we can assume that $\mathfrak{A}_0 = \mathfrak{A}_0' \rceil I_{\mathcal{M}}$ where $\mathfrak{A}_0' \in \mathcal{M}'$. Now by (ii) the existence of an $\mathfrak{A}_1$ for which (3) holds implies the existence of an $\mathfrak{A}_1$ and $\mathfrak{A}_1'$ such that (3′): (3) holds, $\mathfrak{A}_1 = \mathfrak{A}_1' \rceil I_{\mathcal{M}}$, and $\mathfrak{A}_1' \in \mathcal{M}'$. Now we define $\mathfrak{A}_\xi$, $\mathfrak{A}_\xi'$, and $\mathcal{Q}_\xi$ recursively by imposing the same conditions on $\mathcal{Q}_\xi$ as before and requiring that $\mathfrak{A}_\xi = \mathfrak{A}_\xi' \rceil I_{\mathcal{M}}$, $\mathfrak{A}_\delta' = \bigcup(\mathfrak{A}_\eta'/\eta < \delta)$, and $\mathfrak{A}_{\xi+1}'$ is related to $\mathfrak{A}_\xi'$ as $\mathfrak{A}_1'$ is to $\mathfrak{A}_0'$ in (3′). Then clearly $\mathfrak{A} = \mathfrak{A}_{cf\varkappa} = \mathfrak{A}_{cf\varkappa}' \rceil I_{\mathcal{M}}$ is as desired.

Theorem 2.10 (b) below, is an immediate consequence of 2.6, 2.4 (c), and 1.1 (e) if the GCH and $\mathrm{VI}_\varkappa$ (rather than $\mathrm{VI}_{\varkappa+}$) are assumed. The remainder of this section will be devoted to establishing it in the absence of those assumptions. The reader who wishes to may go directly to Section 3.

Lemma 2.9. *Suppose* $\mathcal{M}$ *has* V, $\mathfrak{A}' \in \mathcal{M}$, *and* g *is an isomorphism of* $\mathfrak{A}'$ *into itself. Then there exist* $\mathfrak{A} \in \mathcal{M}$ *and an automorphism* f *of* $\mathfrak{A}$ *such that* $\mathfrak{A}' \subseteq \mathfrak{A}$, $g \subseteq f$, *and* $\overline{\overline{\mathfrak{A}'}} = \overline{\overline{\mathfrak{A}}}$.

Proof. Let $\mathfrak{A}' \cong_g \mathfrak{A}_0$ and put $\mathfrak{A}_1 = \mathfrak{A}'$ and $f_0 = g$. It is immediate that there exist $\mathfrak{A}_2 \supseteq \mathfrak{A}_1$ and $f_1 \supseteq f_0$ such that $\mathfrak{A}_2 \cong_{f_1} \mathfrak{A}_1$. ($\mathfrak{A}_2$ and f_1 could even be explicitly defined.) Proceeding inductively we obtain $\mathfrak{A}_0 \subseteq \ldots \subseteq \mathfrak{A}_n \subseteq \ldots$ and $f_0 \subseteq \ldots \subseteq f_n \subseteq \ldots$ such that $\mathfrak{A}_{n+1} \cong_{f_n} \mathfrak{A}_n$, for each n. Clearly $\mathfrak{A} = \bigcup(\mathfrak{A}_n/n \in \omega)$ and $f = \bigcup(f_n/n \in \omega)$ are as desired.

Theorem 2.10. *Suppose* $\mathcal{M}$ *has* IV, V, *and* $\mathrm{VI}_{\varkappa+}$, $\mathfrak{A}_0 \in \mathcal{M}$, *and* $\overline{\overline{\mathfrak{A}_0}} = \varkappa$.

(a) *If* $\mathcal{G}$ *is a family of isomorphisms between members of* $\mathcal{S}^{\mathcal{M}}(\mathfrak{A}_0)$ *and* $\overline{\overline{\mathcal{G}}} \leq \varkappa$, *then there exists* $\mathfrak{A} \in \mathcal{M}$ *such that* $\mathfrak{A}_0 \subseteq \mathfrak{A}$, $\overline{\overline{\mathfrak{A}}} = \varkappa$, *and every member of* $\mathcal{G}$ *can be extended to an automorphism of* $\mathfrak{A}$.

(b) *If* $\varkappa^\lambda = \varkappa$, *then there exists* $\mathfrak{A} \in \mathcal{M}$ *such that* $\mathfrak{A}_0 \subseteq \mathfrak{A}$, $\overline{\overline{\mathfrak{A}}} = \varkappa$, *and* $\mathfrak{A}$ *is* $\mathcal{M}$*-homogeneous of degree* λ.

(c) *If* $\mathcal{M}'$ *has* V *and* $\mathrm{VI}_{\varkappa+}$ *and* 2.8′ (i), (ii) *hold, then in* (a) *and* (b) *we can find* $\mathfrak{A} \in \mathcal{M}' \rceil I_{\mathcal{M}}$.[9]

Proof. (a) We shall first assume that $\mathcal{G}$ has a single member f_0. Let f_0 be on $\mathfrak{A}_0'$ and let $\mathfrak{A}_1' = \mathfrak{A}_0$. By IV and $\mathrm{VI}_{\varkappa+}$ there exists $\mathfrak{A}_2' \in \mathcal{M}_{\varkappa+}$

[9] For the class $\mathcal{M}$ of all groups and with $\overline{\overline{\mathcal{G}}} = 1$, 2.10 (a) was proved (in a different way) by G. Higman, B. H. Neumann and H. Neumann, J. London Math. Soc. 24 (1949), pp. 247-254. Lemma 2.9 and the special case of 2.10 (a) in (6) below were obtained by the first author in 1955. (An alternative "back and forth" proof of (6), avoiding 2.9, was suggested by A.Ehrenfeucht; the proof above of 2.10 (a) for $\overline{\overline{\mathcal{G}}} = 1$ could also be thus modified.) Svenonius [20] found (6) independently; using his result Craig [3] found independently some special cases of 2.10 (b).

and an isomorphism f_1 of $\mathfrak{A}_1'$ into $\mathfrak{A}_2'$ such that $\mathfrak{A}_1' \subseteq \mathfrak{A}_2'$ and $f_0 \subseteq f_1$. By recursion we easily obtain $\mathfrak{A}_0' \subseteq \ldots \subseteq \mathfrak{A}_n' \subseteq \ldots$ and $f_0 \subseteq \ldots \subseteq f_n \subseteq \ldots$ such that, for each n, $\mathfrak{A}_n' \in \mathscr{M}$ and f_n is isomorphism of $\mathfrak{A}_n'$ into $\mathfrak{A}_{n+1}'$. Hence $g = \bigcup (f_n / n \in \omega)$ is an isomorphism of $\mathfrak{A}' = \bigcup (\mathfrak{A}_n' / n \in \omega)$ into itself. By applying 2.9 the desired $\mathfrak{A}$ is obtained.

