MATH. SCAND. 11 (1962), 79–90

CONVEX CONES WITH PROPERTIES RELATED TO WEAK LOCAL COMPACTNESS

OTTE HUSTAD

Introduction.

The present paper originated from the problem to characterize the polar cone of a rich cone, where a convex cone P in a topological vector space is called *rich* if $p+P$ intersects every dense linear subspace whenever $p \in P$ (see [5] and [6]). During the work with this problem, we found some general properties concerning convex cones. These results are collected in §§ 1 and 2. The main result in § 1 (Proposition 3) states that a convex cone P is weakly locally compact if and only if P satisfies a Borel–Lebesgue property with respect to open half-spaces. We also show (Proposition 7) that P has this property if and only if the polar cone has a certain separation property. Here we make use of the concept of non-support point introduced by Floyd and Klee [4]. Such a point of P is virtually the same as a strictly positive functional on the polar cone. We show in § 2 that if Q is a locally compact convex cone and P admits a strictly positive functional, then $P+Q$ admits a strictly positive functional if (and only if) $P+Q$ is proper. In § 3 we prove some properties about rich cones. If P intersects every linear variety which separates in a certain sense every point pair of the polar cone, then P is rich (Proposition 9), and a partial converse is also valid. We use this result to show that under some not too restrictive conditions the intersection of two rich cones is again a rich cone.

Notation and terminology. Set-theoretic difference between sets A and B is denoted $A \sim B$. The letter E shall always denote a real locally convex Hausdorff topological vector space, and P a *convex cone* in E, that is $P+P \subset P$ and $\lambda P \subset P$ for each $\lambda \geqq 0$. The cone P is *proper* if P contains no line through zero. A *base* of P is a convex set $K \subset E \sim \{0\}$ such that P is the convex cone generated by K. A linear functional f on E is (*strictly*) *P-positive* or shorter (*strictly*) *positive* if $f(p) \geqq 0$ ($f(p) > 0$) for each $p \in P \sim \{0\}$. The *polar* cone P° of P consists of all positive and continuous linear functionals on E. More generally we define (slightly

Received December 10, 1961.

different from [1]) the *polar* set K° of $K \subset E$ as the set consisting of all continuous linear functionals f such that $f(k) \geqq -1$ for each $k \in K$. The topological dual of E is denoted E'. An *open half-space* is a set of the form $\{x : f(x) > 0\}$, where $f \in E' \sim \{0\}$. We note that the closure of this set is $\{x : f(x) \geqq 0\}$, i.e. a *closed half-space*. For each $x \in E$ we define $\hat{x}$ on E' by $\hat{x}(f) = f(x)$. E' is always equipped with the weak topology, and extensive use will be made of the identification of E with the topological dual of E'. If $\mathcal{U} = \{U_\gamma\}$ is a family of subsets of a topological space, the *closure* of $\mathcal{U}$ is the family $\{\overline{U}_\gamma\}$. If A, B are subsets of R, the real number field, then $A \leqq B$ ($A < B$) means that $\alpha \leqq \beta$ ($\alpha < \beta$) for each $\alpha \in A$ and each $\beta \in B$. Otherwise our terminology follows Bourbaki [1].

1. The half-space Borel-Lebesgue property.

DEFINITION 1. *A subset K of a topological vector space has the (weakened) half-space Borel–Lebesque property if each family of open half-spaces which covers $K \sim \{0\}$ contains a finite subfamily (the closure of) which covers $K \sim \{0\}$.*

PROPOSITION 1. *Suppose that P is closed. Then the following two statements are equivalent.*

(i) *Whenever $X \subset E$ is non-empty, convex and disjoint from P, there exists a closed hyperplane separating P and X.*

(ii) *P° has the weakened half-space Borel–Lebesgue property.*

PROOF. (i) $\Rightarrow$ (ii). Suppose that (ii) is not satisfied. Then there exists a non-empty subset $A \subset E$ with the following two properties. 1° Whenever $f \in P^\circ \sim \{0\}$, there exists an $a \in A$ such that $f(a) > 0$. 2° If $a_1, \ldots, a_n \in A$ are given, there exists an $f \in P^\circ$ such that $f(a_i) < 0$, $i = 1, \ldots, n$. Let X be the convex hull of A and choose $x \in X$. Then we can find $a_1, \ldots, a_n \in A$ and $\lambda_i \geqq 0$ such that

$$x = \sum_{i=1}^{n} \lambda_i a_i \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i = 1 .$$

