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EXTENSION OF POSITIVE LINEAR FUNCTIONALS

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Introduction.

Let P be a convex cone in a real locally convex Hausdorff topological vector space E, and let F be a closed support space of P. Suppose that $P \cap F$ admits a *supplement* Q (Definition 1) in P such that Q is a closed, locally compact convex cone. Then we prove (Proposition 2) that every positive and continuous linear functional defined on F admits a positive and continuous extension to E. A considerable part of the rest of the present paper is concerned with the problem of finding conditions which ensure that $P \cap F$ admits such a supplement Q. When P is locally compact we use the Krein–Milman theorem to obtain a condition expressed by topological properties of the set of extreme points of a base of P. We use this condition to prove that, among the closed finite dimensional convex cones, only the pyramids have the property that such a supplement exists for all closed support spaces. In § 2 we attack the supplement problem in another way. First we prove that if $P \cap F$ is locally compact and P satisfies a decomposition property, then there exists for every $p \in P$ a $q \in P \cap F$ given as the greatest element in $P \cap F$ which is less than p. The mapping $p \to q$ is a projection, and the inverse image of zero is a supplement of $P \cap F$. With an additional assumption on P we prove that this supplement is closed. A consequence of these results is that if P is a locally compact "topological semi-lattice", then $P \cap F$ always has a closed supplement. Motivated by the above mentioned results we deal in § 3 with the problem of extending a positive and continuous linear functional defined on a closed subspace M to the closed support space generated by M. One of our results here states that every finite dimensional pyramid has the property that its linear sum with any closed subspace is closed. This gives a counter-example to a conjecture of A. Bastiani [1, p. 283].

Notation. E and P shall always be as above, and all sets considered shall be subsets of E. The line, the segment and the open segment between two different points x and y shall be the sets consisting of all points of the form $\lambda x + (1-\lambda)y$, where respectively $-\infty < \lambda < \infty$, $0 \leqq \lambda \leqq 1$,

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and $0<\lambda<1$. A subspace is always a linear subspace, a linear variety is a translation of a subspace. If K is a convex subset and $x \in K$, then the facet of x in K, $F_x(K)$, is the set consisting of x and of points $y \neq x$ in K such that the line between x and y contains an open segment contained in K and containing x. The set of all extreme points of K will be denoted $e(K)$. The linear variety L is called a support variety of K if L intersects K and if L contains the facet of x in K whenever $x \in K \cap L$, or equivalently if L contains every open segment contained in K and intersecting L. A support variety which is a subspace will be called a support space. All convex cones considered shall contain 0, and have 0 as vertex. The convex cone P will be called proper if $p, -p \in P$ implies that $p=0$. Otherwise we use the same notation as in [8].

1. Use of a general extension theorem.

Proposition 1. *Suppose that f is a positive and continuous linear functional on the subspace F of E, and that*

$$\overline{(f^{-1}(0)+P)} \cap F \subset \overline{(f^{-1}(0)+P) \cap F}\,. \tag{1}$$

Then f admits a positive and continuous extension to E.

Proof. According to [8, Theorem 2] it will suffice to show that f is non-negative on the left hand side of (1). Since f is continuous on F we get from (1)

$$\begin{aligned} f\big(\overline{(f^{-1}(0)+P)} \cap F\big) &\subset f\big(\overline{(f^{-1}(0)+P) \cap F} \cap F\big) \\ &\subset \overline{f\big((f^{-1}(0)+P) \cap F\big)}\,. \end{aligned}$$

This gives the desired result since F is a positive linear functional and therefore non-negative on

$$(f^{-1}(0)+P) \cap F\,.$$

We shall say that a linear functional g on E is *strictly* positive provided $g(p)>0$ whenever $p \in P \sim \{0\}$.

Corollary. *Suppose that P and F are closed and that P is locally compact. Then a strictly positive and continuous linear functional f on F admits a positive and continuous extension to E.*

Proof. The subspace $f^{-1}(0)$ is closed. Since f is strictly positive, we have $f^{-1}(0) \cap P=\{0\}$. Hence we conclude from a theorem of V. L. Klee [9, (7, 5), p. 452] that $f^{-1}(0)+P$ is closed, and thus (1) is satisfied.

The following lemma is a slight extension of the above cited result of V. L. Klee.

LEMMA 1. *Let S be a locally compact closed convex cone and F a closed subspace such that $F \cap S = \{0\}$. If M is a subset of F, then*

(i) $\overline{M+S} = \bar{M}+S$

(ii) $\overline{(M+S)} \cap F = \bar{M}$.

PROOF. Let $x \in \overline{M+S}$. Then there exist nets $\{m_\gamma\} \subset M$ and $\{s_\gamma\} \subset S$ such that $m_\gamma + s_\gamma \to x$. Since $\{m_\gamma\} \subset F$, we conclude from the proof of [9, (7, 5), p. 452] that there exists a subnet $\{s_\iota\}$ of $\{s_\gamma\}$ such that $s_\iota \to s \in S$. Since $m_\iota + s_\iota \to x$, we infer that $m_\iota \to m \in \bar{M}$, and thus $x = m+s \in \bar{M}+S$. Hence $\overline{M+S} \subset \bar{M}+S$. The converse inclusion being clear, we have proved (i). Hence we have

$$\overline{(M+S)} \cap F = (\bar{M}+S) \cap F = \bar{M}+S \cap F = \bar{M} .$$

DEFINITION 1. Let Q and S be two convex cones such that Q and S are subsets of P, $Q \cap S = \{0\}$, and $Q+S=P$. Then we shall say that Q and S are *supplementary* subcones of P and that S is a *supplement* of Q in P.

