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THE RESOLUTIONS OF THE IDENTITY FOR SUMS AND PRODUCTS OF COMMUTING SPECTRAL OPERATORS

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1. Introduction.

Dunford [1] proved that if T_1 and T_2 are commuting spectral operators on a reflexive complex Banach space with the resolutions of the identity $E(\cdot)$ and $F(\cdot)$, respectively, and if the Boolean algebra of projections generated by $E(\cdot)$ and $F(\cdot)$ is bounded, then T_1+T_2 and T_1T_2 are spectral operators.

Later this has been improved by Foguel [3] who proved that the same result holds on a weakly complete complex Banach space and that the resolution of the identity $G(\cdot)$ of T_1+T_2 is determined on Borel sets α with $G(\text{boundary of } \alpha)x=0$ by

$$G(\alpha)x = \int E(\alpha-\mu)F(d\mu)x ,$$

where the integral exists in the sense of Riemann. For the product T_1T_2 a similar formula was obtained.

The purpose of this note is to prove that the restriction on α can be removed by defining convolutions of $E(\cdot)$ and $F(\cdot)$ differently (see theorem 2).

2. Notation.

If X is a complex Banach space, a spectral measure is a set function $E(\cdot)$, defined on the set $\mathscr{B}$ of all Borel sets in the complex plane π, whose values are projections on X, satisfying:

1. If $\sigma,\delta \in \mathscr{B}$, then $E(\sigma)E(\delta)=E(\sigma\cap\delta)$.
2. $E(\pi)=I$, $E(\emptyset)=0$, where $\emptyset$ denotes the empty set.
3. The vector valued set function $E(\cdot)x$ is countably additive for every $x \in X$.
4. There exists a compact set $\varkappa \neq \emptyset$ such that $E(\sigma)=0$ if $\sigma\cap\varkappa=\emptyset$.

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From the principle of uniform boundedness it follows that if $E(\cdot)$ is a spectral measure, then there exists a constant K such that $\|E(\alpha)\| \leqq K$ for every $\alpha \in \mathscr{B}$.

A bounded operator T is a spectral operator with resolution of the identity $E(\cdot)$, if it satisfies:

a) $E(\cdot)$ is a spectral measure.

b) If $\alpha \in \mathscr{B}$, then $TE(\alpha) = E(\alpha)T$.

c) If $T \mid E(\alpha)X$ is the restriction of T to the subspace $E(\alpha)X$, then

$$\sigma(T \mid E(\alpha)X) \subset \bar{\alpha} \qquad \text{for every } \alpha \in \mathscr{B},$$

where $\sigma(A)$ denotes the spectrum af A.

If T is a spectral operator, then the operator $S = \int \lambda E(d\lambda)$ is called its scalar part and $N = T - S$ its radical. The operators T, S, N and $E(\alpha)$, $\alpha \in \mathscr{B}$, commute. The operator T is called a scalar operator if $T = S$.

3. The product measure of two commuting spectral measures.

In this section we shall prove the existence of a product measure of two commuting spectral measures. We first state some lemmas without proof.

LEMMA 1. *Any bounded complex measure defined on $\mathscr{B}$ is a regular measure.*

For a proof, see [2, pp. 170].

LEMMA 2. *If $E(\cdot)$ is a spectral measure, then for every $x \in X$ there exists a positive, finite, and regular measure ν_x defined on the Borel sets in the complex plane such that*

$$\lim_{\nu_x(\alpha) \to 0} \|E(\alpha)x\| = 0 .$$

For a proof, see [2, pp. 320–321].

COROLLARY. *If $E(\cdot)$ is a spectral measure, then for every $x \in X$ the set function $E(\cdot)x$ is a regular vector valued measure.*

By $\mathscr{C}$ we denote the algebra generated by all sets in $\pi \times \pi$ of the form $\sigma \times \delta$ with $\sigma \in \mathscr{B}$, $\delta \in \mathscr{B}$, and by $\mathscr{B}^*$ the σ-algebra generated by $\mathscr{C}$.