Because of 2° we can find an $f \in P^\circ$ such that

$$f(x) = \sum_{i=1}^{n} \lambda_i f(a_i) < 0 .$$

Hence $x \notin P$, and consequently $X \cap P = \emptyset$. In view of (i) there exists $\alpha \in R$ and $g \in E' \sim \{0\}$ such that $g(X) \leqq \alpha \leqq g(P)$. Since P is a cone we obtain $g(X) \leqq 0 \leqq g(P)$. Hence $g \in P^\circ \sim \{0\}$ and $g(A) \leqq 0$, which contradicts 1°.

(ii) $\Rightarrow$ (i). Suppose that (i) is not satisfied. Then there exists a non-empty convex subset X of E such that X is disjoint from P and if $f \in P^\circ \sim \{0\}$, then we can find an $x \in X$ such that $f(x) > 0$. Hence, by (ii), we can find $x_1, \ldots, x_n \in X$ such that

$$(1) \qquad \max_{i=1,\ldots,n} \{f(x_i)\} \geqq 0, \qquad \forall f \in P^\circ .$$

Let K be the convex hull of $x_1, \ldots, x_n$. Then K is compact and disjoint from P. Hence there exists [1, p. 73] a closed hyperplane separating K and P strictly. This implies that there exists an $f \in P^\circ$ such that $f(x_i) < 0$, $i = 1, \ldots n$. Since this contradicts (1), the proof is finished.

Proposition 2. *Suppose that P satisfies* (i) *in Proposition* 1, *and let A be a non-empty convex subset of E such that $A-P$ is dense in E. Then $E = A - P$.*

Proof. Let $x \in E$. Then $A-P-x$ is convex and dense in E, and must intersect P, for otherwise we could find an $f \in E' \sim \{0\}$ such that $f(A-P-x) \leqq 0 \leqq f(P)$, which is impossible since $A-P-x$ is dense. Let q belong to $P \cap (A-P-x)$. Then $q = a-p-x$, and thus $x = a-(p+q) \in A-P$.

Corollary. *Suppose that P satisfies* (i) *in Proposition* 1. *Then every positive linear functional g is continuous.*

Proof. If g is discontinuous, then $g^{-1}(0)$ is dense, and therefore $E = g^{-1}(0) - P$. This implies that $g(x) \leqq 0$ whenever $x \in E$. Consequently $g = 0$ and this is impossible since we assumed g discontinuous.

Lemma 1. *Suppose that $E' = P^\circ - P^\circ$ and that $P \sim \{0\}$ is covered by a finite family of open half-spaces. Then P admits a strictly positive continuous linear functional.*

Proof. By assumption we can find $g_i, g_i' \in P^\circ$, $i = 1, \ldots, n$, such that

$$(2) \qquad \max_{i=1,\ldots,n} \{(g_i - g_i')(p)\} > 0, \qquad \forall p \in P \sim \{0\} .$$

Put $g = g_1 + \ldots + g_n$ and let $p \in P \sim \{0\}$. Then $g(p) \geqq 0$. If $g(p) = 0$, then $g_i(p) = 0$ for all $i = 1, \ldots, n$. In view of (2) there exists an i such that $-g_i'(p) > 0$. This is impossible since $g_i' \in P^\circ$. Hence $g(p) > 0$.

The following simple lemma is certainly well known.

Lemma 2. *$P-P$ is dense in E if and only if P° is proper. If P is closed, then P is proper if and only if $P^\circ - P^\circ$ is dense in E'.*

Proof. If $P-P$ is not dense in E, there exists an $f \in E' \sim \{0\}$ such that f is zero on $P-P$. Hence $f \in P^\circ \cap -P^\circ$. Conversely, if $P-P$ is dense and $g \in P^\circ \cap -P^\circ$, then g is zero on $P-P$ and therefore $g=0$. This proves the first assertion. The second statement is a consequence of the first one, since $P=(P^\circ)^\circ$ when P is closed.

