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ON THE DOUBLE TANGENTS OF PLANE CLOSED CURVES

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1.

Let c denote a closed differentiable oriented curve in an affine plane. We assume that c is composed of a finite number of convex arcs which do not touch one another. This implies that c has a *finite* number of double points, inflectional points and double tangents. The double points and the double tangents are assumed to be *simple*, i.e., through each double point pass two and only two branches of the curve, and analogously, each double tangent has two and only two points of contact with the curve. Further, it is assumed that no double tangent is tangent at an inflectional point; consequently each point of contact is an interior point of a convex arc belonging to the curve.

A double tangent r shall be called *exterior* or *interior* according as the convex arcs in the neighbourhood of the points of contact R_1 and R_2 are on the *same* side of r or on *opposite* sides. Hence a double tangent intersects each of the arcs R_1R_2 in an even or odd number of points according as it is exterior or interior. The positive half-tangents at R_1 and R_2 may have the *same* direction, or *opposite* directions which may point *towards* one another or *away* from one another. Corresponding to these possibilities we distinguish between 3 types of exterior double tangents, denoted by E_1, E_2 and E_3, and 3 types of interior double tangents, denoted by I_1, I_2 and I_3. If we change the orientation of the curve, the types E_1 and I_1 are invariant while the types E_2 and E_3 are interchanged and likewise the types I_2 and I_3.

2.

If a point P traverses the curve c, the number of points common to c and either the positive half-tangent $p+$ or the negative half-tangent $p-$ will remain unchanged except when P passes a double point or an inflectional point, or if the tangent p passes a double tangent. In the first two cases a common point S of c and p will go from $p+$ to $p-$,

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so that one point on $p+$ is being lost and one point on $p-$ is gained. When p passes a double tangent of the type E_1 (resp. I_1), two points S are gained (resp. lost) on $p+$ by the one passing, and two points are lost (resp. gained) on $p-$ by the other passing. For the type E_2 (resp. I_2) two points S are gained (resp. lost) on $p+$ by each passing, and for the type E_3 (resp. I_3) two points are lost (resp. gained) on $p-$ by each passing. These statements are intuitively clear, but exact proofs can of course be given.

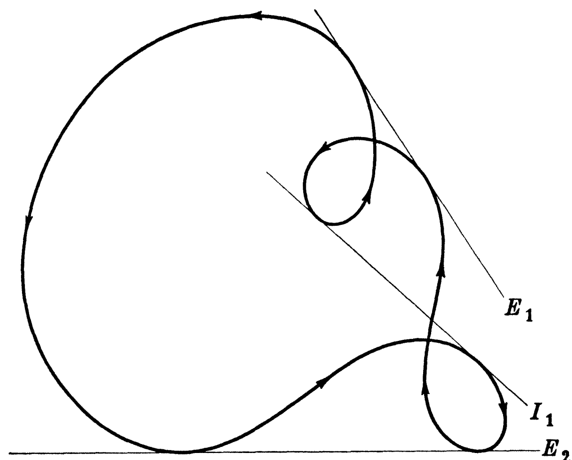

Fig. 1.

3.

Now, let the curve c have d double points, $2i$ inflectional points, t_1, t_2, t_3, exterior double tangents of the types E_1, E_2, E_3, respectively, and s_1, s_2, s_3 interior double tangents of the types I_1, I_2, I_3, respectively. From the above remarks it follows that when P traverses the curve c once, $2t_1+4t_2$ points S are *gained* on the positive half-tangent $p+$ while $2s_1+4s_2+2d+2i$ points S are *lost* on the same half-tangent. This gives the equation

$$2t_1+4t_2 = 2s_1+4s_2+2d+2i .$$

If we consider the negative half-tangent $p-$, we obtain the analogous equation

$$2t_1+4t_3 = 2s_1+4s_3+2d+2i .$$

Addition of these equations gives

$$4(t_1+t_2+t_3)-4(s_1+s_2+s_3) = 4d+4i$$

or

(1) $$t-s = d+i ,$$

where $t=t_1+t_2+t_3$ is the *total number* of exterior double tangents and $s=s_1+s_2+s_3$ the *total number* of interior double tangents. Thus we have shown

THEOREM 1. *For the closed curve c, the difference between the numbers of exterior and interior double tangents is equal to the sum of the number of double points and half the number of inflectional points. Hence, the number of exterior double tangents is greater than or equal to this sum. Equality holds if and only if c has no interior double tangents.*

(The figure illustrates an example, where $t_1=1$, $t_2=t_3=2$, $s_1=2$, $s_2=s_3=0$, $d=2$, $i=1$ and thus $t-s=d+i=3$.)

From (1) it follows that the total number of exterior *and* interior double tangents $t+s$ has the *same parity* as the number $d+i$. For a closed curve of even order in the projective plane (and for $d=0$) this property has been established by A. Kneser [3].

4.

For curves without interior double tangents the equation (1) reduces to

$$t = d+i \,. \tag{2}$$

Consider in the projective plane an algebraic or non-algebraic curve c_4 of order 4 with at least one double tangent r. The line r has only the points of contact in common with the curve. This implies that there exist lines in the neighbourhood of r which have not point in common with c_4, and hence the curve may be considered as placed in an affine plane. Moreover, it follows that the double tangents to c_4 are exterior. Thus, any curve c_4 of this type has the minimum number of double tangents, and the equation (2) is satisfied.

For these curves of order 4 the equation (2) was first stated by C. Juel [2, p. 161]. It may be noted that if a curve of order 4 has no double tangent the equation (2) does not hold. Juel has shown (l.c.) that for a curve of order 4 *without* double tangents, either $d=i=2$ or $d=3$, $i=1$, and hence in both cases $d+i=4$, whereas $t=0$. Furthermore, any line in the plane intersects the curve in at least two points, and the curve cannot be placed in an affine plane.

5.

If the curve c has no interior double tangents, and if, in addition, c is *locally convex*, i.e., has no inflectional points, we have

THEOREM 2. *A locally convex curve c with the minimum number of double tangents has equally many double points and double tangents, that is, $t=d$.*

A special class of locally convex curves for which $t=d$ has been investigated in a recent paper [1].

6.

Finally we shall establish a connection between the number of double tangents for the curve c and the so-called *rotation number*. If the tangent p to c turns an angle $2\pi\nu$ while the point of contact traverses the curve once, the number ν is called the rotation number of the curve. H. Whitney [4] has shown that if $\nu=0$, the curve has at least one double point, and if $\nu \geqq 1$, the curve has at least $\nu-1$ double points. Since a curve with rotation number $\nu=0$ has at least two inflectional points we get from the above theorems

THEOREM 3. *A curve c with rotation number $\nu=0$ has at least two double tangents.*

A curve c with rotation number $\nu \geqq 1$, has at least $\nu-1$ or ν double tangents according as the curve is locally convex or not.

The lemniscate is an example of a curve with the minimum number of double tangents $t=2$ and where $\nu=0$. In [1] it is shown how to construct a locally convex curve with rotation number $\nu \geqq 1$ and with exactly $\nu-1$ double points (see [1, p. 51 and fig. 2]). This curve has also the minimum number of double tangents.

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