Let $E(\cdot)$ and $F(\cdot)$ be commuting spectral measures, that is, $E(\sigma)F(\delta) = F(\delta)E(\sigma)$ for any two Borel sets σ, δ. Every $\alpha \in \mathscr{C}$ may be represented in the following way:

$$\alpha = \bigcup_{i=1}^{n} (\sigma_i \times \delta_i)$$

with $\sigma_i \in \mathscr{B}$, $\delta_i \in \mathscr{B}$, $i=1,2,\ldots,n$, and with $(\sigma_i \times \delta_i) \cap (\sigma_j \times \delta_j) = \emptyset$ for $i \neq j$. If $\alpha \in \mathscr{C}$ is written in this way, we define

$$P_0(\alpha) = \sum_{i=1}^{n} E(\sigma_i) F(\delta_i) .$$

It is easily seen that this definition is independent of the representation of α and that the set function $P_0(\cdot)$ is finitely additive on $\mathscr{C}$ and satisfies

$$P_0(\alpha \cap \beta) = P_0(\alpha) P_0(\beta)$$

for any two sets α and β in $\mathscr{C}$. Also $P_0(\pi \times \pi) = I$, and there exists a non-empty compact set $\varkappa_1 \times \varkappa_2$ such that $\alpha \cap (\varkappa_1 \times \varkappa_2) = \emptyset$ implies $P_0(\alpha) = 0$.

These notations will be used throughout this and the following section.

LEMMA 3. *If there exists a constant K such that $\|P_0(\alpha)\| \leqq K$ for every $\alpha \in \mathscr{C}$, that is, if the Boolean algebra of projections generated by $E(\cdot)$ and $F(\cdot)$ is bounded, then for every $x \in X$ the set function $P_0(\cdot)x$ is regular and countably additive.*

PROOF. Let $\sigma \in \mathscr{B}$ and $\delta \in \mathscr{B}$. From the corollary it follows that there exist two closed sets $\varkappa_1$ and $\varkappa_2$ and two open sets γ_1 and γ_2 with $\varkappa_1 \subset \sigma \subset \gamma_1$ and $\varkappa_2 \subset \delta \subset \gamma_2$ such that

$$\sup_{\alpha \subset \gamma_1 \setminus \varkappa_1} \|E(\alpha)x\| < \varepsilon/2K^2 \quad \text{and} \quad \sup_{\alpha \subset \gamma_2 \setminus \varkappa_2} \|F(\alpha)x\| < \varepsilon/2K^2 .$$

It follows that $\varkappa_1 \times \varkappa_2 \subset \sigma \times \delta \subset \gamma_1 \times \gamma_2$ and that for $\alpha \subset \gamma_1 \times \gamma_2 \setminus \varkappa_1 \times \varkappa_2$ and $\alpha \in \mathscr{C}$ we have

$$\begin{aligned}
\|P_0(\alpha)x\| &= \|P_0(\gamma_1 \times \gamma_2 \setminus \varkappa_1 \times \varkappa_2) P_0(\alpha)x\| \\
&\leqq \|P_0(\alpha)\| \, \|P_0(\gamma_1 \times \gamma_2 \setminus \varkappa_1 \times \varkappa_2)x\| \\
&\leqq K \|P_0\big((\gamma_1 \setminus \varkappa_1) \times \gamma_2 \cup (\gamma_1 \cap \varkappa_1) \times (\gamma_2 \setminus \varkappa_2)\big)x\| \\
&= K \|P_0\big((\gamma_1 \setminus \varkappa_1) \times \gamma_2\big)x + P_0\big(\varkappa_1 \times (\gamma_2 \setminus \varkappa_2)\big)x\| \\
&\leqq K\big(\|E(\gamma_1 \setminus \varkappa_1) F(\gamma_2)x\| + \|E(\varkappa_1) F(\gamma_2 \setminus \varkappa_2)x\|\big) \\
&< K\big(\|F(\gamma_2)\| \varepsilon/2K^2 + \|E(\varkappa_1)\| \varepsilon/2K^2\big) \leqq \varepsilon ,
\end{aligned}$$

so that $P_0(\cdot)x$ is regular.

In order to prove that $P_0(\cdot)x$ is countably additive, let $\{\sigma_n\}$ be a decreasing sequence of sets from $\mathscr{C}$ whose intersection is void, and let $\varepsilon > 0$ be given. Since $P_0(\cdot)x$ is regular, there exists a closed set $\varkappa_1$, which we may assume compact and in $\mathscr{C}$, such that

$$\varkappa_1 \subset \sigma_1 \quad \text{and} \quad \|P_0(\alpha)x\| < \tfrac{1}{2}\varepsilon \quad \text{for} \quad \alpha \subset \sigma_1 \setminus \varkappa_1, \ \alpha \in \mathscr{C} .$$