PROPOSITION 2. *Suppose that the subspace F is closed and that there exists a closed locally compact convex cone S such that S is a supplement of $P \cap F$ in P. Then a positive and continuous linear functional f on F admits a positive and continuous extension to E.*

PROOF. According to Proposition 1 we are through if we can prove that

$$\overline{(f^{-1}(0)+P)} \cap F \subset \overline{(f^{-1}(0)+P) \cap F} .$$

Putting $M = f^{-1}(0) + P \cap F$, we have $f^{-1}(0)+P = M+S$. Since $S \cap F = \{0\}$ and $M = (f^{-1}(0)+P) \cap F$, the desired result follows from Lemma 1 (ii).

PROPOSITION 3. *Let F and S be as in Proposition 2. Then $F+P$ is closed.*

PROOF. We have

$$F+P = F+F \cap P+S = F+S ,$$

and hence the result follows from Lemma 1.

PROPOSITION 4. *If F is a support space of P, then $P \cap F$ admits a supplement in P. Conversely, if P is proper and F is a subspace such that $P \cap F$ admits a supplement in P, then F is a support space of P.*

PROOF. To prove the first assertion we shall show that the set $S = (P \sim F) \cup \{0\}$ is a supplement of $P \cap F$ in P. This is clear if we can prove that S is a convex cone. It is easy to see that if $\lambda \geqq 0$ and $s \in S$, then $\lambda s \in S$. Therefore it suffices to prove that

$$\text{if} \quad s,s' \in S, \quad \text{then} \quad x = \tfrac{1}{2}(s+s') \in S .$$

We may and shall suppose that $s \neq s'$ and $s,s' \neq 0$. Hence $s,s' \in P \sim F$. Suppose $x \notin S$. Then $x \in F$, and since F is a support space we obtain the contradiction $s,s' \in F$. Suppose conversely that P is proper and that Q is a supplement of $P \cap F$ in P. Let a,b be two different points in P, and suppose that $x = \lambda a + (1-\lambda)b \in F$, where $0 < \lambda < 1$. We then have to prove that $a,b \in F$. We can find $p,p' \in P \cap F$ and $q,q' \in Q$ such that $a = p+q$, $b = p'+q'$. Hence

$$x = \lambda p + (1-\lambda)p' + \lambda q + (1-\lambda)q' .$$

Since x, $\lambda p + (1-\lambda)p' \in F$ we get

$$\lambda q + (1-\lambda)q' \in Q \cap F = \{0\} .$$

From this it follows that

$$q = (\lambda-1)\lambda^{-1}q' \in Q \cap -Q \subset P \cap -P = \{0\} .$$

Hence $q = q' = 0$,and therefore $a = p$ and $b = p'$.

Definition 2. Let K be a convex set and suppose that A and B are convex subsets of K such that $A \cap B = \emptyset$ and such that K is the convex hull of $A \cup B$. Then we shall say that A and B are *complementary*, and that B is a *complement* of A in K.

Definition 3. If K is a non-empty convex subset of P such that $P = [0,\infty\rangle K$ and K is contained in a hyperplane not containing zero, then K is called a *base* of P.

We remark that if P admits a base, then P is proper. We also observe that if P is proper, then P is locally compact if and only if P admits a compact base, and that P is closed in this case [11, p. 341].

Proposition 5. *If F is a subspace and K is a base of P, then $P \cap F$ admits a supplement in P if and only if $K \cap F$ admits a complement in K.*

Proof. Let B be a complement of $K \cap F$ in K. If $B = \emptyset$, then $K \subset F$ and therefore $P \subset F$, so that $\{0\}$ is a supplement of $P \cap F$ in P. In case $B \neq \emptyset$ it is easy to see that $[0,\infty\rangle B$ is a supplement of $P \cap F$ in P. Suppose conversely that S is a supplement of $P \cap F$ in P. We shall show that $K \cap S$ is a complement of $K \cap F$ in K, and in order to do this we only need to verify that K is the convex hull of $(K \cap F) \cup (K \cap S)$. Let $k \in K$. Then $k = p_1 + p_2$ where $p_1 \in P \cap F$, $p_2 \in S$. We can find $\lambda_1, \lambda_2 \geqq 0$ and $k_1, k_2 \in K$ such that $p_1 = \lambda_1 k_1$, $p_2 = \lambda_2 k_2$. If $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$k \in (K \cap S) \cup (K \cap F)$. We can therefore assume that $\lambda_1, \lambda_2 > 0$. Then $k_1 = \lambda_1^{-1} p_1 \in K \cap F$, $k_2 = \lambda_2^{-1} p_2 \in K \cap S$, and $k = \lambda_1 k_1 + \lambda_2 k_2$. Consider

$$k' = \frac{\lambda_1}{\lambda_1 + \lambda_2} k_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} k_2 .$$

Then k' lies in the convex hull of $(K \cap S) \cup (K \cap F)$ and $k = (\lambda_1 + \lambda_2) k'$. If $k \neq k'$, then the line joining k and k' contains zero and this contradicts the fact that K is contained in a hyperplane not containing zero.