LEMMA 3. *If P contains an interior point p_0, then P° is weakly locally compact.*

PROOF. Since $E=P-P$, we have that P° is proper. It is therefore sufficient [9, proof of (2) on p. 341] to show that there exists a weak zero-neighborhood V in E' such that $V \cap P^\circ$ is weakly compact. $P-p_0$ is, by assumption, a zero-neighborhood, and therefore $(P-p_0)^\circ$ is weakly compact. Now we observe that if $q=p-p_0$ with $p \in P$, then $q=\lim q_n$ where

$$q_n = \frac{1}{n}(np) + \left(1-\frac{1}{n}\right)(-p_0)$$

belongs to the convex hull of $P \cup \{-p_0\}$. From this we conclude that the closure of $P-p_0$ equals the closed convex hull of $P \cup \{-p_0\}$. Hence we obtain, by general properties of polar sets,

$$(P-p_0)^\circ = (\{-p_0\} \cup P)^\circ = \{-p_0\}^\circ \cap P^\circ .$$

Since $\{-p_0\}^\circ$ is a weak zero-neighborhood, the proof is finished.

We shall adhere to usual terminology and call an element e of P an *order unit* provided there exists for every $x \in E$ a $\lambda > 0$ such that $\lambda e - x \in P$. This means that the set $(P-e) \cap (e-P)$ is absorbing. Hence e is an order unit if and only if e is an interior point of P, when E is equipped with the finest locally convex topology. In view of Lemma 3, we can therefore infer that if P admits an order unit and every positive linear functional is continuous, then P° is weakly locally compact. We make use of this result in the following proposition.

PROPOSITION 3. *Suppose that P is proper and closed. Then the following three statements are equivalent.*

(i) *P is weakly locally compact.*

(ii) *P satisfies the half-space Borel–Lebesgue property.*

(iii) *P satisfies the weakened half-space Borel–Lebesgue property and admits a strictly positive continuous linear functional.*

PROOF. (i) $\Rightarrow$ (ii). This is a simple consequence of the fact [9, p. 341] that P admits in case (i) a weakly compact base.

(ii) $\Rightarrow$ (iii). For every $p \in P \sim \{0\}$ we can find an $f \in E'$ such that $f(p) > 0$. Hence it follows from Definition 1 that there exists a finite family of open half-spaces covering $P \sim \{0\}$. Applying Propositions 1 and 2 with P° instead of P, we infer, by Lemma 2, that $E' = P^\circ - P^\circ$. Hence it follows from Lemma 1 that P admits a strictly positive continuous linear functional.

(iii) $\Rightarrow$ (i). According to the remark preceding the proposition it is sufficient to prove that P° admits an order unit and that every P°-positive linear functional is continuous. That P° has the last mentioned property follows at once from the corollary of Proposition 2. Let f_0 be a strictly P-positive continuous linear functional. We assert that f_0 is an order unit of P°. Let Q be the set of all λf_0 with $\lambda \geqq 0$. Then $Q - P^\circ$ is dense in E'. For otherwise we could find an $x \in E \sim \{0\}$ such that $g(x) \leqq 0$ whenever $g \in Q - P^\circ$. But this implies $x \in P \sim \{0\}$ and $f_0(x) \leqq 0$, contrary to the hypothesis on f_0. Thus, by Proposition 2, $E' = Q - P^\circ$, and from this it follows that f_0 is an order unit.

COROLLARY. *Suppose that P is closed. Then P° is proper and weakly locally compact if and only if P admits an order unit and every P-positive linear functional is continuous.*

PROOF. The remark preceding Proposition 3 contains the "if"-part. The converse follows from the proof of Proposition 3 when applied to P° instead of P.