Now suppose $\mathscr{G}$ is arbitrary. Since 2.10 (a) is trivial when $\mathscr{G} = 0$, we may put $\mathscr{G} = \{g_\xi / \xi < \varkappa\}$. By recursion we shall define $\mathfrak{C}_\eta$ and h_η for $\eta < \varkappa \cdot \omega$. Write $\eta = \varkappa \cdot n + \xi$, where $\xi < \varkappa$. If $n = 0$, put $\mathfrak{C}_\eta = \mathfrak{B}$ and $h_\eta = g_\xi$. If $n = m+1$, then for $\mathfrak{C}_\eta$ we take the "first" member of $\mathscr{M}_{\varkappa^+}$, with $\bigcup (\mathfrak{C}_\zeta / \zeta < \eta) \subseteq \mathfrak{C}_\eta$, having an automorphism $f \supseteq h_{\varkappa \cdot m + \xi}$; and for h_η we take the "first" such f. Let $\mathfrak{A} = \bigcup (\mathfrak{C}_\eta / \eta < k \cdot \omega)$. Then, for each $\xi < \varkappa$, $\bigcup (h_{\varkappa \cdot n + \xi} / n \in \omega)$ is an automorphism of $\mathfrak{A}$ extending g_ξ.

(b) By recursion we shall define, for $1 \leqq \xi \leqq cf\varkappa$, a system $\mathfrak{A}_\xi$ and functions $\langle f_{\xi,\eta,\zeta} \rangle_{\eta < \xi, \zeta < \varkappa}$. If $\xi < cf\varkappa$, then by 2.10 (a) we can take for $\mathfrak{A}_{\xi+1}$ the "first" member of $\mathscr{M}_{\varkappa^+}$ such that $\mathfrak{A}_\xi \subseteq \mathfrak{A}_{\xi+1}$ and, whenever $\eta < \xi$ and $\zeta < \varkappa$, there is an automorphism g of $\mathfrak{A}_{\xi+1}$ extending $f_{\xi,\eta,\zeta}$; we take for $f_{\xi+1,\eta,\zeta}$ the "first" such g. Moreover (cf. 1.1 (d)) we let $\langle f_{\xi+1,\xi,\zeta} \rangle_{\zeta < \varkappa}$ be the "first" list (with repetitions) of all isomorphisms between members of $\mathscr{S}_\lambda^{\mathscr{M}}(\mathfrak{A}_{\xi+1})$. (We can assume there are some, as the theorem is trivial if $\mathscr{S}_\lambda^{\mathscr{M}}(\mathfrak{A}_0) = 0$.) Put $\mathfrak{A}_\delta = \bigcup (\mathfrak{A}_\gamma / \gamma < \delta)$ and $f_{\delta,\eta,\zeta} = \bigcup (f_{\gamma,\eta,\zeta} / \gamma < \delta)$ for $n < \delta$ and $\zeta < \varkappa$. Then $\mathfrak{A} = \mathfrak{A}_{cf\varkappa}$ is as desired. Indeed, suppose g is an isomorphism of $\mathfrak{B} \in \mathscr{S}_\lambda^{\mathscr{M}}(\mathfrak{A})$ into $\mathfrak{A}$. By 1.1 (e), $\overline{\overline{\mathfrak{B}}} < cf\varkappa$, and hence $\mathfrak{B} \subseteq \mathfrak{A}_\xi$ for some $\xi < cf\varkappa$. Therefore $g = f_{\xi+1,\xi,\zeta}$, for some $\zeta < \varkappa$, and $f_{cf\varkappa,\xi,\zeta}$ is an automorphism of $\mathfrak{A}$ extending g.

(c) is obtained by modifying the above arguments in a way analogous to the proof of 2.8′.

3. Elementary types.

If φ is a formula of $L_{t\mathfrak{A}}$, and m is the smallest number such that the free variables of φ are among $v_0, \ldots v_{m-1}$, then we denote by $\varphi^{\mathfrak{A}}$ the m-ary relation such that $\varphi^{\mathfrak{A}} x_0 \ldots x_{n-1}$ if and only if $\vdash_{\mathfrak{A}} \varphi[x_0, \ldots, x_{m-1}]$. For any relational system $\mathfrak{A}$, we put $\mathfrak{A}^* = (\mathfrak{A}, \varphi^{\mathfrak{A}})_{\varphi \in F}$, where F is the set of all formulas of $L_{t\mathfrak{A}}$. We also write $\mathscr{M}^* = \{\mathfrak{A}^* / \mathfrak{A} \in \mathscr{M}\}$. One easily verifies the following lemma:

LEMMA 3.1. (a) *The following are equivalent*:

$$\mathfrak{A} \prec \mathfrak{B},\ \mathfrak{A}^* \subseteq \mathfrak{B}^*,\ \mathfrak{A}^* \prec \mathfrak{B}^*.$$

(b) *If* $X \subseteq |\mathfrak{A}|$, $Y \subseteq |\mathfrak{B}|$, *and* f *is a function on* X *onto* Y, *then* $\mathfrak{A}^* | X \cong_f \mathfrak{B}^* | Y$ *if and only if* $(\mathfrak{A}, x)_{x \in X} \equiv (\mathfrak{B}, f(x))_{x \in X}$.

THEOREM 3.2. *Suppose* $\mathscr{N} \in \boldsymbol{ET}$. *Then*: $\mathscr{N}$ *has arbitrarily large members.* $\mathscr{N}^* \in \boldsymbol{ET}$ *and* $\mathscr{S}(\mathscr{N}^*) \in \boldsymbol{UC}_\Delta$. $\mathscr{N}^*$ *has* III, IV, V, *and* $\mathrm{VI}_{\varkappa^+}$ *where* $\varkappa = \max(\omega, \bar{\bar{I}}_{\mathscr{N}})$. $\mathscr{S}(\mathscr{N}^*)$ *has* III, IV, V, *and* VI_ω.

