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# THE FUBINI THEOREM AND CONVOLUTION FORMULA FOR REGULAR MEASURES

KAREL DELEEUW[1]

## 1. Introduction.

Let $G$ be a compact abelian group; $\lambda$ and $\mu$ finite regular Borel measures on $G$. (For all measure theoretic terminology, see [3].) Their convolution $\lambda * \mu$ is a finite regular Borel measure on $G$ which can be defined in two equivalent ways (see [4].) In the first definition, for each Borel subset $D$ of $G$, $\lambda * \mu(D)$ is defined to be $\lambda \otimes \mu(E)$, where $E$ is the Borel subset $\{(x,y):\ x+y \in D\}$ of $G \times G$ and $\lambda \otimes \mu$ is the unique finite regular Borel measure on $G \times G$ satisfying

$$\int_{G \times G} g d(\lambda \otimes \mu) = \int_G \left( \int_G g(x,y)\, d\lambda(x) \right) d\mu(y)$$

for all continuous functions $g$ on $G \times G$. In the second definition, $\lambda * \mu$ is taken to be the unique finite regular Borel measure on $G$ satisfying

$$\int_G f d(\lambda * \mu) = \int_G \left( \int_G f(x+y)\, d\lambda(x) \right) d\mu(y)$$

for all continuous functions $f$ on $G$.

From either of these definitions it is simple to deduce (cf. [2, p. 53]) that if $D$ is a Baire subset of $G$,

$$y \to \lambda(-y+D) \tag{1.1}$$

will be a Baire function on $G$ and

$$\lambda * \mu(D) = \int_G \lambda(-y+D)\, d\mu(y) . \tag{1.2}$$

Although the corresponding result for Borel subsets of $G$ is probably widely believed, we have not been able to find a proof in the literature.[2]

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[2] Added in proof: This result for Borel sets has been established in M. Heble and M. Rosenblatt, *Idempotent measures on a compact topological semigroup*, Proc. Amer. Math. Soc. 14 (1963), 177–184.

(For example, the formula (1.2), for $D$ Borel, is stated in [4]. No assertion about the measurability of (1.1) is made and no proof is given.)

The purpose of this note is to establish a Fubini theorem for regular measures from which the Borel measurability of (1.1) and the equality (1.2) for Borel $D$ will follow. For simplicity we restrict ourselves to compact spaces and bounded functions, the extension to locally compact spaces and integrable functions being routine.

## 2. The Fubini theorem.

In this section $X$ and $Y$ are compact Hausdorff spaces and $\lambda$ and $\mu$ finite regular complex Borel measures on $X$ and $Y$ respectively. By the Riesz representation theorem there is a unique finite regular complex Borel measure $\lambda \otimes \mu$ on $X \times Y$ satisfying

$$\int_{X \times Y} g d(\lambda \otimes \mu) = \int_Y \left( \int_X g(x,y) d\lambda(x) \right) d\mu(y)$$

for all continuous functions $g$ on $X \times Y$.

$\lambda \otimes \mu$ is an extension of the usual product measure $\lambda \times \mu$ (see [3]), which is the unique complex measure on the $\sigma$-algebra generated by sets of the form $A \times B$, $A$ Borel in $X$, $B$ Borel in $Y$, satisfying

$$\lambda \times \mu(A \times B) = \lambda(A)\mu(B)$$

for such sets. The reason that the results of [3] do not directly yield the Fubini theorem we prove, is that this $\sigma$-algebra (and even its completion with respect to $\lambda \times \mu$) may not contain all the Borel sets of $X \times Y$ if $X$ and $Y$ are not metrizable.

THEOREM 1. *Let $h$ be a bounded Borel function on $X \times Y$. Then*

$$y \to \int_X h(x,y)\, d\lambda(x) \tag{2.1}$$

*is a Borel function on $Y$ and*

$$\int_{X \times Y} h d(\lambda \otimes \mu) = \int_Y \left( \int_X h(x,y) d\lambda(x) \right) d\mu(y) . \tag{2.2}$$

Before we proceed to the proof of this result we should point out that the Fubini theorem of [1] shows that (2.1) is measurable with respect to the completion of $\mu$ and that the equality (2.2) holds. Thus our only original contribution here is the proof of the Borel measurability of (2.1). However for completeness we also establish the equality (2.2).

The proof of Theorem 1 proceeds by a sequence of lemmas. Because of the Jordan decomposition, we may assume that $\lambda$ and $\mu$ are non-negative measures. We further assume $\lambda$ and $\mu$ normalized to have total mass 1.