PROPOSITION 4. *Suppose that P_1 and P_2 are two closed convex cones in E, both with an order unit and such that their positive linear functionals are continuous. Then $P = P_1 \cap P_2$ has the same two properties if and only if $P - P$ is dense in E.*

PROOF. The "only if" part is clear. Suppose therefore that $P - P$ is dense in E. Then P° is proper. Both $P_1{}^\circ$ and $P_2{}^\circ$ are, by the corollary of Proposition 3, proper and weakly locally compact. Let B_1 (B_2) be a weakly compact base of $P_1{}^\circ$ ($P_2{}^\circ$), and let B be the convex hull of $B_1 \cup B_2$. Then B is weakly compact. If $0 \in B$, then $0 = \lambda_1 b_1 + \lambda_2 b_2$ with $b_i \in B_i$, $\lambda_i \geqq 0$ and at least one $\lambda_i > 0$, say $\lambda_1 > 0$. Then

$$0 \neq b_1 = -\frac{\lambda_2}{\lambda_1} b_2 \in P_1{}^\circ \cap -P_2{}^\circ \subset P^\circ \cap -P^\circ ,$$

contrary to the fact that P° is proper. From this it follows that B is a weakly compact base of $P_1{}^\circ + P_2{}^\circ$, and therefore we can conclude that $P_1{}^\circ + P_2{}^\circ$ is closed and weakly locally compact. Since

$$P^\circ = (P_1 \cap P_2)^\circ = \overline{P_1{}^\circ + P_2{}^\circ} = P_1{}^\circ + P_2{}^\circ ,$$

the desired conclusion follows from the corollary of Proposition 3.

A convex set K in E is called *linearly bounded* if K contains no half-line.

PROPOSITION 5. *Let $K \subset E \sim \{0\}$ be convex and closed. Then K is weakly compact if and only if K is linearly bounded and satisfies the weakened half-space Borel–Lebesgue property.*

PROOF. The "only if" part is clear. To prove the converse, let P be the convex cone generated by K. Since we can separate K and $\{0\}$ strictly, it follows that P admits a strictly positive continuous linear functional. It is easy to see that P has the weakened half-space Borel–Lebesgue property. Furthermore it follows from [7, p. 26] that P is closed. Thus by Proposition 3, P is weakly locally compact. Hence K is weakly locally compact, since K is weakly closed. From [9, p. 343] we infer that K is weakly compact.

LEMMA 4. *Suppose that $M \subset E$ satisfies the weakened half-space Borel–Lebesgue property, and that $Q \neq \{0\}$ is a subset of M. Then Q satisfies the same property if the following condition is fulfilled.*

If $x \in M \sim Q$, there exists an $f_x \in E'$ such that $f_x(Q \sim \{0\}) < 0 < f_x(x)$.

PROOF. Let $\{H_\gamma\}$ be a family of open half-spaces covering $Q \sim \{0\}$. Then the family
$$\{H_\gamma\} \cup \{f_x^{-1}(\langle 0, \infty\rangle\}_{x \in M \sim Q}$$
covers $M \sim \{0\}$. Hence we can find a finite subfamily $\mathscr{F}$ such that the closure of $\mathscr{F}$ covers $M \sim \{0\}$. Then the closure of $\mathscr{F} \cap \{H_\gamma\}$ covers $Q \sim \{0\}$.

PROPOSITION 6. *Suppose that K is a convex and closed subset of E, and that K is contained in a hyperplane H with $0 \notin H$. Then K is weakly compact if and only if K has the weakened half-space Borel–Lebesgue property.*

PROOF. By Proposition 5, we have only to prove that K is linearly bounded if K has the weakened half-space Borel–Lebesgue property. Suppose that this is not true, and let $S = \{\lambda a + (1-\lambda)b : \lambda \geqq 0\}$ be a half-line contained in K. Since S is a subset of H, it is easy to see that S cannot have the weakened half-space Borel–Lebesgue property. Hence, by Lemma 4, we have obtained a contradiction if we can prove that whenever $k \in K \sim S$, there exists an $f \in E'$ such that $f(S) < 0 < f(k)$. Let L be the vector space generated by a, b and k. Then either a, b and k are linearly independent or $k = \alpha a + (1-\alpha)b$, with $\alpha < 0$. In the first case we define f_0 on L by letting $f_0(k) = 1$, $f_0(a) = f_0(b) = -1$, in the second case by letting $f_0(k) = 1, f_0(b) = -1$. In both cases we have $f_0(S) < 0 < f_0(k)$. An extension of f_0 to E has the desired property.

Following Floyd and Klee [4] we call a point $p_0 \in P$ a *non-support point* of P if $f \in E'$ and $f(p_0) = \sup f(P)$ implies that f is constant on P.