By induction we now construct a sequence of closed sets $\{\varkappa_n\}$ from $\mathscr{C}$ such that

$$\varkappa_n \subset \sigma_n \cap \varkappa_{n-1} ,$$
$$\|P_0(\alpha)x\| < \varepsilon/2^n \quad \text{for} \quad \alpha \subset (\sigma_n \cap \varkappa_{n-1})\setminus\varkappa_n .$$

Since $\varkappa_n \subset \sigma_n$ and $\sigma_n \searrow \emptyset$ it follows that $\varkappa_n \searrow \emptyset$; but then there exists a number n_0 such that $\varkappa_n = \emptyset$ for $n > n_0$, that is, $P_0(\varkappa_n)x = 0$ for $n > n_0$. From the identity

$$\sigma_n \setminus \varkappa_n = (\sigma_n \setminus \varkappa_1) \cup \bigcup_{i=1}^{n-1} (\sigma_n \cap \varkappa_i)\setminus \varkappa_{i+1} ,$$

where the union is disjoint and $\sigma_n \setminus \varkappa_1 \subset \sigma_1 \setminus \varkappa_1$, we then get

$$\begin{aligned} \|P_0(\sigma_n \setminus \varkappa_n)x\| &= \left\| P_0(\sigma_n \setminus \varkappa_1)x + \sum_{i=1}^{n-1} P_0\big((\sigma_n \cap \varkappa_i)\setminus \varkappa_{i+1}\big)x \right\| \\ &\leqq \|P_0(\sigma_n \setminus \varkappa_1)x\| + \sum_{i=1}^{n-1} \|P_0\big((\sigma_n \cap \varkappa_i)\setminus \varkappa_{i+1}\big)x\| \\ &\leqq \tfrac{1}{2}\varepsilon + \sum_{i=1}^{n-1} \varepsilon/2^{i+1} < \varepsilon . \end{aligned}$$

Hence, for $n > n_0$, we have

$$\|P_0(\sigma_n)x\| < \varepsilon ,$$

which proves the countable additivity of $P_0(\cdot)x$.

The following theorem now follows from [4, theorem 2.14 and corollary 2.17].

THEOREM 1. *If X is a weakly complete complex Banach space and $E(\cdot)$ and $F(\cdot)$ are commuting spectral measures whose values are projections on X, then there exists a unique set function $P(\cdot)$, defined on the σ-algebra $\mathscr{B}^*$ generated by $\mathscr{C}$, satisfying:*

1. *$P(\cdot)$ is an extension of $P_0(\cdot)$.*
2. *$P(\sigma \cap \delta) = P(\sigma)P(\delta)$ for every two sets σ, δ in $\mathscr{B}^*$.*
3. *The vector valued set function $P(\cdot)x$ is countably additive for every $x \in X$.*

The set function $P(\cdot)$ is called the *product measure* of $E(\cdot)$ and $F(\cdot)$.

4. Sums and products of commuting spectral operators.

We are now able to prove our main theorem.

THEOREM 2. *Let T_1 and T_2 be commuting spectral operators on a weakly complete complex Banach space X with the resolutions of the identity $E(\cdot)$ and $F(\cdot)$, respectively, and let the Boolean algebra of projections generated by*

$E(\cdot)$ and $F(\cdot)$ be bounded. Then T_1+T_2 and T_1T_2 are spectral operators, and their resolutions of the identity, $G(\cdot)$ and $H(\cdot)$ respectively, are determined by

$$G(\alpha) = P(\{(\mu,\lambda) \mid \mu+\lambda \in \alpha\}), \qquad \alpha \in \mathscr{B},$$
$$H(\alpha) = P(\{\mu,\lambda) \mid \mu\lambda \in \alpha\}), \qquad \alpha \in \mathscr{B},$$

where $P(\cdot)$ is the product measure of $E(\cdot)$ and $F(\cdot)$.

Remark. The product T_1T_2 can be dealt with in the same manner as the sum T_1+T_2. Therefore only the sum is considered in the following proof.

Proof. It follows from [1, theorem 5], that $E(\cdot)$ and $F(\cdot)$ are commuting spectral measures. Next we have from theorem 1 that the product measure $P(\cdot)$ exists and that the set function $G(\cdot)$ is a spectral measure. It is sufficient to prove that T_1+T_2 is a spectral operator, in case T_1 and T_2 are of scalar type, i.e.

$$T_1 = \int \lambda E(d\lambda) \qquad \text{and} \qquad T_2 = \int \lambda F(d\lambda)\,.$$

Thus, under this assumption we have to prove that

$$\int \lambda G(d\lambda) = \int \lambda E(d\lambda) + \int \lambda F(d\lambda)\,.$$