PROOF. The first assertion is well-known (cf. [22]). Clearly $\mathscr{N}^* \in \boldsymbol{ET}$: it follows, by a theorem of Łoś [14] and Tarski [21], that $\mathscr{S}(\mathscr{N}^*) \in \boldsymbol{UC}_\Delta$ and $\mathscr{S}(\mathscr{N}^*)$ has V. The statement that $\mathscr{N}^*$ has V is translated by 3.1 (a) into a theorem of Tarski [22] on unions of "$\prec$-chains". That $\mathscr{S}(\mathscr{N}^*)$ has VI_ω is trivial. That $\mathscr{N}^*$ has $\mathrm{VI}_{\varkappa^+}$ is, by 3.2 (a) the Löwenheim–Skolem theorem, as formulated by Tarski [22]. Since $\mathscr{N}^* \in \boldsymbol{ET}$, 1.2 (a) tells us that $\mathscr{N}^*$ has III; it follows trivially that $\mathscr{S}(\mathscr{N}^*)$ also has III. That each of $\mathscr{N}^*$ and $\mathscr{S}(\mathscr{N}^*)$ has IV also follows at once from 1.2 (a), applied now to systems of the form $(\mathfrak{A}^*, x)_{x \in X}$, where $\mathfrak{A} \in \mathscr{N}$ and $X \subseteq |\mathfrak{A}|$.

We shall say that $\mathfrak{A}$ is simply *universal*, *homogeneous* (*of degree* λ), or special if $\mathfrak{A}$ is infinite and if $\mathfrak{A}^*$ is $\mathscr{S}(\mathscr{N}^*)$-universal, $\mathscr{S}(\mathscr{N}^*)$-homogeneous (of degree λ), or $\mathscr{S}(\mathscr{N}^*)$-special, respectively, where $\mathscr{N}$ is the elementary type of $\mathfrak{A}$. By means of 3.1, each of these notions has an obvious translation which avoids the passage to $\mathfrak{A}^*$. Thus, for example, if $\bar{\bar{\mathfrak{A}}} = \varkappa$, then $\mathfrak{A}$ is special if and only if

$$(5)\quad \begin{cases} \textit{There exists } \mathscr{Q} \subseteq \mathscr{S}_\varkappa(|\mathfrak{A}|) \textit{ such that } \mathscr{S}(\mathscr{Q}) \subseteq \mathscr{Q}, \mathscr{Q} \textit{ is cf}\varkappa\textit{-directed, } |\mathfrak{A}| \\ \textit{is the union of a chain of members of } \mathscr{Q}, \textit{ and whenever } X \in \mathscr{Q}, \\ X \subseteq Y \in \mathscr{S}_\varkappa(|\mathfrak{B}|), \textit{ and } (\mathfrak{A}, x)_{x\in X} \equiv (\mathfrak{B}, x)_{x \in X}, \textit{ there exists a function} \\ f \supseteq Id_X \textit{ on } Y \textit{ onto a member of } \mathscr{Q} \textit{ such that } (\mathfrak{B}, y)_{y \in Y} \equiv (\mathfrak{A}, f(y))_{y \in Y}. \end{cases}$$

3.3 and 3.4, below, elucidate the relationship between some of these notions and certain apparently weaker conditions. 3.4 is due to Keisler (cf. [11]).

THEOREM 3.3. *Suppose* $\bar{\bar{\mathfrak{A}}} = \varkappa > \omega, \bar{\bar{I}}_{\mathfrak{A}}$, *and let* $\mathscr{N}$ *be the elementary type of* $\mathfrak{A}$. *Then* $\mathfrak{A}$ *is special* (*homogeneous-universal*) *if and only if* $\mathfrak{A}^*$ *is* $\mathscr{N}^*$*-special* ($\mathscr{N}^*$*-homogeneous-universal*).

PROOF. "Only if" is obvious for "homogeneous-universal"; for "special" a straightforward argument is needed, which we leave to the reader. For the converse, suppose $\mathfrak{A}^*$ is $\mathscr{N}^*$, $\mathscr{Q}'$-homogeneous and $\mathscr{N}_\varkappa^* \subseteq \mathscr{I}(\mathscr{Q}')$. Let $\mathscr{Q}$ be the set of all X such that for some $\mathfrak{C} \in \mathscr{Q}'$, $X \subseteq |\mathfrak{C}|$. Then $\mathscr{Q}$ obviously fulfills all but the last clause of (5). To see that it is also fulfilled, let $(\mathfrak{A}, x)_{x\in X} \equiv (\mathfrak{B}, x)_{x \in X}$, $X \subseteq Y \in \mathscr{S}_\varkappa(|\mathfrak{B}|)$, and $X \in \mathscr{Q}$, i.e., $X \subseteq |\mathfrak{C}|$, where $\mathfrak{C}^* \in \mathscr{Q}'$. Since $\mathscr{N}^*$ has $\mathrm{VI}_\varkappa$ by 3.3, we can clearly assume $\bar{\bar{\mathfrak{B}}} < \varkappa$. By 1.2 (a), there exist $\mathfrak{B}' \succ \mathfrak{C}$ and an isomorphism $f \supseteq Id_X$ of $\mathfrak{B}$ into an elementary subsystem of $\mathfrak{B}'$. By $\mathrm{VI}_\varkappa$ for $\mathscr{N}^*$ we can take $\bar{\bar{\mathfrak{B}}}' < \varkappa$. By hypothesis, there exist $\mathfrak{C}'^* \in \mathscr{Q}'$ and $g \supseteq Id_{|\mathfrak{C}|}$ such that $\mathfrak{B}' \cong_g \mathfrak{C}'$. Thus

g restricted to Y is as desired in (5). The parenthetical case follows by taking $\mathscr{Q}' = \mathscr{S}_\varkappa^{\mathscr{N}*}(\mathfrak{A}^*)$.

THEOREM 3.4. *If $\overline{\overline{\mathfrak{A}}} = \varkappa$, then $\mathfrak{A}$ is homogeneous-universal if and only if the following condition holds: whenever $X \in \mathscr{S}_\varkappa(|\mathfrak{A}|)$, $\mathfrak{B} = (\mathfrak{A}, x)_{x \in X}$, and Σ is a set of formulas of $L_{t\mathfrak{B}}$ each having only v_0 free, if every finite subset of Σ is satisfiable in $\mathfrak{B}$ then so is Σ.*

PROOF. Using the completeness theorem and 3.1 (and the observation at the end of the proof of 1.2) one easily checks that the condition above is equivalent to the condition in 2.5 (b) (with $\mathscr{S}(\mathscr{N}^*)$ for "$\mathscr{M}$").