It is evident that if $P-P$ is dense in E, then P cannot be separated from a set which contains a non-support point of P. We now propose to characterize those convex cones which can be separated from every convex set which does not contain a non-support point of the cone.

We omit the easy proof of the following lemma.

LEMMA 5. *Suppose that $P-P$ is dense in E, and let $p_0 \in P$. Then p_0 is a non-support point of P if and only if $\hat{p}_0$ is a strictly positive linear functional on P°.*

LEMMA 6. *An order unit e of P is a non-support point of P.*

PROOF. Let $f \in E'$ be such that $f(e) = \sup f(P)$. Then $f(P) \leqq 0 = f(e)$. Let $x \in E$. Then there exists a $\lambda > 0$ such that $\lambda e - x \in P$. Hence $0 \geqq f(\lambda e - x) = -f(x)$, and therefore $f = 0$.

PROPOSITION 7. *Suppose that P is closed and that $P-P$ is dense in E. Then the following two statements are equivalent.*

(i) *Whenever X is a non-empty convex subset of E such that X contains no non-support point of P, there exists a closed hyperplane separating P and X.*

(ii) *P° satisfies the half-space Borel–Lebesgue property.*

PROOF. (i) $\Rightarrow$ (ii). By applying Lemma 5, the proof proceeds in much the same way as the proof of the first part of the Proposition 1, and is therefore omitted.

(ii) $\Rightarrow$ (i). P° is proper, since $P-P$ is dense. Hence, by Proposition 3 and its corollary, P admits an order unit and every P-positive functional is continuous. Let $X \subset E$ be non-empty, convex and without any non-support point of P. Hence, by Lemma 6, X is disjoint from the set of all order units of P. This set is the same as the interior of P when E is equipped with the finest locally convex topology. By Eidelheits separation theorem [2, p. 22] there exists a linear functional g on E such that $g(X) \leqq g(P)$. Hence g is P-positive and therefore continuous. This shows that P and X can be separated.

2. Strict separation.

Klee shows in [8] that if P and Q are proper, closed convex cones in E, $P \cap Q = \{0\}$, and Q is locally compact, then P and Q can be separated by a closed hyperplane. He further shows that if P is locally compact or E is a separable normed space, then it is possible to separate P and Q *strictly*, i.e. there exists an $f \in E'$ such that $f(Q \sim \{0\}) < 0 < f(P \sim \{0\})$.

Several times in the preceding, especially in Lemma 4, we have encountered problems of this sort. Now we have the following result.

PROPOSITION 8. *Suppose that P and Q are proper, closed convex cones in E, that Q is weakly locally compact, and that $P \cap Q = \{0\}$. Then it is possible to separate P and Q strictly if and only if P admits a strictly positive continuous linear functional.*

PROOF. Since the "only if" part is trivial, let us assume that $g \in E'$ is strictly P-positive. Then $M = P \cap g^{-1}(1)$ is a closed base of P. Let K be a weakly compact base of Q, and let B be the closed convex hull of $M \cup (-K)$. Since, by [8, p. 313], $P-Q$ is closed, one proves easily that $P-Q$ is the convex cone generated by B. We are going to prove that $0 \notin B$. Suppose that this is not true. Then there exist nets $\{m_\gamma\} \subset M$, $\{k_\gamma\} \subset K$ and non-negative numbers λ_γ, μ_γ with $\lambda_\gamma + \mu_\gamma = 1$, such that $\lambda_\gamma m_\gamma - \mu_\gamma k_\gamma \to 0$ weakly. Since K is weakly compact, there exists a subnet $\{k_\iota\}$ of $\{k_\gamma\}$ such that $k_\iota \to k \in K$. Since $\{\mu_\iota\} \subset [0,1]$, there also exists a subnet $\{\mu_\alpha\}$ of $\{\mu_\iota\}$ such that $\mu_\alpha \to \mu \in [0,1]$. Hence $k_\alpha \to k$, $\lambda_\alpha = 1 - \mu_\alpha \to 1 - \mu$ and $\lambda_\alpha m_\alpha - \mu_\alpha k_\alpha \to 0$. Suppose first that $\mu = 0$. Then $\mu_\alpha k_\alpha \to 0k = 0$, and therefore $\lambda_\alpha m_\alpha \to 0$. Hence $m_\alpha = \lambda_\alpha^{-1}(\lambda_\alpha m_\alpha) \to 1 \cdot 0 = 0$, which contradicts the fact that M is closed and $0 \notin M$. And if $\mu > 0$, then $\lambda_\alpha m_\alpha \to \mu k$ and therefore $\mu^{-1}\lambda_\alpha m_\alpha \to k$. Since $\mu^{-1}\lambda_\alpha m_\alpha \in P$, this gives the contradiction $0 \neq k \in P \cap Q$. Hence $0 \notin B$. Therefore there exists an $f \in E'$ such that $0 < f(B)$, and one verifies directly that $f(Q \sim \{0\}) < 0 < f(P \sim \{0\})$.