Corollary. *If K is a compact base of P, then $P \cap F$ admits a closed locally compact supplement in P if and only if $K \cap F$ admits a closed complement in K.*

Proof. This follows from the preceding reasoning and the fact that if B is a compact convex set not containing zero, then $[0, \infty\rangle B$ is closed and locally compact [11, p. 341].

Proposition 6. *Let K be a convex compact set, and suppose that M is a closed support variety of K. Then $K \cap M$ admits a closed complement in K if and only if*

$$\overline{e(K) \sim M} \cap M = \emptyset . \tag{2}$$

Proof. Suppose that B is a closed complement of $K \cap M$ in K. Since K is the convex hull of $(K \cap M) \cup B$, we conclude from a theorem of Milman [3, Chap. 2, p. 84] that $e(K) \subset (K \cap M) \cup B$. Hence $e(K) \sim M \subset B$, and so

$$\overline{e(K) \sim M} \cap M \subset B \cap M = \emptyset .$$

Suppose conversely that (2) is satisfied. Let C be the closed convex hull of $e(K) \sim M$. Since $e(K) \subset C \cup (K \cap M)$, it follows from the Krein–Milman theorem that K is the closed convex hull of $C \cup (K \cap M)$. Now, since C and $K \cap M$ are compact convex sets, it follows by a proposition in [3, Chap. 2, p. 80] that the convex hull of $C \cup (K \cap M)$ is compact, and hence equals K. Therefore C will be a closed complement of $K \cap M$ if we can verify that $C \cap M = \emptyset$. Again we conclude from the theorem af Milman that $\overline{e(C)} \subset \overline{e(K) \sim M}$. Since C is a subset of K we have on the other hand

$$e(K) \sim M \subset e(K) \cap C \subset e(C) .$$

Hence $\overline{e(C)} = \overline{e(K) \sim M}$, and therefore $\overline{e(C)} \cap M = \emptyset$. The conclusion then follows from the following lemma, and the fact that if $C \cap M \neq \emptyset$, then $e(C \cap M) \neq \emptyset$.

Lemma 2. *Let K be a convex set, M a support variety of K, and C a convex subset of K. Then*

$$e(M \cap C) = M \cap e(C) . \tag{3}$$

PROOF. We have

$$M \cap e(C) = M \cap C \cap e(C) \subset e(M \cap C) .$$

Suppose on the other hand that $x \in e(M \cap C)$. If x is not an extreme point of C, we can find two different points $a,b \in C$ such that x is contained in the open segment between a and b. Since $a,b \in K$, $x \in M \cap K$ and M is a support variety of K, it follows that $a,b \in M \cap C$. This contradicts the fact that x is an extreme point of $M \cap C$.

COROLLARY 1. *Suppose that P admits a compact base K and that F is a closed support space of P. Then $P \cap F$ admits a closed locally compact supplement in P, if and only if*

$$\overline{e(K) \sim F} \cap F = \emptyset . \tag{4}$$

PROOF. If $P \cap F = \{0\}$, the assertion is obvious. In case $P \cap F \neq \{0\}$ the conclusion follows from Proposition 6 and the corollary of Proposition 5, together with the fact that F in this case is also a support space of K.

A half-line extending from zero and passing through a point of P different from zero is called a *generatrix* of P. The generatrix is called *extreme* if the line generated by it is a support space of P. We remark that if P admits a base K, then a generatrix is extreme if and only if its intersection with K is an extreme point of K [11, p. 341].

COROLLARY 2. *Suppose that P admits a compact base K. Then an extreme generatrix G of P admits a closed locally compact supplement in P if and only if its intersection p with K is an isolated point of $e(K)$.*

PROOF. If F is the line generated by G, then $P \cap F = G$ and

$$\overline{e(K) \sim F} \cap F = \overline{e(K) \sim \{p\}} \cap \{p\} .$$

The result then follows from Corollary 1.

COROLLARY 3. *Suppose that P admits a compact base K, that $e(K)$ is closed, and that F is a closed support space of P. Then $P \cap F$ admits a closed locally compact supplement in P if and only if $F \cap e(K)$ is an open-closed subset of $e(K)$ in the induced topology.*

PROOF. Since $e(K)$ is closed, the condition (4) is equivalent with

$$\overline{e(K) \sim F \cap e(K)} \cap F \cap e(K) = \emptyset .$$

This condition expresses that $F \cap e(K)$ is an open subset of $e(K)$ in the induced topology. On the other hand, $F \cap e(K)$ is always closed in this topology since F is closed.

COROLLARY 4. *Suppose that P admits a compact base K and that $e(K)$ is closed. Then a finite intersection of closed support spaces, each having the property that its intersection with P admits a closed locally compact supplement in P, is a support space with the same property.*

PROOF. This follows immediately from Corollary 3.

DEFINITION 4. A convex cone P in E is said to have the *supplement property in E* if $P \cap F$ admits a closed supplement in P whenever $F \subset E$ is a closed support space of P.

PROPOSITION 7. *If P is locally compact and has the supplement property in E, then every positive and continuous linear functional defined on a closed support space of P admits a positive and continuous extension to E.*

PROOF. This is an immediate consequence of Proposition 2.
In § 2 we shall show that some "semi-lattice" cones have the supplement property. At this point we deal with the finite dimensional case.