By 3.2 we can apply each result of Section 2 to an elementary type $\mathscr{N}$, either taking $\mathscr{N}^*$ for "$\mathscr{M}$" or else taking $\mathscr{S}(\mathscr{N}^*)$ for "$\mathscr{M}$" and $\mathscr{N}^*$ for "$\mathscr{M}'$" (in 2.8′ and 2.10 (c)). Thus, for example, by applying 2.10 (a), (c) (with $\overline{\overline{\mathscr{G}}} = 1$) in the second way, we obtain at once:

(6) $\begin{cases} \textit{If } (\mathfrak{A}, x)_{x \in X} \equiv (\mathfrak{A}, f(x))_{x \in X}, \textit{ where } f \in |\mathfrak{A}|^X \textit{ and } \overline{\overline{\mathfrak{A}}} = \varkappa \geqq \overline{\overline{I}}_{\mathfrak{A}}, \textit{ then there} \\ \textit{is a system } \mathfrak{A}' \succ \mathfrak{A} \textit{ of power } \varkappa \textit{ which has an automorphism extending } f. \end{cases}$

(The second way is sometimes available when the first is not, since $\mathscr{S}(\mathscr{N}^*)$ has $VI_\varkappa$ for every $\varkappa$; hence we have followed it in defining "universal", "homogeneous", etc.) We shall not enumerate all such consequence of Section 2 and 3.2, but we will state explicitly the following theorem, which is the principal result of the paper.

THEOREM 3.5. *Suppose $\mathscr{N} \in \boldsymbol{ET}$ and $\overline{\overline{I}}_{\mathscr{N}} \leqq \varkappa = \varkappa^*$. Suppose that the set of isomorphism types of $\mathscr{S}_\varkappa(\mathscr{N}^*)$ has power $\leqq \varkappa$ or, a fortiori, that ω, $\overline{\overline{I}}_{\mathscr{N}} < \varkappa$. Then there is up to isomorphism exactly one special system $\mathfrak{A} \in \mathscr{N}$ of power $\varkappa$. $\mathfrak{A}$ is universal and is homogeneous of degree $cf\varkappa$.*[10]

In the remainder of this section we shall describe some additional properties which are possessed by special, or homogeneous-universal, systems.

[10] In the first two abstracts of [23], only $\mathscr{M}^*$ was considered, while in the third and in [24], the case $\varkappa = \omega$ of 3.5 was derived directly (rather then being inferred from Fraïssé-Jónsson type results). Using (6), proved directly, we saw that an $\mathscr{N}^*$-homogeneous-universal system $\mathfrak{A}$, of power $\varkappa > \omega \geqq \overline{\overline{I}}_{\mathfrak{A}}$, did in fact have the homogeneity property for arbitrary subsets. Later Keisler [10], [11], proceeding directly, constructed at once such a system C with the stronger homogeneity. He also obtained the alternative characterization 3.4 with "one element at a time" property (which, as he noted, also allows a simplification in the existence proof). We had erroneously thought this was a speciality of $\varkappa = \omega$. Recently we found that $\mathscr{S}(\mathscr{N}^*)$ could be used in place of $\mathscr{N}^*$, thus ensuring the same advantages for the indirect method. Only then did we realize that by using $\mathscr{S}(\mathscr{N}^*)$ the case $\varkappa = \omega$ of 3.5, dealt with in [24], could also be obtained indirectly.

THEOREM 3.6. *To 3.5 we can add the assertion: Moreover, $\mathfrak{A}$ is relation-universal, i.e., whenever $\mathfrak{B} \upharpoonright I_{\mathfrak{A}} \equiv \mathfrak{A}$ and $\bar{I}_{\mathfrak{B}} \leqq \varkappa$, there exists $\mathfrak{C} \equiv \mathfrak{B}$ such that $\mathfrak{C} \upharpoonright I_{\mathfrak{A}} = \mathfrak{A}$.*

PROOF. Given any such $\mathfrak{B}$ let $\mathscr{N}'$ be the elementary type of $\mathfrak{B}$. By 1.2 (b) (and 3.2), the hypothesis of 2.8′ hold with $\mathscr{S}(\mathscr{N}^*)$ for "$\mathscr{M}$" and with the class $\mathscr{N}'^*$ for "$\mathscr{M}'$". Therefore, by 2.8′, there is a special system $\mathfrak{A}'$ with $\overline{\overline{\mathfrak{A}'}} = \varkappa$ and $\mathfrak{A}' \in \mathscr{N}' \upharpoonright I_{\mathfrak{A}}$. But $\mathfrak{A}' \cong \mathfrak{A}$, so also $\mathfrak{A} \in \mathscr{N}' \upharpoonright I_{\mathfrak{A}}$, as desired.

3.6 can be regarded as a strengthening of Robinson's Theorem [16]: *If $\mathscr{N}, \mathscr{N}', \mathscr{N}'' \in$ **ET** and $\mathscr{N}' \upharpoonright I_{\mathscr{N}}$, $\mathscr{N}'' \upharpoonright I_{\mathscr{N}} \subseteq \mathscr{N}$, then $(\mathscr{N}' \upharpoonright I_{\mathscr{N}}) \cap (\mathscr{N}'' \upharpoonright I_{\mathscr{N}}) \neq 0$.*

THEOREM 3.7. *If $\mathfrak{A}$ is special or homogeneous universal, the same applies to any reduct of $\mathfrak{A}$ and to any infinite relativization of $\mathfrak{A}$ (and the latter must have the same power as $\mathfrak{A}$).*

PROOF. Suppose $\overline{\overline{\mathfrak{A}}} = \varkappa$ and $J \subseteq I_{\mathfrak{A}}$. If $\mathfrak{A}$ is homogeneous-universal then it is immediate from 3.4 that $\mathfrak{A} \upharpoonright J$ is, also. We could argue similarly for $\mathfrak{A}$ special if we had formulated the obvious analogue of 3.4 for the notion "special". Alternatively, if $\mathscr{Q}$ and $\mathfrak{A}$ are as in (5), then one can show that $\mathscr{Q}$ and $\mathfrak{A} \upharpoonright J$ also satisfy the last clause of (5) by a straightforward argument using 1.2 (b) (the rest of (5) for $\mathscr{Q}$ and $\mathfrak{A} \upharpoonright J$ is obvious).

A well-known argument using the completeness theorem shows that if $R_j^{\mathfrak{A}}$ is singulary and infinite, then, for some $\mathfrak{B} \equiv \mathfrak{A}$, $\overline{\overline{R_j^{\mathfrak{B}}}} = \varkappa$. Since $\mathfrak{A}$ is universal it follows that $\overline{\overline{R_j^{\mathfrak{A}}}} = \varkappa$. That $\mathfrak{A} \mid R_j^{\mathfrak{A}}$ is special or homogeneous-universal can be proved in a way analogous to the above, or it can rather easily be inferred from the result condering reducts.