COROLLARY. *Suppose that P is closed and admits a strictly positive continuous linear functional. Let Q be a proper weakly locally compact convex cone in E. Then $P+Q$ admits a strictly positive continuous linear functional if and only if $P+Q$ is proper.*

PROOF. If $P+Q$ is proper, then $P \cap -Q = \{0\}$, and every $g \in E'$ with the property $g(-Q \sim \{0\}) < 0 < g(P \sim \{0\})$ is strictly positive on $P+Q$.

3. Some properties of rich cones.

Another formulation of the weakened half-space Borel–Lebesgue property of a subset K of E is the following. If M is a subset of E' such that there exists for each $k \in K \sim \{0\}$ an $f \in M$ such that $f(k) > 0$, then there exist $f_1, \ldots, f_n \in M$ such that

$$\max_{i=1,\ldots,n} \{f_i(k)\} \geqq 0, \qquad \forall k \in K .$$

A weakening of this property has bearing on the richness of a convex cone.

DEFINITION 2. *Let K be a subset of E and M a subset of E'. We say that M separates positively the points of K if, whenever $p, q \in K \sim \{0\}$ and $p \neq q$, there exists an $f \in M$ such that $0 < f(p) \neq f(q) > 0$.*

LEMMA 7. *Let M be a dense linear subspace of E', $g \in E'$ and $p, q \in E \sim \{0\}$ with $p \neq q$. Suppose that the convex cone Q generated by p and q is proper. Then, whenever $x \in E \sim (-Q)$, there exists an $f \in M$ such that*

$$(f+g)(x) = 1 \qquad \textit{and} \qquad 0 < (f+g)(p) \neq (f+g)(q) > 0 .$$

PROOF. Let L be the vector space generated by x, p and q, and let $M \mid L$ denote the set of all $f \mid L$, where $f \in M$ and $f \mid L$ is the restriction of f to L. Since $M \mid L$ is a dense linear subspace of L', we have $L' = M \mid L$ and hence $M \mid L + g \mid L = L'$. Therefore the proof reduces to showing that there exists an $h \in L'$ such that $h(x) = 1$ and $0 < h(p) \neq h(q) > 0$. This is clear if the dimension of L is one or three. Suppose therefore $\dim L = 2$. We can assume that p and q are linearly independent, since the case when p and q are linearly dependent is easily settled. Hence we have $x = \alpha p + \beta q$, with at least one α or β positive, since $-x \notin Q$. Therefore we may and shall assume that $\alpha > 0$. Define h on L by first choosing

$$0 < h(q) < \min\{|\beta|^{-1}, |\alpha+\beta|^{-1}\}$$

and then

$$h(p) = \frac{1-\beta h(q)}{\alpha}.$$

One then verifies readily that h has the desired properties.

LEMMA 8. (Cf. [3, p. 618].) *Suppose that P is closed and that $f_1, \ldots, f_n \in E'$ are given. Then the condition*

$$\max_{i=1,\ldots,n} \{f_i(p)\} \geqq 0, \qquad \forall p \in P ,$$

is satisfied if and only if the convex hull Γ of $\{f_1, \ldots, f_n\}$ intersects P°.

PROOF. The "if" part is clear. Hence suppose that $P^\circ \cap \Gamma = \emptyset$. Since Γ is compact and P° is closed, there exists an $x \in E$ such that $\hat{x}(\Gamma) < \hat{x}(P^\circ)$. Hence $x \in P$ and $f_i(x) < 0$, $i = 1, \ldots, n$.