LEMMA 3. *Suppose that P has the supplement property in E and that F is a closed support space of P. Then $P \cap F$ has the supplement property in F.*

PROOF. Let $L \subset F$ be a closed support space of $P \cap F$. Then L is a closed support space of P. Hence $P \cap L$ admits a closed supplement Q in P. The cone $Q \cap F$ is a closed supplement of $(P \cap F) \cap L$ in $P \cap F$.

PROPOSITION 8. *Suppose that P is a proper, closed, finite dimensional convex cone in E. Then P has the supplement property in E, if and only if P has a finite number of extreme generatrices.*

PROOF. First we observe that, since P is locally compact, P admits a compact base K. If P has a finite number of extreme generatrices, then $e(K)$ is a finite set. Hence it follows from Proposition 6, Corollary 1 that P has the supplement property in E. Suppose conversely that P has the supplement property in E. We use induction with respect to the dimension of P. Assume therefore that whenever Q has dimension $\leqq n$ and Q has the supplement property in some vector space, then Q has a finite number of extreme generatrices. Suppose that P has dimension $n+1$, and that P has an infinite number of extreme generatrices. This means that $e(K)$ is an infinite set. Hence we can find a $k \in \overline{e(K)}$ such that every neighborhood of k contains a point from $e(K)$ different from k. It is easy to see that every extreme point of K is a boundary point of K. Hence k belongs to the boundary of K. We may and shall assume that K is contained in a linear variety H not containing zero, and of dimension n. It is well known [6, p. 20] that there exists a support variety L of K such that k belongs to L, and such that L is contained in H and has dimension $n-1$. Let F be the vector space generated by L. Then F is a support space of P. For let $x=\lambda a+(1-\lambda)b \in F$, where $a,b \in P$ and $0<\lambda<1$. If a or $b=0$, then $a,b \in F$. We can therefore assume $a,b \neq 0$. Then there exists $a',b' \in K$ and $\alpha,\beta>0$, such that $a=\alpha a'$, $b=\beta b'$. Let $x'=(\lambda\alpha+(1-\lambda)\beta)^{-1}x$. Here we have that x' belongs to the open segment between a' and b', and that $x' \in H \cap F$. Since $H \cap F = L$ it follows that $a',b' \in L$ and consequently that $a,b \in F$. By Lemma 3, $P \cap F$ has the supplement property in F. Since $P \cap F$ has dimension $\leqq n$ it follows from the induction assumption that $P \cap F$ has only a finite number of extreme generatrices. Since $K \cap L$ is a base of $P \cap F$, it further follows that $e(K \cap L)$ is finite. By Lemma 2 we conclude that $L \cap e(K)$ is finite. Since P has the supplement property in E, it follows from Proposition 6, Corollary 1 that

$$\emptyset = \overline{e(K) \sim F \cap e(K)} \cap F = \overline{e(K) \sim L \cap e(K)} \cap L . \tag{5}$$

This is a contradiction, since every neighborhood of k contains an infinite number of points from $e(K)$, such that k belongs to the right hand side of (5).

EXAMPLES

(i) Let X be a compact space, let $\mathscr{M}(X)$ denote the vector space of all Radon measures on X, $\mathscr{M}^+(X)$ the convex cone of all positive Radon measures, and $\mathscr{M}_1{}^+(X)$ the convex set of all positive Radon measures μ such that $\mu(1)=1$. We equip $\mathscr{M}(X)$ with the vague topology. Then $\mathscr{M}_1{}^+(X)$ is a compact base of $\mathscr{M}^+(X)$ and its set of extreme points is the set $\{\varepsilon_x : x \in X\}$, where ε_x is the measure with the unit mass placed at x [11, p. 337]. This set is homeomorphic with X by the correspondence $x \to \varepsilon_x$ and is thus compact. Now let $F \subset \mathscr{M}(X)$ be a closed support space of $\mathscr{M}^+(X)$. From Proposition 6, Corollary 3 it follows that $F \cap \mathscr{M}^+(X)$ admits a closed supplement in $\mathscr{M}^+(X)$ if and only if the set $\{x \in X : \varepsilon_x \in F\}$ is open-closed in X.

(ii) Let $\mathscr{F}$ denote the vector space of all real valued functions on $[0,1]$, $\mathscr{I}$ the convex cone consisting of all non-negative, increasing functions on $[0,1]$, and let $\mathscr{I}_1$ denote the convex set of those $f \in \mathscr{I}$ such that $f(1)=1$. We equip $\mathscr{F}$ with the topology of pointwise convergence. Then $\mathscr{I}_1$ is a compact base of $\mathscr{I}$, and $e(\mathscr{I}_1)$ is compact and consists of all functions of the form f_A, where