We could also modify 3.6 by strengthening the notion of relation-universal to include reducts of relativizations. But one can see directly that any relation-universal system $\mathfrak{A}$ has also the new property. (One passes from a system $\mathfrak{B}$ with larger universe than $\mathfrak{A}$ to one with the same universe as $\mathfrak{A}$ plus an inner universe, relations like those of $\mathfrak{B}$ on both universes and an isomorphism function as still another relation.) 3.7 (and 3.8 ,below) obviously can be extended to systems whose universes and relations are of the form $\varphi^{\mathfrak{A}}$, as one sees at once by passing from $\mathfrak{A}$ to $\mathfrak{A}^*$.

Theorem 3.7 can be viewed as a strengthening of the following theorem, essentially due to Specker:

(7) $\begin{cases} \textit{If } j_1, j_2 \in I_{\mathfrak{A}}, R_{j_1}{}^{\mathfrak{A}} \textit{ and } R_{j_2}{}^{\mathfrak{A}} \textit{ are singular } f_1, f_2 \in I_{\mathfrak{A}}{}^J, \textit{ and } \langle R_{j_1}{}^{\mathfrak{A}}, R_{f_1(j)}{}^{\mathfrak{A}}\rangle_{j\in J} \\ \equiv \langle R_{j_2}{}^{\mathfrak{A}}, R_{f_2(j)}{}^{\mathfrak{A}}\rangle_{j\in J}, \textit{ then there exists } \mathfrak{B} \equiv \mathfrak{A} \textit{ such that } \langle R_{j_1}{}^{\mathfrak{B}}, R_{f_1(j)}{}^{\mathfrak{B}}\rangle_{j\in J} \cong \\ \langle R_{j_2}{}^{\mathfrak{B}}, R_{f_2(j)}{}^{\mathfrak{B}}\rangle_{j\in J}. \end{cases}$[11]

To prove (7) when $R_{j_1}{}^{\mathfrak{A}}$ is infinite, find a special $\mathfrak{B} \equiv \mathfrak{A}$, by 3.5. Then, by 3.7 both $\langle R_{j_1}{}^{\mathfrak{B}}, R_{f_1}{}^{\mathfrak{B}}{}_{(j)}\rangle_{j\in J}$ and $\langle R_{j_2}{}^{\mathfrak{B}}, R_{f_2}{}^{\mathfrak{B}}{}_{(j)}\rangle_{j\in J}$ are special and have the power of $\mathfrak{A}$, so they are isomorphic. If $R_{j_1}{}^{\mathfrak{A}}$ is finite, then one can take $\mathfrak{B} = \mathfrak{A}$, by [21, Theorem 1.4].

Any universal system $\mathfrak{A}$ obviously has an isomorphism f with a proper elementary subsystem of itself. As the first author noted in 1955, it follows, by 3.5 and the Löwenheim–Skolem theorem applied to $(\mathfrak{A}, f)$, that:

If $\overline{\overline{I}}_{\mathfrak{B}} \leqq \varkappa$, then there is a system $\mathfrak{A} \equiv \mathfrak{B}$, of power $\varkappa$, isomorphic to a proper elementary subsystem of itself.

D. Scott raised the question whether there is a system elementarily equivalent to $\mathfrak{C}_1 = \langle \omega, +, \cdot, < \rangle$ which is isomorphic to a proper initial segment of itself. The general form of (7), above, was formulated by the second author just in order to give an affirmative answer. Indeed, let $\mathfrak{C}_2$ be a proper elementary extension of $\mathfrak{C}_1$ and, considering $\mathfrak{C}_1$ and $\mathfrak{C}_2$ as relativized reducts of $\mathfrak{A} = (\mathfrak{C}_2, \omega)$, apply (7).

The following theorem is due to Keisler [11]:

THEOREM 3.8. *If $\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{A}}}'$, $\mathfrak{A}$ and $\mathfrak{A}'$ are special (and similar), and every positive sentence true in $\mathfrak{A}$ is true in $\mathfrak{A}'$, then $\mathfrak{A}'$ is a homomorphic image of $\mathfrak{A}$.*

Roughly speaking, 3.8 is established by a generalization of the method of proof of 2.3 (b). We refer the reader to a forthcoming paper by Keisler for the proof (and to [12] for the definition of "positive sentence" and "homomorphism").[12]

As Keisler has shown in [12], the following immediate consequence of 3.8 and 3.5 can be considered as a form of a well-known theorem of R. Lyndon: *If every positive sentence true in $\mathfrak{A}$ is true in (the similar system) $\mathfrak{A}'$, then some elementary equivalent of $\mathfrak{A}'$ is a homomorphic image of an elementary equivalent of $\mathfrak{A}$.*

[11] (7) was established by Specker [18], [19] in a slightly less general form. There are a number of different proofs of (7). One is similar to Ehrenfeucht's proof of (6), described in footnote 9. (Note that (6) is a special case of (7)). Scott has found a very short proof, directly from Robinson's Theorem, which is presented in [19].

[12] Keisler proves only the case when $\mathfrak{A}$ and $\mathfrak{A}'$ are homogeneous-universal, but it is completely straightforward to deal similarly with $\mathfrak{A}$ and $\mathfrak{B}$ special.

4. Elementary classes.

For classes $\mathscr{M} \in \boldsymbol{EC}_\Delta$, we can apply 3.5 to obtain the following improvement of the results of [8] (cf. 2.8 above) concerning the existence of $\mathscr{M}$-universal systems of power $\varkappa > \omega$. (Cf. footnote [4]).

THEOREM 4.1. *If $\mathscr{M} \in \boldsymbol{EC}_\Delta$ and $\omega, \bar{\bar{I}}_{\mathscr{M}} < \varkappa = \varkappa^*$, then there is an $\mathscr{M}$-universal system of power $\varkappa$ if and only if $\mathscr{M}$ has* III.

PROOF. Let Σ be a set of sentences whose class of models is $\mathscr{M}$, and let $\lambda = \max(\omega, \bar{\bar{J}}_{\mathscr{M}})$. Since there are at most 2^λ sets of sentences (of $L_{t\mathscr{M}}$), the family $E = \{\mathscr{N} \in \boldsymbol{ET} / \mathscr{N} \subseteqq \mathscr{M}\}$ has power at most $2^\lambda \leqq \varkappa$. From 3.5 we infer that there are (pairwise disjoint) systems $\mathfrak{A}_{\mathscr{N}}$ for $\mathscr{N} \in E$ such that $\mathfrak{A}_{\mathscr{N}} \in \mathscr{N}$ and, if $\bar{\bar{\mathfrak{A}}}_{\mathscr{N}} \geqq \omega$, then $\bar{\bar{\mathfrak{A}}}_{\mathscr{N}} = \varkappa$ and $\mathfrak{A}_{\mathscr{N}}$ is universal (and hence certainly, $\mathscr{N}$-universal). If $\mathscr{M}$ has III, then clearly $\Sigma \cup \bigcup (D(\mathfrak{A}_{\mathscr{N}}) / \mathscr{N} \in E)$ is consistent and hence has a model $\mathfrak{A}$ of power $\varkappa$. $\mathfrak{A}$ is obviously $\mathscr{M}$-universal.