PROPOSITION 9. *Suppose that P is closed and that $P-P$ is dense in E. Then P is rich if the following condition is satisfied.*

Whenever $M \subset E$ is a linear variety which separates positively the points of P°, there exist $x_1, \ldots, x_n \in M$ such that

$$\max_{i=1,\ldots,n} \{f(x_i)\} \geqq 0, \qquad \forall f \in P^\circ .$$

PROOF. Let F be a dense linear subspace of E and let $p \in P$. Choose $f, g \in P^\circ \sim \{0\}$ with $f \neq g$. Since P° is proper, it follows from Lemma 7 that we can find a $y \in F$ such that $0 < (\hat{y}-\hat{p})(f) \neq (\hat{y}-\hat{p})(g) > 0$. Hence $F-p$ separates positively the points of P°. Consequently there exist $y_1, \ldots, y_n \in F$ such that

$$\max_{i=1,\ldots,n} \{(\hat{y}_i - \hat{p})(f)\} \geqq 0, \qquad \forall f \in P^\circ .$$

This means, by Lemma 8, that there exists a $y \in F$ such that $y-p \in P$, and hence $(P+p) \cap F \neq \emptyset$.

REMARK. It follows from this proposition that if P° satisfies the weakened half-space Borel–Lebesgue property, then P is rich. (This is also a simple consequence of Propositions 1 and 2.) If for instance P admits an order unit and every P-positive linear functional is continuous, then P is rich, since, by the corollary of Proposition 3, P° is weakly locally compact in this case.

By Lemma 8, a converse of Proposition 9 would have the form: If P is rich, then $P \cap M \neq \emptyset$ whenever $M \subset E$ is a linear variety which separates positively the points of P°. If this converse were true, then, since P is rich in the finest locally convex topology, $P \cap M \neq \emptyset$ whenever M separates positively the points of the cone $P^\square$ consisting of all P-positive linear functionals. And using Proposition 9 it would follow that P were rich in the weakest topology $\sigma(E, P^\square - P^\square)$ which renders every $g \in P^\square$ continuous, provided P was closed and $P-P$ was dense in this topology. What we now actually can prove is the following.

PROPOSITION 10. *Suppose that E is equipped with the topology $\sigma(E, P^\square - P^\square)$ and suppose that P is rich. Then $M \cap P \neq \emptyset$ whenever $M \subset E$ is a linear variety which separates positively the points of P°.*

PROOF. Let us assume $M \cap P = \emptyset$. Then we have $M = F + x$, where F is a linear subspace of E and $x \in E \sim F$. Let $f \in E' \sim \{0\}$. Then $f = h - g$, with $h, g \in P^\circ$. If $h, g \neq 0$, there exists an $m \in M$ such that $h(m) - g(m) = f(m) \neq 0$. And if for instance $g = 0$, then $2h \neq h$ and the existence of m remains valid also in this case. Hence the linear subspace $F + Rx$ generated by M is dense in E. Define f_0 on $F + Rx$ by $f_0(y + \lambda x) = -\lambda$. We note that if $y + \lambda x \in P$ and $y \in F$, then $\lambda \leqq 0$, for otherwise $\lambda^{-1} y + x \in M \cap P$. Hence f_0 is a positive linear functional on $F + Rx$. It follows from [5, Proposition 7] that f_0 admits a positive and continuous extension f to E. Since $f \in P^\circ \sim \{0\}$, there exists an $m_0 \in M$, say $m_0 = y_0 + x$, such that $f(m_0) > 0$. But $f(m_0) = f_0(y_0 + x) = -1$, and this contradiction proves our assertion.