Let $\varepsilon > 0$ be given and let $\varkappa_0$ be a compact set in the complex plane π such that $G(\sigma)=0$ if $\sigma \cap \varkappa_0 = \emptyset$. Now, let

$$(\alpha_i)_{i=1}^{N}, \qquad (\delta_j)_{j=1}^{N_1}, \qquad (\sigma_k)_{k=1}^{N_2}$$

be partitions of $\varkappa_0$, $\sigma(T_1)$, and $\sigma(T_2)$, respectively, in Borel sets such that $\operatorname{diam} \alpha_i < \frac{1}{2}\varepsilon$ for $i=1,2,\ldots,N$, and correspondingly for the two other partitions, and choose $\lambda_i \in \alpha_i$, $\mu_j \in \delta_j$, $\nu_k \in \sigma_k$. By [1, theorem 7], we then have

$$\left\| \int \lambda G(d\lambda) - \sum_{i=1}^{N} \lambda_i G(\alpha_i) \right\| \leqq 4K_0\varepsilon\,, \tag{1}$$

$$\left\| T_1 - \sum_{j=1}^{N_1} \mu_j E(\delta_j) \right\| = \left\| \int \lambda E(d\lambda) - \sum_{j=1}^{N_1} \mu_j E(\delta_j) \right\| \leqq 4K_1\varepsilon\,, \tag{2}$$

$$\left\| T_2 - \sum_{k=1}^{N_2} \nu_k F(\sigma_k) \right\| = \left\| \int \lambda F(d\lambda) - \sum_{k=1}^{N_2} \nu_k F(\sigma_k) \right\| \leqq 4K_2\varepsilon\,, \tag{3}$$

where

$$K_0 = \sup_{\alpha\in\mathscr{B}} \|G(\alpha)\|, \qquad K_1 = \sup_{\alpha\in\mathscr{B}} \|E(\alpha)\|, \qquad K_2 = \sup_{\alpha\in\mathscr{B}} \|F(\alpha)\|\,.$$

Let now x be a fixed element in X. By the corollary in section 3 there exists for every $i=1,2,\ldots,N$ a compact set $\varkappa_i \subset \alpha_i$ such that

$$\|G(\alpha_i)x - G(\varkappa_i)x\| < \varepsilon/N , \tag{4}$$

from which it follows that

$$\left\| \sum_{i=1}^{N} \lambda_i G(\alpha_i)x - \sum_{i=1}^{N} \lambda_i G(\varkappa_i)x \right\| \leqq \varepsilon M , \tag{5}$$

where $M = \sup_{\lambda \in \varkappa_0} |\lambda|$. Furthermore, it follows from (2), (3), and (4) that

$$\left\| \sum_{i,j} \mu_j E(\delta_j)[G(\alpha_i)x - G(\varkappa_i)x] \right\| \leqq (\|T_1\| + 4K_1\varepsilon)\varepsilon , \tag{6}$$

$$\left\| \sum_{i,k} \nu_k F(\sigma_k)[G(\alpha_i)x - G(\varkappa_i)x] \right\| \leqq (\|T_2\| + 4K_2\varepsilon)\varepsilon . \tag{7}$$

For each $n = 0, 1, 2, \ldots$ and each pair of integers p, q, let $\beta_n[p,q]$ denote the square of all z with

$$2^{-n}p < \operatorname{Re} z \leqq 2^{-n}(p+1), \qquad 2^{-n}q < \operatorname{Im} z \leqq 2^{-n}(q+1) .$$

With this notation we have for every closed subset $\varkappa$ of π that

$$\bigcup_{p,q} \beta_n[p,q] \times (\varkappa - \beta_n[p,q]) \searrow \{(\mu,\lambda) \mid \mu + \lambda \in \varkappa\} \quad \text{as } n \to \infty ,$$

and hence

$$\begin{aligned} G(\varkappa)x &= \lim_{n\to\infty} P\left(\bigcup_{p,q} \beta_n[p,q] \times (\varkappa - \beta_n[p,q]) \right) x \\ &= \lim_{n\to\infty} \sum_{p,q} E(\beta_n[p,q]) F(\varkappa - \beta_n[p,q]) x . \end{aligned}$$