The converse is easily derived from the completeness and Löwenheim–Skolem theorems.

The rest of this section is devoted to two metamathematical results concerning the general classes $\mathscr{M}$ of Section 2.

THEOREM 4.2. *Suppose $\mathscr{M}$ has* III, IV, V, *and* $\mathrm{VI}_\varkappa$. $\bar{\bar{\mathfrak{A}}} = \varkappa$. *$\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{M}$-special, and $\mathfrak{A} \subseteqq \mathfrak{B}$. Then $\mathfrak{A} \prec \mathfrak{B}$.*

PROOF. First assume that $\mathfrak{B}$ is $\mathscr{M}$-homogeneous of degree $\varkappa$, rather than $\mathscr{M}$-special. Let $\mathscr{Q}$ be as in 2.2 (d) for $\mathfrak{A}$. Suppose $X \in \mathscr{S}_\omega(|\mathfrak{A}|)$ and $y \in |\mathfrak{B}| - |\mathfrak{A}|$. By [22, Theorem 3.1], in order to establish that $\mathfrak{A} \prec \mathfrak{B}$, it will suffice to show that there is an automorphism $f \supseteqq Id_X$ of $\mathfrak{B}$ such that $f(y) \in |\mathfrak{A}|$. To this end, we first choose $\mathfrak{A}_1 \in \mathscr{Q}$ so that $X \subseteqq |\mathfrak{A}_1|$ (by 2.2 (iii)). Next, by $\mathrm{VI}_\varkappa$, we find $\mathfrak{B}_1 \in \mathscr{S}_\varkappa^{\mathscr{M}}(\mathfrak{B})$ such that $|\mathfrak{A}_1| \cup \{y\} \subseteqq |\mathfrak{B}_1|$. By 2.2 (d), there exist $\mathfrak{A}_2 \in \mathscr{Q}$ and $g \supseteqq Id_{|\mathfrak{A}_1|}$ such that $\mathfrak{B}_1 \cong_g \mathfrak{A}_2$. Since $\mathfrak{B}$ is homogeneous of degree $\varkappa$, g can be extended to an automorphism f of $\mathfrak{B}$; and f is as desired.

Now assume the hypothesis as stated, and let $\bar{\bar{\mathfrak{B}}} = \lambda$. By 2.8 there is an $\mathscr{M}$-special system $\mathfrak{C}$ of power, say, $\beth_{\omega\lambda+}$, and $\mathfrak{C}$ is homogeneous of degree λ and universal. Since $\mathfrak{C}$ is universal we can in fact take $\mathfrak{C} \supseteqq \mathfrak{B}$. But then, by what we proved above, $\mathfrak{A} \prec \mathfrak{C}$ and $\mathfrak{B} \prec \mathfrak{C}$. It easily follows (cf. [22, Theorem 1.8 (iii)]) that $\mathfrak{A} \prec \mathfrak{B}$.[13]

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[13] Let $\mu = \min$ (cf. $\mathfrak{A}$, cf. $\mathfrak{B}$). 4.2 can be strenghtened to assert that $\mathfrak{B}$ is an elementary extension in the sense of the language L^μ having expressions of arbitrary length $< \mu$. The argument is the natural extension of that above. First one must extend in a straightforward way to L^μ [22, Theorem 3.1] (already extended by Tarski to the "weak second-order" language). A weaker form of 4.2 follows directly from [7, Theorem I]. (See [7] for definitions and references.)

Thus, for example, if $\mathscr{G}$ is the class of all groups, then by 4.2, there is a certain elementary type $\mathscr{H} \subseteq \mathscr{G}$ containing all non-denumerable $\mathscr{G}$-special systems. By 2.6 and 3.5 there exist in each power $\varkappa = \varkappa^* > \omega$ groups $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathscr{H}$ of power $\varkappa$ such that $\mathfrak{G}_1$ is $\mathscr{G}$-special while $\mathfrak{G}_2$ is special. We do not know whether or not $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are the same (up to isomorphism). However, the first author has found an example of an $\boldsymbol{EC}_\Delta$-class $\mathscr{M}$ with $\bar{\bar{I}}_{\mathscr{M}} = \omega$, having III, IV, V (and $\mathrm{VI}_{\omega+}$) such that the $\mathscr{M}$-special system of a certain power $> \omega$ is not special. On the other hand, we can prove:

THEOREM 4.3. *Suppose $\mathscr{M} \in \boldsymbol{EC}_\Delta$, $\mathscr{M}$ has* III, IV, V, *and* VI_ω, $\bar{\bar{I}}_{\mathscr{M}} < \omega$, *and $\mathscr{N}$ is the elementary type of $\mathscr{M}$-special systems. Then $\mathscr{N}$ is ω-categorical and model-complete; and hence, for any $\mathfrak{A} \in \mathscr{N}$, $\mathfrak{A}$ is $\mathscr{M}$-special if and only if $\mathfrak{A}$ is special.*