$$f_A(x) = \begin{cases} 0, & x \in A \\ 1, & x \notin A \end{cases} .$$

and where either $A=[0,a]$ with $0\leqq a<1$, or $A=[0,a\rangle$ with $0\leqq a\leqq 1$ (and where $[0,0\rangle=\emptyset$) [5, p. 240]. We identify $e(\mathscr{I}_1)$ with the set consisting of all such intervals A, by identifying f_A with A. Now we observe that whenever $0\leqq x\leqq y\leqq 1$ then the set of all A such that $[0,x]\subset A\subset[0,y\rangle$ is open, and that the same is true for the set consisting of the one element $[0,0\rangle$. We want to show that every closed hyperplane F which is a support space of $\mathscr{I}$ has the property that $\mathscr{I}\cap F$ admits a closed supplement in $\mathscr{I}$. If we have proved this, we also know by Proposition 6, Corollary 4 that a finite intersection of such hyperplanes has this property. There exists a positive and continuous linear functional Φ on $\mathscr{F}$ such that $F=\Phi^{-1}(0)$. According to [3, Chap. 4, p. 75] we can find points $0\leqq x_1\leqq\ldots\leqq x_n\leqq 1$, and real numbers $\varphi_1,\ldots,\varphi_n$ such that

$$\Phi(f) = \sum_{i=1}^{n} \varphi_i f(x_i), \qquad \forall f\in\mathscr{F} .$$

Let $0\leqq k_1<\ldots<k_r\leqq n$ be those numbers k such that

$$\sum_{i>k} \varphi_i = 0 .$$

In particular we have $k_r=n$. Now one finds that f_A belongs to F if and only if either $[0,x_{k_j}]\subset A\subset[0,x_{k_j+1}\rangle$ for some $j=1,\ldots,r$, or in case $k_1=0$, $A=[0,0\rangle$. (By definition $x_0=0$ and $x_{n+1}=1$.) Hence the set $F\cap e(\mathscr{I}_1)$ is open-closed in $e(\mathscr{I}_1)$, and thus, by Proposition 6, Corollary 3, $F\cap\mathscr{I}$ admits a closed supplement in $\mathscr{I}$. We can use this result in the following situation: Let $y_1,\ldots,y_m\in[0,1]$ and $\psi_1,\ldots,\psi_m\in R$ be given, and define Ψ on $\mathscr{F}$ by

$$\Psi(f) = \sum_{i=1}^{m} \psi_i f(y_i), \quad f\in\mathscr{F} .$$

Suppose that $\Psi(f)\geqq 0$ whenever $f\in\mathscr{I}\cap F$. Then the restriction of Ψ to F is a positive and continuous linear functional, and admits therefore, by Proposition 2, a positive and continuous extension $\overline{\Psi}$ to $\mathscr{F}$. Since $\overline{\Psi}$ and Ψ coincide on $F=\Phi^{-1}(0)$, there exists a real number α such that $\overline{\Psi}=\Psi-\alpha\Phi$. In other words: If

$$\sum_{i=1}^{m} \psi_i f(y_i) \geqq 0$$

whenever $f\in\mathscr{I}$ and

$$\sum_{i=1}^{n} \varphi_i f(x_i) = 0 ,$$

then there exists a real α such that

$$\sum_{i=1}^{m} \psi_i f(y_i) \geqq \alpha \sum_{i=1}^{n} \varphi_i f(x_i), \qquad \forall f\in\mathscr{I} .$$

In concluding we remark that the condition of being positive imposed on Φ is equivalent with the condition

$$\sum_{i=k}^{n} \varphi_i \geqq 0, \qquad k = 1,\ldots,n .$$

This can be proved either by an elementary induction argument, or by appealing to the Krein–Milman theorem.

2. Cones with a decomposition property.

We recall that if $x,y \in E$, then $x \geqq y$ means that $x-y \in P$. In the sequel a support space shall always mean a support space of P.

PROPOSITION 8. *Let F be a subspace of E. Then F is a support space if and only if $0 \leqq q \leqq p \in F$, implies that $q \in F$.*

PROOF. Suppose that F has the stated property. Let $a,b \in P$ and $x=\lambda a+(1-\lambda)b \in F$, where $0<\lambda<1$. Since $0 \leqq \lambda a \leqq x$, we obtain $\lambda a \in F$. Therefore $a \in F$, and in the same way $b \in F$. Hence F is a support space. The converse statement is a consequence of the following.

LEMMA 4. *If $0 \leqq q \leqq p$, then q belongs to the facet $F_p(P)$ of p in P.*

PROOF. The case $q=p$ being trivial, we may assume $q \neq p$. Let $a=2p-q$ and $b=\frac{1}{2}(p+q)$. Then $a,b \in P$, $a \neq b$ and a,b belong to the line joining p and q. Since $p=\frac{1}{3}a+\frac{2}{3}b$ we have $q \in F_p(P)$.

COROLLARY. *Let M be a subspace of E. Then the set $s(M)$ defined by*

$$s(M) = \{x \in E : m \leqq x \leqq m' \text{ for some } m,m' \in M\}$$

is the support space generated by M.

PROOF. By Proposition 8 it is clear that $s(M)$ is a support space containing M. Let F be a support space containing M and let $m \leqq x \leqq m'$, with $m,m' \in M$. Then $0 \leqq x-m \leqq m'-m \in F$. Hence $x-m \in F$ and $x=(x-m)+m \in F$.

DEFINITION 5. The convex cone P has the *decomposition* property if $p+q=r+s$, where $p,q,r,s \in P$, implies that there exist $a,b,c,d \in P$ such that $a+b=p$, $c+d=q$, $a+c=r$, $b+d=s$.