From this we can conclude that there exists an integer n_0 such that

$$\left\| G(\varkappa_i)x - \sum_{p,q} E(\beta_n[p,q]) F(\varkappa_i - \beta_n[p,q]) x \right\| \leqq \varepsilon/N$$

for all $n > n_0$ and $i = 1, 2, \ldots, N$. If we choose $n > n_0$ such that

$$\operatorname{diam}(\beta_n[p,q]) = 2^{-n}\sqrt{2} < \min\left(\tfrac{1}{2}\varepsilon,\ \tfrac{1}{2} \min_{j \neq k} \operatorname{dist}(\varkappa_j, \varkappa_k)\right) ,$$

and write β_{pq} instead of $\beta_n[p,q]$, we have

$$\left\| \sum_i \lambda_i \left(G(\varkappa_i)x - \sum_{p,q} E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x \right) \right\| \leqq \varepsilon M , \tag{8}$$

$$\left\| \sum_j \mu_j E(\delta_j) \left[\sum_i \left(\sum_{p,q} E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x - G(\varkappa_i)x \right) \right] \right\| \leqq (\|T_1\| + 4K_1\varepsilon)\varepsilon , \tag{9}$$

$$\left\| \sum_k \nu_k F(\sigma_k) \left[\sum_i \left(\sum_{p,q} E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x - G(\varkappa_i)x \right) \right] \right\| \leqq (\|T_2\| + 4K_2\varepsilon)\varepsilon . \tag{10}$$

Choose now $\lambda_{pq} \in \beta_{pq}$. Then, for each bounded functional x^* in the dual space X^* of X we get

$$\begin{aligned}\Big| x^* \sum_{i,p,q} (\lambda_i - \lambda_{pq}) E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x - x^* \sum_k \nu_k F(\sigma_k) \sum_{i,p,q} E(\beta_{pq}) F((\varkappa_i - \beta_{pq}) x \Big| \\ \leqq \sum_{i,k,p,q} |\lambda_i - \lambda_{pq} - \nu_k| \cdot \big| x^* E(\beta_{pq}) F\big(\sigma_k \cap (\varkappa_i - \beta_{pq})\big) x \big| \\ \leqq 2\varepsilon \cdot \operatorname{var} x^* P(\cdot) x \leqq 8\varepsilon \|x\| \, \|x^*\| K ,\end{aligned}$$

where $K = \sup_{\alpha \in \mathscr{B}} \|P(\alpha)\|$. Since this inequality is valid for all $x^* \in X^*$, we have

$$\text{(11)} \quad \begin{aligned}\Big\| \sum_{i,p,q} (\lambda_i - \lambda_{pq}) E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x - \sum_k \nu_k F(\sigma_k) \sum_{i,p,q} E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x \Big\| \\ \leqq 8\varepsilon K \|x\| .\end{aligned}$$

In the same way we can prove that

$$\text{(12)} \quad \begin{aligned}\Big\| \sum_{i,p,q} \lambda_{pq} E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x - \sum_j \mu_j E(\delta_j) \sum_{i,p,q} E(\beta_{pq}) F(\varkappa_i - \beta_{pq}) x \Big\| \\ \leqq 4\varepsilon K \|x\| .\end{aligned}$$

If we now put

$$M_0 = 4(K_0 + K_1 + K_2)\|x\| + 2M + 2\|T_1\| + 2\|T_2\| + 8(K_1 + K_2) + 12K\|x\| ,$$

then M_0 is independent of ε, and using (1)–(3) and (5)–(12) we get, for $\varepsilon < 1$,

$$\Big\| \int \lambda G(d\lambda) x - \int \lambda E(d\lambda) x - \int \lambda F(d\lambda) x \Big\| \leqq \varepsilon M_0 .$$

Thus, we have proved that

$$\int \lambda G(d\lambda) x = \int \lambda E(d\lambda) x + \int \lambda F(d\lambda) x = T_1 x + T_2 x ,$$

but since all the integrals

$$\int \lambda G(d\lambda), \qquad \int \lambda E(d\lambda), \qquad \int \lambda F(d\lambda) ,$$

exist in the uniform topology, we also have

$$\int \lambda G(d\lambda) = \int \lambda E(d\lambda) + \int \lambda F(d\lambda) = T_1 + T_2 ,$$

that is, $G(\cdot)$ is the resolution of the identity of $T_1 + T_2$.

REFERENCES

1. N. Dunford, *Spectral operators*, Pacific J. Math. 4 (1954), 321–354.
2. N. Dunford and J. T. Schwartz, *Linear operators* I, New York, 1958.
3. S. R. Foguel, *Sums and products of commuting spectral operators*, Ark. Mat. 3 (1957), 449–461.
4. D. McGarvey, *Operators commuting with translating by one*, J. Math. Anal. Appl. 4 (1962), 366–410.

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