PROOF. By 2.8 there is a denumerable $\mathscr{M}$-homogeneous-universal system $\mathfrak{A}_0$. As is well-known, we can construct, for each $\mathfrak{B} \in \mathscr{M}_\omega$, a sentence $\sigma_{\mathfrak{B}}$ which is true in an arbitrary $\mathfrak{C} \in \mathscr{M}$ if and only if $\mathfrak{B} \in \mathscr{I}\mathscr{S}(\mathfrak{C})$. Similarly, if $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathscr{M}_\omega$ and $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ we easily construct a sentence $\sigma'_{\mathfrak{B}_1, \mathfrak{B}_2}$ which is true in a system $\mathfrak{C} \in \mathscr{M}$ if and only if, whenever $\mathfrak{B}_1 \cong_f \mathfrak{B}_1' \subseteq \mathfrak{C}$, there exists an isomorphism $g \supseteq f$ of $\mathfrak{B}_2$ into $\mathfrak{C}$. By 2.4 (b), (e), all these sentences are true in $\mathfrak{A}_0$, and all their models belonging to $\mathscr{M}_{\omega+}$ are isomorphic; thus $\mathscr{N}$ is ω-categorical. By 4.2 it follows that if $\mathfrak{A}, \mathfrak{B} \in \mathscr{N}_{\omega+}$ and $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A} \prec \mathfrak{B}$. This implies that $\mathscr{N}$ is model complete (since if $\mathfrak{A}', \mathfrak{B}' \in \mathscr{N}$, $\mathfrak{A}' \subseteq \mathfrak{B}'$, and $\mathfrak{A}' \nprec \mathfrak{B}'$, then $\mathrm{Th}((\mathfrak{B}', |\mathfrak{A}'|))$ would have a denumerable model). Hence, by 3.3, the last assertion of the theorem obviously holds if $\bar{\bar{\mathfrak{A}}} > \omega$. On the other hand, since $\mathscr{I}\mathscr{S}_\omega(\mathscr{N}^*) \subseteq \mathscr{I}\mathscr{S}_\omega(\mathfrak{A}_0^*)$, certainly the number of isomorphism types of $\mathscr{S}_\omega(\mathscr{N}^*)$ has power $\leqq \omega$. Hence, by 3.5, some and hence all denumerable members of $\mathscr{M}$ are special, as well as $\mathscr{M}$-special.

5. $\varkappa$-categorical elementary types.

A number of such elementary types are familiar, for example (for $\varkappa > \omega$) the class of all algebraically closed fields of characteristic zero: (For other examples, see, e.g., [13].) All of these exhibit in common a number of strong properties, but to establish corresponding general theorems appears to be difficult. For instance, each known $\mathscr{N} \in \boldsymbol{ET}$ which is $\varkappa$-categorical for some $\varkappa > \omega$ is so for all $\varkappa > \omega$; Łoś [13] raised the question whether this is a general theorem. Similarly, in known cases, every member of $\mathscr{N}$ is homogeneous, but no general theorem is known. Using the results of Section 3, we can make some (limited) progress on these problems. For the sake of simplicity, *in the remainder of the paper we consider only relational systems with countable index sets.*

From 3.2 and 2.10 (b), (c) follows at once

THEOREM 5.1. *If an elementary type* $\mathscr{N}$ *is* $\varkappa$*-categorical,* $\mathfrak{A} \in \mathscr{N}$*, and* $\overline{\overline{\mathfrak{A}}} = \varkappa$*, then* $\mathfrak{A}$ *is* λ*-homogeneous whenever* $\varkappa^{\underline{\lambda}} = \varkappa$.[14]

Thus, by 1.1 (g), (b), $\mathfrak{A}$ is μ^+-homogeneous if $\varkappa = 2^\mu$, $cf\varkappa$-homogeneous if $\varkappa = \varkappa^*$, and ω-homogeneous in any case. However, even if the GCH is assumed, 5.1 leaves open whether $\mathfrak{A}$ is (fully) homogeneous when $\varkappa$ is singular. The next theorem throws some light on this case, and also makes a slight contribution toward the problem of Łoś mentioned above.

THEOREM 5.2. *Suppose* $\mathscr{N} \in \boldsymbol{ET}$, $\varkappa = \beth_\delta$*, and there are arbitrarily large* $\xi < \delta$ *such that* $\mathscr{N}$ *is* $\beth_{\xi+1}$*-categorical. Then* $\mathscr{N}$ *is* $\varkappa$*-categorical, and its members of power* $\varkappa$ *are homogeneous.*

PROOF. It will suffice to show that an arbitrary member $\mathfrak{A}$ of $\mathscr{N}$ of power $\varkappa$ is homogeneous-universal. By the Löwenheim–Skolem theorem, it is clear that $\mathscr{N}_\varkappa^* \subseteq \mathscr{I}\mathscr{S}(\mathfrak{A}^*)$. Hence by 3.3, 3.4, and 2.4, we only need to show that, if $\mathfrak{B}_1 \prec \mathfrak{C}_1 \prec \mathfrak{A}$, $\mathfrak{B}_2 \prec \mathfrak{A}$, $\mathfrak{B}_1 \cong_f \mathfrak{B}_2$, and $\overline{\overline{\mathfrak{C}}}_1 < \varkappa$, then f can be extended to an isomorphism of $\mathfrak{C}_1$ onto an elementary subsystem of $\mathfrak{A}$. By hypothesis, $\mathscr{N}$ is $\beth_{\eta+1}$-categorical for some η such that $\overline{\overline{\mathfrak{C}}}_1 \leqq \beth_\eta$. By the Löwenheim–Skolem theorem, there exists $\mathfrak{A}' \prec \mathfrak{A}$ such that $\overline{\overline{\mathfrak{A}'}} = \beth_{\eta+1}$ and $\mathfrak{C}_1, \mathfrak{B}_2 \subseteq \mathfrak{A}'$. But by 5.1, $\mathfrak{A}'$ is homogeneous of degree $\beth_\eta{}^+$, so f can clearly be extended as desired.

Ehrenfeucht [5] has established the following result:

(8) $\left\{\begin{array}{l}\textit{If an elementary type } \mathscr{N} \textit{ is } \varkappa\textit{-categorical for some } \varkappa, \textit{ then the set of} \\ \textit{isomorphism types of } \mathscr{S}_\omega(\mathscr{N}^*) \textit{ has power } \leqq \omega.\end{array}\right.$

From (8) and 3.5 follows at once:

THEOREM 5.3. *If an elementary type* $\mathscr{N}$ *is* $\varkappa$*-categorical for some* $\varkappa$*, then there is a denumerable homogeneous-universal system* $\mathfrak{A} \in \mathscr{N}$.[15]

(8) may also be applied in conjunction with 5.1. For example, suppose $\varkappa = \varkappa^\omega$, $\mathscr{N}$ is $\varkappa$-categorical, $\mathfrak{A} \in \mathscr{N}$, $X \subseteq |\mathfrak{A}|$, and $\overline{\overline{X}} \leqq \omega$. Then, by 5.1, the elementary type $\mathscr{N}'$ of $(\mathfrak{A}, x)_{x \in X}$ is also $\varkappa$-categorical, and hence, by (8), the set of isomorphism types of $\mathscr{S}_\omega(\mathscr{N}'^*)$ has power $\leqq \omega$.[16]

[14] For $\varkappa = \omega$ this is immediate from Ryll–Nardzewski's characterization of ω-categorical elementary types (cf. [24]).

[15] That $\mathscr{N}$ has a denumerable universal member (if $\varkappa = 2^\lambda$) was already shown in [4].

[16] We noted this originally only for $\varkappa = 2^\lambda$. A remark of Dana Scott showed us the extension to $\varkappa = \varkappa^\omega$, which in turn led us to the general form of 2.10 (b).