PROPOSITION 11. *Suppose that E is equipped with the topology $\sigma(E, P^{\square} - P^{\square})$, that P is closed and rich and that $Q \subset E$ is a convex, closed cone such that Q° satisfies the weakened half-space Borel–Lebesgue property. Then $Q \cap P$ is rich, provided $Q \cap P - Q \cap P$ is dense in E.*

PROOF. Let F be a dense linear subspace of E and let $x \in E$. Denote the linear variety $F - x$ by M. Choose $h \in Q^{\circ} \sim \{0\}$. Then $-h \notin P^{\circ}$, for otherwise $h, -h \in Q^{\circ} + P^{\circ} \subset (Q \cap P)^{\circ}$, and this is impossible since $(Q \cap P)^{\circ}$ is proper. The linear variety $M \cap h^{-1}(1)$ separates positively the points of P°. In fact, let $f, g \in P^{\circ} \sim \{0\}$ with $f \neq g$ be given. Since the convex cone generated by f and g is contained in P°, this cone is proper and cannot contain $-h$. Hence, by Lemma 7, we can find a $y \in F$ such that $h(y-x) = 1$, $0 < f(y-x) \neq g(y-x) > 0$, and this proves our assertion. It follows from Proposition 10 that there exists an element

$$p_h \in P \cap M \cap h^{-1}(1).$$

By the condition on Q° we infer that there exist $p_1, \ldots, p_n \in P \cap M$ such that

$$\max_{i=1,\ldots,n} \{h(p_i)\} \geqq 0, \qquad \forall h \in Q^{\circ}.$$

Hence, by Lemma 8, there exists $p \in P \cap M \cap Q$, and therefore

$$(P \cap Q + x) \cap F \neq \emptyset.$$

COROLLARY. *Let the hypotheses on E and P remain unchanged. Let*

$$Q = \bigcap_{i=1}^{n} f_i^{-1}([0, \infty\rangle),$$

where $f_1, \ldots, f_n \in E'$, and assume that $P \cap Q - P \cap Q$ is dense in E. Then, whenever F is a dense linear subspace of E and $x \in E$, there exists an element $y \in (P + x) \cap F$, such that $f_i(y) \geqq f_i(x)$, $i = 1, \ldots, n$.

PROOF. Since $Q - Q$ is dense in E, Q° is proper and weakly locally compact. Hence Q° satisfies the weakened half-space Borel–Lebesgue property. From the proof of the proposition we infer that there exists an element $p \in P \cap Q \cap (F - x)$. Thus $p = y - x$ with $y \in F$, and this y has the desired properties.

It is easy to give an example which shows that Proposition 11 is not valid without some sort of restriction on $P \cap Q - P \cap Q$. Then we must of course assume that E admits at least one discontinuous linear functional, say g. Suppose further that $f_0 \in E'$ is such that $S = f_0^{-1}([0, \infty\rangle) \cap P$

is a half-line, say $S = \{\lambda q : \lambda \geqq 0\}$. Then S cannot be rich. For there exists an $f \in E'$ such that $f(q) \neq -g(q)$, and therefore $F = (f+g)^{-1}(0)$ is a dense subspace such that $(S+q) \cap F = \emptyset$.

REFERENCES

1. N. Bourbaki, *Espaces vectoriels topologiques*, Chap. 1–2 et 3–4 (Act. Sci. Ind. 1189 et 1229), Paris 1953 et 1955.
2. M. M. Day, *Normed linear spaces* (Ergebnisse d. Math., N. F., Heft 21), Berlin · Göttingen · Heidelberg, 1958.
3. K. Fan, I. Glicksberg and A. J. Hoffman, *Systems of inequalities involving convex functions*, Proc. Amer. Math. Soc. 8 (1957), 617–622.
4. E. E. Floyd and V. L. Klee, *A characterization of reflexivity by the lattice of closed subspaces*, Proc. Amer. Math. Soc. 5 (1954), 655–661.
5. O. Hustad, *On positive and continuous extension of positive functionals defined over dense subspaces*, Math. Scand. 7 (1959), 392–404.
6. O. Hustad, *Linear inequalities and positive extension of linear functionals*, Math. Scand. 8 (1960), 333–338.
7. V. L. Klee, *Boundedness and continuity of linear functionals*, Duke Math. J. 22 (1955), 263–269.
8. V. L. Klee, *Separation properties of convex cones*, Proc. Amer. Math. Soc. 6 (1955), 313–318.
9. G. Köthe, *Topologische lineare Räume* I, Berlin · Göttingen · Heidelberg, 1960.

UNIVERSITY OF OSLO, NORWAY