We observe that the argument given in [4, p. 20] shows that if P is proper and satisfies

(i) $\inf\{a,b\}$ exists in P whenever $a,b \in P$,

(ii) $\inf\{a+b,a+c\}=a+\inf\{b,c\}$ whenever $a,b,c \in P$,

in the induced ordering, then P has the decomposition property. We shall call P a *semi-lattice cone* if P is proper and satisfies the conditions (i) and (ii) above. (P is "une monoide semiréticulé" [2, p. 24].)

LEMMA 5. *Suppose that P has the decomposition property and that F is a support space of P. Then the following is true. If $p,q \in P \cap F$ and $p,q \leqq u$ for some $u \in P$, then there exists a $v \in P \cap F$ such that $p,q \leqq v \leqq u$.*

PROOF. Let $p_0 = u-p$ and $q_0 = u-q$. Then $p_0, q_0 \in P$ and $u = p+p_0 = q+q_0$. By the decomposition property there exist $a,b,c,d \in P$ such that $a+b=p$, $c+d=p_0$, $a+c=q$, $b+d=q_0$. Since $p,q \in F \cap P$ and $a,b \leqq a+b=p$, $c \leqq a+c=q$, we have $a,b,c \in F \cap P$. Hence $v=a+b+c \in F \cap P$. Since $p,q \leqq v \leqq a+b+c+d=p+p_0=u$, the proof is finished.

When $p,q \in P$ and $p \leqq q$ we denote the set of all x such that $p \leqq x \leqq q$ by $[p,q]$. We have $[p,q]=(p+P)\cap(q-P)$ and hence, $[p,q]$ is closed if P is closed.

PROPOSITION 9. *Suppose that $p \in P$, where P is proper, closed and has the decomposition property. Let F be a support space of P such that $P \cap F$ is locally compact. Then the set of all $q \in P \cap F$ such that $q \leqq p$ has a unique greatest element.*

PROOF. Since P is proper, the uniqueness is obvious. To prove the existence we first note that $[0,p]\cap F$ is closed since $P \cap F$ is a proper locally compact convex cone, and therefore closed. Hence $[0,p]\cap F$ is a locally compact, convex set which contains zero. If we can prove that $[0,p]\cap F$ contains no ray issuing from zero, then it follows from [10, p. 736] that $[0,p]\cap F$ is compact. Let $0 \neq y \in [0,p]\cap F$. If $ny \in [0,p]\cap F$ for all $n=1,2,\ldots$, then $y-n^{-1}p \leqq 0$ for all n. Since P is closed, we obtain the contradiction $0<y\leqq 0$. Hence $[0,p]\cap F$ is compact. If now $v_1,\ldots,v_n \in [0,p]\cap F$, then it follows by induction from Lemma 5, that there exists a $v \in [0,p]\cap F$ such that $v_1,\ldots,v_n \leqq v$. This implies that the family $\{[v,p]\cap F : v \in [0,p]\cap F\}$ has the finite intersection property. Since each $[v,p]\cap F$ is a closed subset of the compact set $[0,p]\cap F$, there exists an element $q \in [0,p]\cap F$ such that $q \in [v,p]\cap F$ whenever $v \in [0,p]\cap F$. This q is the desired greatest element.

In the sequel p^F shall denote this unique greatest element of $[0,p]\cap F$, and we shall denote the mapping $p \to p^F$ from P to $P \cap F$ by Φ. A *projection* on P is a mapping T from P to P such that $T^2=T$, and such that $T(\lambda p+\mu q)=\lambda T(p)+\mu T(q)$ whenever $p,q \in P$, $\lambda,\mu \geqq 0$. T is called *order preserving* if $p \leqq q$ implies $T(p) \leqq T(q)$.

PROPOSITION 10. *Let P and F be as in Proposition 9. Then the mapping $\Phi : p \to p^F$ is an order preserving projection on P.*

Proof. The property $\Phi^2=\Phi$ follows from the very definition. Further it is easy to see that $(\lambda p)^F=\lambda p^F$, whenever $\lambda \geqq 0$. Now let $p,q \in P$. Then $p^F \leqq p$, $q^F \leqq q$, and hence $p^F+q^F \leqq p+q$. Therefore $p^F+q^F \leqq (p+q)^F$. Suppose on the other hand that $r \in P \cap F$ and $r \leqq p+q$. Then there exists a $r_0 \in P$ such that $r+r_0=p+q$. Let $a,b,c,d \in P$ be such that $a+b=r$, $a+c=p$, $b+d=q$. Then $a \leqq p$, $b \leqq q$, and $a,b \leqq r$. From Proposition 8 it follows that $a,b \in F$. Hence $a \leqq p^F$, $b \leqq q^F$ and thus $r=a+b \leqq p^F+q^F$. This shows that $(p+q)^F=p^F+q^F$. If we assume $p \leqq q$, then $p^F \leqq q$ and by definition $p^F \leqq q^F$.

Corollary 1. *The set $\Phi^{-1}(0)=\{p \in P : p^F=0\}$ is a convex cone with the property that if $0 \leqq q \leqq p \in \Phi^{-1}(0)$, then $q \in \Phi^{-1}(0)$.*

Proof. Obvious.