We shall denote by $C(X)$ and $C(X\times Y)$ the spaces of real valued continuous functions on $X$ and $X\times Y$ respectively. If $y$ is an element of $Y$, for each subset $E$ of $X\times Y$, the slice $\{x\colon\ x\in X,\ (x,y)\in E\}$ will be denoted by $E_y$. And the regular Borel measure $\lambda_y$ is defined on $X\times Y$ by $\lambda_y(E)=\lambda(E_y)$, all $E$ Borel in $X\times Y$.

We shall denote the characteristic function of any set $E$ by $\chi_E$.

LEMMA 1. *Let $E$ be a closed subset of $X\times Y$. Then, for each $y$ in $Y$, $\lambda(E_y)$ is equal to*

$$\inf\left\{\int_{X\times Y} g\,d\lambda_y\colon\ g\in C(X\times Y),\ \chi_E\leqq g\leqq 1\right\}. \tag{2.3}$$

PROOF. Since $\lambda_y$ is non-negative, it is clear that (2.3) is no smaller than $\lambda_y(E)$, which equals $\lambda(E_y)$. For the reverse inequality, choose any $\varepsilon>0$. By the regularity of $\lambda$ and the fact that $E_y$ is a closed subset of $X$, there is (see [3, p. 248]) a function $f$ in $C(X)$ satisfying $\chi_{E_y}\leqq f\leqq 1$ and

$$\int_X f\,d\lambda < \lambda(E_y)+\varepsilon . \tag{2.4}$$

Define the function $h$ on $(X\times\{y\})\cup E$ by $h(x,y)=f(x)$ and $h\equiv 1$ on $E$. $h$ is continuous and thus by the Tietze extension theorem has a continuous extension $g$ mapping all of $X\times Y$ into the interval $[0,1]$. The function $g$ will satisfy $\chi_E\leqq g\leqq 1$ and

$$\int_{X\times Y} g\,d\lambda_y = \int_X f\,d\lambda .$$

So by (2.4) and the fact that $\varepsilon$ was arbitrary, (2.3) can be no larger than $\lambda(E_y)$. This completes the proof of the lemma.

LEMMA 2. *Let $E$ be a Borel subset of $X\times Y$. Then the function*

$$y\to\int_X \chi_E(x,y)\,d\lambda(x) \tag{2.5}$$

*is Borel on $Y$.*

PROOF. Let us assume first that $E$ is closed. For each $g$ in $C(X\times Y)$, define $F_g$ on $X$ by

$$F_g(y) = \int_{X \times Y} g\, d\lambda_y\,.$$

Each $F_g$ is continuous on $Y$. By Lemma 1, since $E$ is assumed closed, the function (2.5) is equal to

$$\inf\{F_g\colon\ g \in C(X \times Y),\ \chi_E \leqq g \leqq 1\}\,,$$

and is thus Borel, being the infimum of a collection of continuous functions. Now let $\mathscr{B}$ be the collection of all Borel subsets $E$ of $X \times Y$ for which (2.5) is Borel. $\mathscr{B}$ is clearly closed under disjoint unions and proper differences, and by the monotone convergence theorem, is a monotone class in the sense of [3, p. 26]. We know that $\mathscr{B}$ contains all closed subsets of $X \times Y$, so by Theorem B, p. 27 and Theorem F, p. 223 of [3], $\mathscr{B}$ contains all of the Borel subsets of $X \times Y$.

Since each bounded Borel function on $X \times Y$ is a uniform limit of linear combinations of $\chi_E$ for $E$ Borel, the first assertion of Theorem 1 is an immediate consequence of Lemma 2.

We now proceed to the proof of the second assertion. The reasoning of the preceding paragraph shows that it suffices to establish the equality (2.2) for $h$ the characteristic function of a Borel subset $E$ of $X \times Y$. For such a set $E$ we define $\lambda\cdot\mu(E)$ by

$$\lambda\cdot\mu(E) = \int_Y \left(\int_X \chi_E(x,y)\, d\lambda(x)\right) d\mu(y)\,.$$

This definition is justified by Lemma 2.

To complete the proof of Theorem 1 it remains to show that $\lambda\cdot\mu(E) = \lambda \otimes \mu(E)$ for all Borel sets $E$.

LEMMA 3. *Let $E$ be a closed subset of $X \times Y$. Then $\lambda\cdot\mu(E) \leqq \lambda \otimes \mu(E)$.*

PROOF. Choose any $\varepsilon > 0$. Since $E$ is closed and $\lambda \otimes \mu$ regular, there is a function $f$ in $C(X \times Y)$ satisfying $\chi_E \leqq f$ and

$$\int_{X \times Y} f d(\lambda \otimes \mu) < \lambda \otimes \mu(E) + \varepsilon\,.$$

Then

$$\lambda\cdot\mu(E) \leqq \int_Y \left(\int_X f(x,y)\, d\lambda(x)\right) d\mu(y) = \int_{X \times Y} f d(\lambda \otimes \mu) < \lambda \otimes \mu(E) + \varepsilon\,.$$

Since $\varepsilon$ was arbitrary, the lemma is proved.

