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BRIEF PROOF OF A THEOREM OF BAXTER

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In [1] Baxter proved a theorem that can be stated as follows.

THEOREM. *Let $\mathscr{X}$ be a commutative Banach algebra with identity e. Suppose that P is a bounded linear transformation on $\mathscr{X}$ which, for some fixed $b \in \mathscr{X}$, satisfies*

$$2P(x\cdot Px) = (Px)^2+P(bx^2) \tag{1}$$

for all $x \in \mathscr{X}$, or, equivalently

$$P(x\cdot Py+y\cdot Px) = (Px)(Py)+P(bxy)\,. \tag{2}$$

Then, for given $f \in \mathscr{X}$ and sufficiently small complex z, the equation

$$g = e+zP(fg) \tag{3}$$

has the unique solution

$$g = \exp \sum_1^\infty \frac{z^n}{n} P(b^{n-1}f^n)\,. \tag{4}$$

Baxter's proof was heavily combinatorial. It is the purpose of this note to present a brief proof, of a more analytical character. Before proceeding to the proof we remark that the equivalence of (1) and (2) follows trivially upon applying (1) separately to x, y, and $x+y$.

PROOF OF THE THEOREM. The key idea is to exploit the differential equations (and the initial condition $g(0)=e$) implied by (3) and (4).

Clearly (3) has a unique solution $g=g(z)$ for all small z; moreover $g(z)$ is analytic in z. Hence (3) may be differentiated, with the result

$$g'-zP(fg') = P(fg) \tag{5}$$

where $g'=dg(z)/dz$. Now, if f and g are given and z is small, then the equation

$$h-zP(fh) = P(fg) \tag{6}$$

has a unique solution h. Since (6) is linear, h depends linearly on g; indeed, h has the form

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$$h = kg \tag{7}$$

with $k \in \mathscr{X}$ depending on z and f but not on g. Momentarily leaving aside the proof of (7) and the identification of k we note that (5), (6) and (7) imply $g'(z) = k(z)g(z)$, and hence

$$g(z) = \exp \int_0^z k(t)\, dt \,. \tag{8}$$

We claim now that if u is defined by

$$u = \sum_0^\infty (zbf)^n \,, \tag{9}$$

then $k = P(fu)$, i.e., that $h = gP(fu)$ satisfies (6). We must verify the identity

$$gP(fu) - zP[fgP(fu)] = P(fg) \,. \tag{10}$$

Using (2) with $x = fg$, $y = fu$, the left member of (10) becomes

$$gP(fu) + zP[fuP(fg)] - zP(fg)P(fu) - zP(bfgfu) ; \tag{11}$$

on substituting $zP(fg) = g - e$ from (3) and $zbfu = u - e$ (which follows from (9)), the quantity (11) is transformed into

$$gP(fu) + P[fu(g-e)] - (g-e)P(fu) - P\big(fg(u-e)\big)$$

which equals $P(fg)$. This proves (10), (7), and hence (8). Carrying out the indicated integration completes the proof of (4).

REFERENCE

1. Glen Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math. 10 (1960), 731–742.

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