Corollary 2. *Every $p \in P$ can be written uniquely in the form $p=q+q'$, where $q \in P \cap F$, $q' \in \Phi^{-1}(0)$. More precisely, we have $q=p^F$ and $q'=p-p^F$. In particular, $\Phi^{-1}(0)$ is a supplement of $P \cap F$ in P.*

Proof. If $p=q+q'$, with $q \in P \cap F$, $q' \in \Phi^{-1}(0)$, then

$$p^F = q^F+(q')^F = q^F = q .$$

On the other hand we have $p=p^F+r$, where $r=p-p^F \in P$. Hence $p^F=p^F+r^F$, and thus $r \in \Phi^{-1}(0)$.

Proposition 11. *Let P and F be as in Proposition 9. Consider the following statements.*

(i) *The mapping Φ: $p \to p^F$ is continuous.*

(ii) *The graph of Φ is a closed subset of $E \times E$.*

(iii) *$\Phi^{-1}(0)$ is closed.*

Then (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii). *If in addition we suppose that P is locally compact and that F is closed, then* (iii) $\Rightarrow$ (i).

Proof. That (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) is immediate. Suppose therefore that P is locally compact and that F and $\Phi^{-1}(0)$ are closed. Then $\Phi^{-1}(0)$ is a closed locally compact convex cone and $\Phi^{-1}(0) \cap F=\{0\}$. Now let $\{p_\gamma\}$ be a net in P such that $p_\gamma \to p$. We have to prove that $\Phi(p_\gamma) \to \Phi(p)$. It is sufficient to prove that every subnet has a subnet which converges to $\Phi(p)$. Let $\{\Phi(p_\iota)\}$ be a subnet of $\{\Phi(p_\gamma)\}$. It suffices to prove that $\{p_\iota\}$ has a subnet $\{p_\omega\}$ such that $\Phi(p_\omega) \to \Phi(p)$. Now $p_\iota=\Phi(p_\iota)+p_\iota'$, where $\Phi(p_\iota) \in P \cap F$, $p_\iota' \in \Phi^{-1}(0)$. Since $p_\iota \to p$, we conclude as in the proof of Lemma 1 that $\{p_\iota'\}$ has a subnet $\{p_\omega'\}$ converging to $p' \in \Phi^{-1}(0)$. Since $p_\omega=\Phi(p_\omega)+p_\omega' \to p$, we obtain $\Phi(p_\omega) \to q=p-p' \in P \cap F$. Hence $p=q+p'$, and therefore $\Phi(p)=\Phi(q)=q$.

PROPOSITION 12. *Let P and F be as in Proposition 9 and assume in addition that P has the following property*:

(A) *Given the net $\{p_\gamma\} \subset P$, where $p_\gamma \to p \in P$ and given $V \in \mathscr{V}(0)$, $q \in [0,p]$, there exists a γ such that $q \in [0,p_\gamma] + P \cap V$.*

Then $\Phi^{-1}(0)$ is closed.

PROOF. Suppose that $p_\gamma \to p$, where $\{p_\gamma\} \subset \Phi^{-1}(0)$. Let $V \in \mathscr{V}(0)$ and $q \in [0,p] \cap F$. According to the hypothesis there exists a γ such that $q = r + v$, where $r \in [0,p_\gamma]$, $v \in V \cap P$. Since $0 \leqq r \leqq q \in F$, we have $r \in F$. Hence $0 \leqq r \leqq p_\gamma{}^F = 0$, and consequently $q = v \in V \cap P$. Since V is arbitrary, this implies that $q = 0$ and therefore $p^F = \Phi(p) = 0$.

If P is a semi-lattice cone such that the mapping $p \to \inf\{p,q\}$ from P to P is continuous for every $q \in P$, then we shall call P a *topological* semi-lattice cone.

PROPOSITION 13. *If P is a topological semi-lattice cone, then P satisfies the condition* (A) *of Proposition* 12.

PROOF. Let $p_\gamma \to p$, $V \in \mathscr{V}(0)$ and $q \in [0,p]$. Then $\inf\{p_\gamma,q\} \to \inf\{p,q\} = q$. Therefore we can find a γ and a $v \in -V$, such that $\inf\{p_\gamma,q\} = q + v$. Hence $-v = q - \inf\{p_\gamma,q\} \in V \cap P$, and thus

$$q = \inf\{p_\gamma,q\} - v \in [0,p_\gamma] + V \cap P .$$

PROPOSITION 14. *A locally compact topological semi-lattice cone P has the supplement property.*

PROOF. Let F be a closed support space of P. Then P and F satisfy the hypothesis of Proposition 9. Proposition 10, Corollary 2 shows that $\Phi^{-1}(0)$ is a supplement of $P \cap F$, and this supplement is closed by Propositions 12 and 13.

We conclude this section with a result of a negative nature. It shows that in an infinite dimensional disk space (espace tonnelé) there exist no locally compact, generating cones. This result might be well known. but I have been unable to find a reference.

PROPOSITION 15. *Suppose that E is a disk space, that P is locally compact, and that $E = P - P$. Then E is finite dimensional.*

PROOF. Let $V \in \mathscr{V}(0)$ be symmetric, convex, and such that $V \cap P$ is compact. Then $V \cap P - V \cap P$ is symmetric, convex and compact (since $V \cap P - V \cap P$ is the image of the compact set $(V \cap P) \times (V \cap P)$ by the continuous mapping $(x,y) \to x - y$). Now let $x \in E$. Then $x = p - q$, with $p,q \in P$. We can find a $\lambda > 0$ such that $\lambda p, \lambda q \in V \cap P$. Hence $\lambda x \in V \cap P - V \cap P$, and consequently this set is a disk and is therefore a compact neighborhood of zero.