Thus far we have used only the regularity of $\lambda$ and $\lambda \otimes \mu$; the following lemma makes use of the regularity of $\mu$.

LEMMA 4. *Let $E$ be a closed subset of $X \times Y$. Then $\lambda \otimes \mu(E) \leqq \lambda \cdot \mu(E)$.*

PROOF. By Lemma 2, $y \to \lambda(E_y)$ is Borel on $Y$. Choose any $\varepsilon > 0$. $\mu$ is regular, so by Lusin's theorem (see [3], p. 243), there is a closed subset $K$ of $Y$ so that $\mu(Y-K) < \varepsilon$ and $y \to \lambda(E_y)$ is continuous on $K$. For each $y$ in $K$, by Lemma 1, it is possible to find a function $g_y$ in $C(X \times Y)$ satisfying $\chi_E \leqq g_y \leqq 1$ and

$$\int_{X \times Y} g_y d\lambda_y < \lambda(E_y) + \varepsilon . \tag{2.6}$$

Let $U_y$ be a neighborhood of $y$ in $K$ so that, for each $t$ in $U_y$,

$$\int_{X \times Y} g_y d\lambda_t < \lambda(E_t) + \varepsilon . \tag{2.7}$$

Such a neighborhood exists because of (2.6) and the fact that both sides of the inequality (2.7) are continuous for $t$ in $K$. By compactness, there are $y_1, \ldots, y_n$ in $K$ so that $K = U_{y_1} \cup \ldots \cup U_{y_n}$. Let $g = \inf\{g_{y_1}, \ldots, g_{y_n}\}$, so $\chi_E \leqq g \leqq 1$ and

$$\int_{X \times Y} g d\lambda_t < \lambda(E_t) + \varepsilon$$

for all $t$ in $K$. Then

$$\begin{aligned}
\lambda \cdot \mu(E) &= \int_Y \lambda(E_t)\, d\mu(t) \\
&\geqq \int_K \lambda(E_t)\, d\mu(t) \\
&> \int_K \left( \int_{X \times Y} g d\lambda_t - \varepsilon \right) d\mu(t) \\
&\geqq \int_K \left( \int_{X \times Y} g d\lambda_t \right) d\mu(t) - \varepsilon \\
&\geqq \int_Y \left( \int_{X \times Y} g d\lambda_t \right) d\mu(t) - 2\varepsilon \\
&= \int_{X \times Y} g d(\lambda \otimes \mu) - 2\varepsilon \geqq \lambda \otimes \mu(E) - 2\varepsilon .
\end{aligned}$$

Since $\varepsilon$ was arbitrary, this completes the proof.

We are now able to finish the proof of Theorem 1. We had observed that it remained only to show that

$$\lambda \cdot \mu(E) = \lambda \otimes \mu(E) \tag{2.8}$$

for all Borel $E$ in $X \times Y$. We argue as in Lemma 2. Let $\mathscr{B}$ be the collection of all Borel subsets $E$ of $X \times Y$ for which (2.8) holds. $\mathscr{B}$ is clearly closed under disjoint unions and proper differences, and by the monotone convergence theorem is a monotone class in the sense of [3]. We know, because of Lemmas 3 and 4, that $\mathscr{B}$ contains all closed subsets of $X \times Y$, so by Theorem B, p. 27 and Theorem F, p. 223 of [3], $\mathscr{B}$ contains all Borel subsets of $X \times Y$.

## 3. The convolution formula.

The validity of the convolution formula is an easy consequence of Theorem 1.

THEOREM 2. *Let $G$ be a compact abelian group, $\lambda$ and $\mu$ finite regular complex Borel measures on $G$. Then, for each Borel subset $D$ of $G$,*

$$y \to \lambda(-y+D) \tag{3.1}$$

*is a Borel function on $G$ and*

$$\lambda * \mu(D) = \int_G \lambda(-y+D)\, d\mu(y) . \tag{3.2}$$

PROOF. Let $E$ be the subset $\{(x,y)\colon x+y \in D\}$ of $G \times G$. $E$ is Borel since the mapping $(x,y) \to x+y$ of $G \times G$ into $G$ is continuous. Let $h$ be the characteristic function of $E$. Then, since

$$\int_X h(x,y)\, d\lambda(x) = \lambda(-y+D) ,$$

the Borel measurability of (3.1) follows from the first assertion of Theorem 1. Finally (3.2) is a consequence of the second assertion of Theorem 1 and the fact that $\lambda * \mu(D) = \lambda \otimes \mu(E)$.

The analogous result for non-commutative groups and even semigroups is of course equally valid.

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STANFORD UNIVERSITY, CALIF., U.S.A.