6. A Löwenheim-Skolem theorem for two cardinals.

LEMMA 6.1. (a) *If* $\mathscr{N} \in \boldsymbol{ET}$ *and* $J \subseteq I_{\mathscr{N}}$, *then there is a denumerable, homogeneous system* $\mathfrak{A} \in \mathscr{N} \upharpoonright J$.

(b) *If* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are homogeneous and denumerable*, $\mathfrak{A} \equiv \mathfrak{A}'$, *and* $\mathscr{I}\mathscr{S}_\omega(\mathfrak{A}^*) = \mathscr{I}\mathscr{S}_\omega(\mathfrak{A}'^*)$, *then* $\mathfrak{A} \cong \mathfrak{A}'$. (Cf. footnote [5]).

(c) *If* $\mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \ldots \prec \mathfrak{A}_n \prec \ldots$ *and each* $\mathfrak{A}_n$ *is homogeneous and denumerable, then so is* $\bigcup(\mathfrak{A}_n / n \in \omega)$.

PROOF. (a) By 3.3, 1.2 (b), 2.6, and 2.8′. (b) By 3.3 and 2.3 (b). (c) easily follows from 3.3 and 2.4 (d).

With the aid of 6.1 we can establish the following result (announced in [25]):

THEOREM 6.2. *If* $R_j^{\mathfrak{A}}$ *is singulary and* $\omega \leq \overline{\overline{R}}_j^{\mathfrak{A}} < \overline{\overline{\mathfrak{A}}}$, *then there exists a system* $\mathfrak{B} \equiv \mathfrak{A}$ *such that* $\overline{\overline{\mathfrak{B}}} = \omega_1$ *and* $\overline{\overline{R}}_j^{\mathfrak{B}} = \omega_0$.

PROOF. By the Löwenheim–Skolem theorem, there exists $\mathfrak{B}_1 \prec \mathfrak{A}$ such that $R_j^{\mathfrak{A}} \subseteq |\mathfrak{B}_1|$ and $\overline{\overline{\mathfrak{B}}}_1 = \overline{\overline{R}}_j^{\mathfrak{A}}$. By (7) there exists a system $(\mathfrak{B}_2, C) \equiv (\mathfrak{A}, |\mathfrak{B}_1|)$ such that, for some f, $\mathfrak{B}_2 \cong_f \mathfrak{B}_2 \upharpoonright C$. By 6.1 (a), there exists $(\mathfrak{A}_1, A_0, g) \equiv (\mathfrak{B}_2, C, f)$ such that $\mathfrak{A}_1$ is denumerable and homogeneous. Put $\mathfrak{A}_0 = \mathfrak{A}_1 \upharpoonright A_0$. Then $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are isomorphic, $\mathfrak{A}_1$ is a proper elementary extension of $\mathfrak{A}_0$, and $R_j^{\mathfrak{A}_0} = R_j^{\mathfrak{A}_1}$, because all of these facts are expressable in sets of elementary sentences and so are retained from $(\mathfrak{A}, |\mathfrak{B}_1|)$ and $(\mathfrak{B}_2, C, f)$.

By recursion we shall define systems $\mathfrak{A}_\xi \cong \mathfrak{A}_0$ such that $R_j^{\mathfrak{A}_\xi} = R_j^{\mathfrak{A}_0}$, $\mathfrak{A}_\xi \prec \mathfrak{A}_\eta$, and $\mathfrak{A}_\xi \neq \mathfrak{A}_\eta$, if $\xi < \eta < \omega_1$. Indeed, since $\mathfrak{A}_\xi \cong \mathfrak{A}_0$, we may take $\mathfrak{A}_{\xi+1}$, to be a system related to $\mathfrak{A}_\xi$ as $\mathfrak{A}_1$ is to $\mathfrak{A}_0$. If $\delta < \omega_1$, let $\mathfrak{A}_\delta = \bigcup(\mathfrak{A}_\eta / \eta < \delta)$. By 6.1 (c), $\mathfrak{A}_\delta$ is homogeneous (and denumerable). Moreover, $\mathscr{I}\mathscr{S}_\omega(\mathfrak{A}_\delta^*) = \mathscr{I}\mathscr{S}_\omega(\mathfrak{A}_0^*)$, because any member of $\mathscr{S}_\omega(\mathfrak{A}_\delta^*)$ belongs to $\mathscr{S}_\omega(\mathfrak{A}_\eta^*)$, for some $\eta < \delta$, and by hypothesis $\mathfrak{A}_\eta \cong \mathfrak{A}_0$. Consequently, by 6.1 (b), $\mathfrak{A}_\delta \cong \mathfrak{A}_0$. It is now clear that the system $\mathfrak{B} = \bigcup(\mathfrak{A}_\xi / \xi < \omega_1)$ has the desired properties.

In general, we may ask,

(9) *If the hypothesis of* 6.2 *holds and* $\varkappa < \lambda$, *must there exist a system* $\mathfrak{B} \equiv \mathfrak{A}$ *with* $\overline{\overline{\mathfrak{B}}} = \lambda$ *and* $\overline{\overline{R}}_j^{\mathfrak{B}} = \varkappa$?

Raphael Robinson discovered the following examples showing that the answer is sometimes negative: Let X be any infinite set; let Exy if and only if $x \in X$, $y \subseteq X$, and $x \in y$; and put $\mathfrak{A} = \langle X \cup S(X), E, X \rangle$. Then (by the axiom of extensionality), in any system $\langle U, F, Y \rangle \equiv \mathfrak{A}$, we must have $\overline{\overline{U}} \leq 2^{\overline{\overline{Y}}}$. By iterating this construction any finite number of times, analogous situations are obtained having instead the inequalities $\overline{\overline{U}} \leq 2^{2^{\overline{\overline{Y}}}}$, $\overline{\overline{U}} \leq 2^{2^{2^{\overline{\overline{Y}}}}}$, etc.

On the other hand, these examples are, roughly speaking, the only limitations we know on positive answers to (9) (say, assuming the GCH). The Löwenheim–Skolem theorem obviously implies that we can lower $\bar{\bar{\mathfrak{A}}}$ whole keeping $\bar{\bar{R}}_j^{\mathfrak{A}}$ fixed. Recently, by an interesting new method, Chang and Keisler [2] have obtained positive answers to (9) in some additional cases (i.e., other than that just mentioned and 6.2). For many cases, however, the answer is still unknown. For example it is not known whether ω_0 and ω_1 in 6.2 can be replaced by $\varkappa$ and $\varkappa^+$.

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