3. The closed support space generated by a subspace.

In the sequel $s(M)$ denotes the support space generated by a subspace M. We deal in the present section with some rather simple aspects of the following general problem: Let f be a positive and continuous linear functional on M. Find conditions which ensure that f admits a positive and continuous extension to the closed support space generated by M. We split this problem into two new problems in the following way. According to a result of I. Namioka [12, p. 8], f admits a positive extension $\bar{f}$ to $s(M)$. First problem: *Find conditions which ensure that $\bar{f}$ is continuous.* If $\bar{f}$ is continuous, then $\bar{f}$ admits, by a result in [8, p. 336] a positive and continuous extension to $\overline{s(M)}$, provided $P \cap \overline{s(M)} \subset \overline{P \cap s(M)}$. Second problem: *Find conditions which ensure that $\overline{s(M)}$ is a support space.*

Lemma 6. *When M is a subspace of E, then*

$$s(M) = M + P \cap s(M) .$$

Proof. This follows from the corollary of Proposition 8.

Proposition 16. *Suppose that M is a closed subspace, that $s(M) \cap P$ is finite dimensional, and that P is closed. Then $s(M)$ is closed, and a positive and continuous linear functional f on M admits a positive and continuous extension to $s(M)$.*

Proof. $s(M)$ is, by Lemma 6 and the hypothesis, the direct sum of M and a finite dimensional subspace E_0, and is therefore closed. The corollary of Proposition 8 shows that every positive element in $s(M)$ is dominated by an element from M. Hence it follows from [12, p. 8] that f admits a positive extension $\bar{f}$ to $s(M)$. Since E_0 is a topological supplement of M in $s(M)$, and the restriction of $\bar{f}$ to E_0 is continuous, we conclude that $\bar{f}$ is continuous.

Corollary. *Suppose that P is proper, closed and finite dimensional, and that P has only a finite number of extreme generatrices. Let M be a closed subspace of E. Then every positive and continuous linear functional on M admits a positive and continuous extension to E, and $M+P$ is closed.*

Proof. The first statement is a consequence of Propositions 7, 8 and 16. The second statement then follows from [8, p. 336].

Applied to E in the case E is equipped with the finest locally convex topology, this corollary gives a counterexample of a conjecture of A. Bastiani, concerning what she calls "pyramide stricte" [1, p. 283].

We use this opportunity to give a characterization of a $\mathscr{T}$-pyramide as defined by A. Bastiani [1, p. 273].

PROPOSITION 17. *Let P be a convex cone. Then the following statements are equivalent.*

(i) *P is a $\mathscr{T}$-pyramide in the sense of Bastiani.*

(ii) *$P+M$ is closed whenever M is a finite dimensional subspace.*

(iii) *Every positive linear functional defined on some finite dimensional subspace admits a positive and continuous extension to E.*

PROOF. The equivalence of (i) and (ii) follows from a remark in [1, p. 273]. The equivalence of (ii) and (iii) follows by just the same argument as we used in the proof of Proposition 2 in [8].

PROPOSITION 18. *Suppose that E is a disk space, that P is weakly locally compact, and that M is a closed subspace of E, such that $s(M)=E$. Then a positive and continuous linear functional f on M admits a positive and continuous extension to E.*

PROOF. According to [12, p. 8], f admits a positive extension $\bar{f}$ to E. We shall show that $\bar{f}$ is continuous. Let $\varepsilon>0$ be given. Choose V as a weak zero-neighborhood such that V is symmetric, convex, closed, and has the following properties: $V \cap P$ is weakly compact and $|f(x)| \leqq \varepsilon$ whenever $x \in V \cap M$. Let W be the set

$$W = (V \cap M + V \cap P) \cap (V \cap M - V \cap P)\,.$$

Then W is symmetric and convex, and if $y \in W$, then $|\bar{f}(y)| \leqq \varepsilon$. Hence we are through if we can prove that W is a disk. That W is weakly closed follows from the following fact: If K is a compact set and C a closed set, then $K+C$ is closed [11, p. 158]. Therefore it remains to show that W is point absorbing. Let $x \in E$. Then $x=m+p=m'-p'$, where $m,m' \in M$, $p,p' \in P$. There exists a $\lambda>0$ such that λm, $\lambda m'$, λp, $\lambda p' \in V$, and hence $\lambda x \in W$.

PROPOSITION 19. *Suppose that M is a closed subspace fulfilling the following requirements:*

(i) *$P \cap \overline{s(M)}$ is weakly locally compact.*

(ii) $P \cap \overline{s(M)} = P \cap s(M)$,

(iii) *$s(M)$ is a disk space.*

Then $\overline{s(M)}$ is the closed support space generated by M, and a positive and continuous linear functional f defined on M admits a positive and continuous extension to $\overline{s(M)}$.

PROOF. The first assertion follows from (ii) together with Proposition 8. Now we observe that $s(M)$ is the same as the support space generated by M with respect to the cone $P \cap s(M)$. Since $P \cap s(M)$ is assumed weakly locally compact, it follows from Proposition 18, that f admits a positive and continuous extension to $s(M)$. The desired result then follows from (ii) and the argument used in [8, Proposition 3, p. 336].

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